SL(2,R)-geometric phase space and (2+2)-dimensions

R. Flores[a], J. A. Nieto[b,a], J. Tellez[a], E. A. Leon[b], and E. R. Estrada[c]
[a]*Departamento de Investigación en Física de la Universidad de Sonora, 83000, Hermosillo Sonora, México,*
e-mail: rflorese@gauss.mat.uson.mx; jtellez@cajeme.cifus.uson.mx
[b]*Facultad de Ciencias Físico-Matemáticas de la Universidad Autónoma de Sinaloa, 80010, Culiacán Sinaloa, México.*
e-mail: niet@uas.edu.mx; janieto1@asu.edu; ealeon@uas.edu.mx
[c]*Instituto Tecnológico Superior de Eldorado, Eldorado, Sinaloa, Mexico,*
e-mail: profe.emmanuel@gmail.com

Received 23 November 2012; accepted 19 April 2013

We propose an alternative geometric mathematical structure for arbitrary phase space. The main guide in our approach is the hidden SL(2,R)-symmetry which acts on the phase space changing coordinates by momenta and *vice versa*. We show that the SL(2,R)-symmetry is implicit in any symplectic structure. We also prove that in any sensible physical theory based on the SL(2,R)-symmetry the signature of the flat target "spacetime" must be associated with either one-time and one-space or at least two-time and two-space coordinates. We discuss the consequences as well as possible applications of our approach on different physical scenarios.

Keywords: Symplectic geometry; constrained Hamiltonian systems; two time physics.

PACS: 04.20.Gz; 04.60.-Ds; 11.30.Ly

1. Introduction

The importance of the $SL(2,R)$-group in physics and mathematics, specially in string theory [1], two dimensional black holes [2] and conformal field theory [3-4], has been recognized for long time. Recently such a group structure has been considered as the key structure in the development of two-time physics (2t-physics) (see [5-7] and references therein). An interesting aspect is the relevance of the $SL(2,R)$-group in 2t-physics emerging from the Hamiltonian formalism of ordinary classical mechanics. In fact, the $SL(2,R)$-group acts on a phase space, rotating coordinates by momenta and *vice versa*. Requiring this symmetry for the constraint Hamiltonian system leads us to the conclusion that the flat target "spacetime" must have either a (1+1)-signature or at least a (2+2)-signature [8]. However, this result still requires a refined mathematical proof.

Specifically, we prove, in two alternative ways, that in a constraint Hamiltonian formalism, in which the groups $SL(2,R)$ and $SO(t,s)$ are symmetries of a classical system, the possible values for t-time and s-space are $t=1$ and $s=1$ or $t\geqq 2$ and $s\geqq 2$. In the process, we formalize an alternative geometric structure for the phase space based on the $SL(2,R)$-group.

As an application of our formalism, we develop the Dirac type equation in $(2+2)$-dimensions. We show that the $SL(2,R)$-group is relevant to understand such equation.

The structure of this paper is as follows. In Secs. 2 and 3, we develop the necessary steps to highlight the importance of the $SL(2,R)$-group in classical constraint Hamiltonian systems. In Sec. 4, we prove the main proposition mentioned above. In Sec. 5, we construct the Dirac type equation in $(2+2)$-dimensions. Finally, in Sec. 6 we make some additional comments.

2. Lagrange-Hamiltonian system

Let us consider the action

$$S[q]=\int dtL(q,\dot{q}), \tag{1}$$

where the Lagrangian $L=L(q,\dot{q})$ is a function of the q^i-coordinates and the corresponding velocities $\dot{q}^i\equiv dq^i/dt$, with $i,j=1,\ldots,n$.

The canonical momentum p_i conjugate to q^i is defined to be

$$p_i\equiv\frac{\partial L}{\partial\dot{q}^i}, \tag{2}$$

Thus the action (1) can be rewritten in the form

$$S[q,p]=\int dt(\dot{q}^ip_i-H_c), \tag{3}$$

where $H_c=H_c(q,p)$ is the canonical Hamiltonian,

$$H_c(q,p)\equiv\dot{q}^ip_i-L. \tag{4}$$

If one considers m first class Hamiltonian constraints $H_A(q,p)\approx 0$ (here the symbol "≈" means weakly equal to zero [9-11]), with $A=1,2...,m$, then the action (3) can be generalized as follows:

$$S[q,p]=\int dt(\dot{q}^ip_i-H_c-\lambda^AH_A). \tag{5}$$

Here, λ^A are arbitrary Lagrange multipliers.

The Poisson bracket for arbitrary functions $f(q,p)$ and $g(q,p)$ of the canonical variables q and p is defined as usual

$$\{f,g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}. \tag{6}$$

Using (6) we find that

$$\{q^i, q^j\} = 0,$$
$$\{q^i, p_j\} = \delta^i_j, \tag{7}$$
$$\{p_i, p_j\} = 0,$$

where the symbol δ^i_j denotes a Kronecker delta.

3. $SL(2,R)$-Hamiltonian system

It turns out that an alternative possibility to analyze the previous program has emerged in the context of 2t-physics [5-7] (see also Ref. 12 and references therein). The key point in this new approach is the realization that, since in the action (3) there is a hidden invariance $SL(2,R) \sim Sp(2,R) \sim SU(1,1)$, one may work in a unified canonical phase space of coordinates and momenta. Let us recall how such a hidden invariance emerges. Consider first the change of notation

$$q^i_1 \equiv q^i, \tag{8}$$

and

$$q^i_2 \equiv p^i. \tag{9}$$

These two expressions can be unified by introducing the object q^i_a, with $a = 1,2$. The next step is to rewrite (5) in terms of q^i_a rather than in terms of q^i and p^i. One finds that, up to a total derivative, the action (5) becomes [4] (see also Refs. 12 and 13)

$$S = \int_{t_i}^{t_f} dt \left(\frac{1}{2} J^{ab} \dot{q}^i_a q_{bi} - H(q^i_a) \right). \tag{10}$$

Here, $J^{ab} = -J^{ba}$, where $J^{12} = 1$ is the antisymmetric $SL(2,R)$-invariant density (some times denoted with the symbol ε^{ab}) and

$$H(q^i_a) = H_c + \lambda^A H_A. \tag{11}$$

According to Dirac's terminology in the constrained Hamiltonian systems formalism [14] (see also Refs. 9-11), (11) corresponds to a total Hamiltonian. From the action (10) one observes that, while the $SL(2,R)$-symmetry is hidden in (5), now in the first term of (10) it is manifest. Thus, it is natural to require the same $SL(2,R)$-symmetry for the total Hamiltonian $H(q^i_a)$.

Consider the usual Hamiltonian for a free non-relativistic point particle

$$H = \frac{p^i p^j \delta_{ij}}{2m} + V(q), \tag{12}$$

with $i = \{1,2,3\}$. According to the notation (8)-(9) we have

$$H = \frac{q^i_2 q^j_2 \delta_{ij}}{2m} + V(q_1). \tag{13}$$

It is evident from this expression that H in (13) is not $SL(2,R)$-invariant Hamiltonian. Thus, a Hamiltonian of the form (12) does not admit a $SL(2,R)$-invariant formulation. The same conclusion can be obtained by considering a Hamiltonian constraint $H = \lambda(p^i p_i + m^2)$ for the relativistic point particle, where in this case i runs from 0 to 3.

Thus, one finds that the simplest example of $SL(2,R)$-invariant Hamiltonian seems to be [5]

$$H = \frac{1}{2}\lambda^{ab} q^i_a q^j_b \eta_{ij}, \tag{14}$$

which can be understood as the Hamiltonian associated with a relativistic harmonic oscillator in a phase space. Here, we assume that $\lambda^{ab} = \lambda^{ba}$ is a set of Lagrange multipliers and $\eta_{ij} = \text{diag}(-1,-1,...,-1,1,...,1)$. Note that we are considering a signature of the form $n = t + s$, with t time-like and s space-like signature. The reason for this general choice is that the $SL(2,R)$-symmetry requires necessarily a target 'spacetime' with either $t = 1$ and $s = 1$ or $t \geqq 2$ and $s \geqq 2$ as we shall prove below.

Using (14) one sees that (10) can be written in the form

$$S = \frac{1}{2}\int_{t_i}^{t_f} dt \left(J^{ab} \dot{q}^i_a q^j_b \eta_{ij} - \lambda^{ab} H_{ab} \right), \tag{15}$$

where

$$H_{ab} = q^i_a q^j_b \eta_{ij}. \tag{16}$$

Of course $H_{ab} \approx 0$ is the constraint of the theory. Observe that the constraint $H_{ab} \approx 0$ is symmetric in the indices a and b, that is $H_{ab} = H_{ba}$.

Note that using the definitions (8) and (9) we can write the usual Poisson bracket (6), for arbitrary functions $f(q,p)$ and $g(q,p)$ of the canonical variables q and p as

$$\{f,g\} = J_{ab}\eta^{ij}\frac{\partial f}{\partial q^i_a}\frac{\partial g}{\partial q^j_b}. \tag{17}$$

Thus, from (17) one discovers the algebra

$$\{q^i_a, q^j_b\} = J_{ab}\eta^{ij}, \tag{18}$$

which is equivalent to (7).

Moreover, using (17) one finds that H_{ab} satisfies the $SL(2,R)$-algebra

$$\{H_{ab}, H_{cd}\} = J_{ac}H_{bd} + J_{ad}H_{bc} + J_{bc}H_{ad} + J_{bd}H_{ac}, \tag{19}$$

which shows that H_{ab} is a first class constraint. Explicitly, the nonvanishing brackets of the algebra (19) can be decomposed as

$$\{H_{11}, H_{22}\} = 4H_{12}, \tag{20}$$

$$\{H_{11}, H_{12}\} = 2H_{11}, \tag{21}$$

and

$$\{H_{12}, H_{22}\} = 2H_{22}. \tag{22}$$

By writing

$$S_3 = -\frac{1}{2}H_{12}, \qquad S_1 = \frac{1}{4}(H_{11} + H_{22})$$

and

$$S_2 = \frac{1}{4}(H_{11} - H_{22})$$

one finds that

$$\{S_1, S_2\} = S_3, \tag{23}$$

$$\{S_3, S_1\} = S_2, \tag{24}$$

and

$$\{S_2, S_3\} = -S_1, \tag{25}$$

which can be succinctly written as

$$\{S_\mu, S_\nu\} = \epsilon_{\mu\nu}{}^{\alpha} S_\alpha \tag{26}$$

or

$$\{S_{\mu\nu}, S_{\alpha\beta}\} = \eta_{\mu\alpha}S_{\nu\beta} - \eta_{\mu\beta}S_{\nu\alpha} + \eta_{\nu\beta}S_{\mu\alpha} - \eta_{\nu\alpha}S_{\mu\beta}. \tag{27}$$

Here

$$\eta_{\mu\nu} = (-1, 1, 1), \; S_{\mu\nu} = -S_{\nu\mu} \text{ and } S^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha}S_{\nu\alpha},$$

with $\epsilon^{123} = -1$ and $\epsilon_{123} = 1$. This is one way to see that the algebra $sl(2, R)$ is equivalent to the algebra $so(1, 2)$. Furthermore, the group $SL(2, R)$ is double cover of $SO(1, 2)$.

All this developments are relevant for quantization. In this case, one defines the Poisson brackets in classical phase space and then associate operators $\hat{f}(\hat{q}, \hat{p})$ and $\hat{g}(\hat{q}, \hat{p})$ to the functions $f(q, p)$ and $g(q, p)$. Without constraints, the transition from classical to quantum mechanics is made by promoting the canonical Hamiltonian H_c as an operator $\hat{H}_c$ via the nonvanishing commutator

$$[\hat{q}^i, \hat{p}_j] = i\delta^i_j, \tag{28}$$

(with $\hbar = 1$) obtained from the second bracket in (7), and by writing the quantum formula

$$\hat{H}_c|\Psi\rangle = i\frac{\partial}{\partial t}|\Psi\rangle, \tag{29}$$

which determines the physical states $|\Psi\rangle$ (see Refs. 9-11 for details). Here, the bracket $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ denotes the commutator. This is in agreement with the meaning of $\hat{H}_c$ as the generator of temporal evolution for operators in the Hilbert space.

If we have a constrained Hamiltonian system characterized by m first class constraints H_A, one also imposes that the correspondent operators acts on the physical states as $\hat{H}_A|\Psi\rangle = 0$.

4. $SL(2, R)$-symplectic structure and the (2+2)-signature

Applying Noether's procedure to (15) one learns that the angular momentum

$$L^{ij} = q^i p^j - q^j p^i \tag{30}$$

or

$$L^{ij} = J^{ab} q^i_a q^j_b \tag{31}$$

is a conserved dynamic variable. Using (7) and (30) one can show that L^{ij} obeys Lorentz group algebra

$$\{L^{ij}, L^{kl}\} = \eta^{ik}L^{jl} - \eta^{il}L^{jk} + \eta^{jl}L^{ik} - \eta^{jk}L^{il}. \tag{32}$$

Alternatively, one can show that this result also follows from (18) and (31).

We are now ready to write and prove the following proposition:

Proposition: Let $(t+s)$ be the signature of the flat metric η_{ij} associated with a phase space described with coordinates q^i_a which determine the $SL(2, R)$-symplectic structure given by the Poisson brackets (17). Then, only in the cases $t = 1$ and $s = 1$ or $t \geqq 2$ and $s \geqq 2$ there exist coordinates q^i_a different from zero such that

$$H_{ab} = 0 \tag{33}$$

and

$$L^{ij} \neq 0. \tag{34}$$

Proof: Consider a $SL(2, R)$-symplectic structure as in (17). For the η_{ij}-symbol we shall assume the general case of $(t + s)$-signature corresponding to t-time and s-space coordinates q^i. First observe that explicitly, (33) yields

$$q^i q^j \eta_{ij} = 0, \tag{35}$$

$$q^i p^j \eta_{ij} = 0, \tag{36}$$

and

$$p^i p^j \eta_{ij} = 0. \tag{37}$$

Of course a theory with $t = 0$ and $s = 0$ is vacuous, so we shall assume that $t \neq 0$ or $s \neq 0$. From (35) and (37) one finds that if $t = 0$ and $s \neq 0$, that is if η_{ij} is Euclidean, then $q^i = 0$ and $p^i = 0$. This shows the need for at least one time-like dimension, that is $t > 0$. Note that one can multiply (35)-(37) by a minus sign. This changes the signature of η_{ij} from $t + s$ to $s + t$. This means that if one assumes $t \neq 0$ and $s = 0$, it results that the theory should have at least one space-like dimension, that is $s > 0$. So putting together these two results we have a consistent solution of (35)-(37) only if $t \geqq 1$ and $s \geqq 1$.

We shall show that the case $t = 1$ and $s = 1$ is an exceptional solution of (35)-(37). In this case, these expressions become

$$-(q^1)^2 + (q^2)^2 = 0, \tag{38}$$

$$-q^1 p^1 + q^2 p^2 = 0 \tag{39}$$

and

$$-(p^1)^2 + (p^2)^2 = 0, \tag{40}$$

respectively. Using (38) and (40), one can verify that (39) is an identity. Thus (38) and (40) do not lead to any relation between q and p and therefore in this case the angular momentum condition (34) is satisfied.

It remains to explore consistency when $t = 1$ and $s \geqq 2$ (or $t \geqq 2$ and $s = 1$ due to the sign freedom in (35)-(37)). A well known result is that when $t = 1$ and $s \geqq 2$ two light-like orthogonal vectors are necessarily parallel. Hence, in this case we get the expression $q^i = ap^i$ which, according to (30), implies $L^{ij} = 0$. This clearly contradicts our assumption (34). The same result holds for the case $t \geqq 2$ and $s = 1$. Hence, we have shown that (33) and (34) makes sense only if $t = 1$ and $s = 1$ or $t \geqq 2$ and $s \geqq 2$.

Therefore, since (34) is linked to the $SO(t, s)$-symmetry one may concludes a consistent $SL(2, R)$-theory can be obtained only in the cases $SO(1, 1)$ or $SO(t \geqq 2, s \geqq 2)$. From the perspective that $SO(2, 2)$ is a minimal alternative, we have shown that the signatures $(1 + 1)$ and $(2 + 2)$ are exceptional.

An alternative method for arriving at the same result is as follows. Let us separate from (35)-(37) one time variable in the form

$$-(q^1)^2 + q^{i'} q^{j'} \eta_{i'j'} = 0, \tag{41}$$

$$-q^1 p^1 + q^{i'} p^{j'} \eta_{i'j'} = 0, \tag{42}$$

and

$$-(p^1) + p^{i'} p^{j'} \eta_{i'j'} = 0, \tag{43}$$

where the indices i', j', etc. run from 2 to $t + s$. The formula (42) leads to

$$(q^1)^2 (p^1)^2 - q^{i'} p^{j'} \eta_{i'j'} q^{k'} p^{l'} \eta_{k'l'} = 0. \tag{44}$$

Using (41) and (43) we find that (44) becomes

$$q^{i'} q^{j'} \eta_{i'j'} p^{k'} p^{l'} \eta_{k'l'} - q^{i'} p^{j'} \eta_{i'j'} q^{k'} p^{l'} \eta_{k'l'} = 0, \tag{45}$$

which can also be written as

$$(\delta^{j'}_{i'} \delta^{l'}_{k'} - \delta^{l'}_{i'} \delta^{j'}_{k'}) q^{i'} q_{j'} p^{k'} p_{l'} = 0. \tag{46}$$

Observe that this implies that $\frac{1}{2} L^{i'j'} L_{i'j'} = 0$. If $\eta_{k'l'}$ is a Euclidean metric this result in turn implies $L^{i'j'} = 0$ which means that $q^{i'} = \varsigma p^{i'}$, that is $q^{i'}$ and $p^{i'}$ are parallel quantities. The combination of (41) and (43) implies that $q^1 = \varsigma p^1$. This is another way to show that two light-like orthogonal vectors are parallel.

Let us now introduce the completely antisymmetric symbol

$$\varepsilon^{i'_2 \ldots i'_{t+s}}. \tag{47}$$

This is a rank-$t + s - 1$ tensor which values are $+1$ or -1 depending on even or odd permutations of

$$\varepsilon^{2 \ldots t+s}, \tag{48}$$

respectively. Moreover, $\varepsilon^{i'_2 \ldots i'_{t+s}}$ takes the value 0, unless the indices $i'_2 \ldots i'_{t+s}$ are all different.

Relation (46) can be written in terms of $\varepsilon^{i'_2 \ldots i'_{t+s}}$ in the form

$$\varepsilon^{j'l'i'_4 \ldots i'_{t+s}} \varepsilon_{i'k'i'_4 \ldots i'_{t+s}} q^{i'} q_{j'} p^{k'} p_{l'} = 0, \tag{49}$$

where we have dropped the nonzero factor $1/(t + s - 2)!$. Moreover, (49) can be rewritten as

$$\varepsilon^{j'l'i'_4 \ldots i'_{t+s}} \varepsilon_{i'k'i'_4 \ldots i'_{t+s}} L^{i'k'} L_{j'l'} = 0. \tag{50}$$

Here, we used (30) and dropped some numerical factors. Observe that

$$L_{i'_4 \ldots i'_{t+s}} = \frac{1}{2} \varepsilon_{i'k'i'_4 \ldots i'_{t+s}} L^{i'k'} \tag{51}$$

is the dual tensor of $L^{i'k'}$.

The lower dimensional case in which (50) holds is

$$\varepsilon^{j'l'} \varepsilon_{i'k'} L^{i'k'} L_{j'l'} = 0, \tag{52}$$

which implies

$$\varepsilon_{i'k'} L^{i'k'} = 0. \tag{53}$$

Consequently, this gives $L_{j'l'} = 0$. Hence this proves that the signature solutions $(1 + 2)$ or $(2 + 1)$ are not consistent with (34). So, it remains to prove that $(1 + (s > 2))$ is also no consistent with (34). In general we have that (50) and (51) imply

$$L_{i'_4 \ldots i'_{t+s}} L^{i'_4 \ldots i'_{t+s}} = 0. \tag{54}$$

But in the case $(1 + s > 2)$, (54) is an Euclidean expression and therefore $L_{i'_4 \ldots i'_{t+s}} = 0$, which in turn implies $L_{j'l'} = 0$. Thus, a consistent solution is also possible in the case $t \geqq 2$ and $s \geqq 2$. Hence, this is an alternative proof that with two time-like dimensions, the minimal case in which the $SL(2, R)$-symmetry is consistent with Lorentz symmetry, is the $2 + 2$-signature. In principle we may continue with this procedure founding that $3 + 3$ and so on are consistent possibilities. But, considering that (35)-(37) are only three constraints we see that there are not enough constraints to eliminate all additional degrees of freedom in all possible cases with $t \geq 3$ and $s \geq 3$. In fact, one should expect that this will lead to unwanted results at the quantum level [5-7].

Note what happens with the Lorentz Casimir operator

$$C \equiv \frac{1}{2} L^{ij} L_{ij} = \det(H_{ab}). \tag{55}$$

From (31) we have

$$\begin{aligned} C &= \frac{1}{2} L^{ij} L_{ij} = \frac{1}{2} J^{ab} q^i_a q^j_b J^{cd} q_{ci} q_{dj} \\ &= \frac{1}{2} J^{ab} J^{cd} q^i_a q_{ci} q^j_b q_{dj} = \frac{1}{2} J^{ab} J^{cd} H_{ac} H_{bd}. \end{aligned} \tag{56}$$

Hence, when $H_{ab} = 0$ we have $C = 0$ which means that in this case the Lorentz Casimir operator vanishes.

Summarizing, by imposing the $SL(2, R)$-symmetry and the Lorentz symmetry $SO(t, s)$ in the Lagrangian (15) we have shown that there exist q^i_a consistent with these symmetries only in the signatures $1 + 1$ and $t \geqq 2 + s \geqq 2$.

5. The Dirac equation and the (2+2)-signature

As an application of our previous developments, in this section we consider the Dirac equation in (2+2)-dimensions. This type of equation has already be mentioned in Ref. 15, but here we construct it from first principles. For this purpose, let us consider a relativistic point particle described by the action

$$S = -m_0 \int d\tau \left(-\dot{x}^\mu \dot{x}^\nu \xi_{\mu\nu}\right)^{1/2}. \tag{57}$$

In this section, we also use the notation $\dot{x}^\mu = dx^\mu(\tau)/d\tau$, where τ is an arbitrary parameter. The tensor $\xi_{\mu\nu}$ is a flat metric with signature $\xi_{\mu\nu} = \text{diag}(-1,-1,1,1)$.

Starting from the Lagrangian associated with (57)

$$\mathcal{L}_1 = -m_0 \left(-\dot{x}^\mu \dot{x}^\nu \xi_{\mu\nu}\right)^{1/2}, \tag{58}$$

one finds that the canonical moments associated with x^μ, namely

$$P_\mu = \frac{\partial \mathcal{L}_1}{\partial \dot{x}^\mu}, \tag{59}$$

lead to

$$P_\mu = \frac{m_0 \dot{x}^\nu \xi_{\mu\nu}}{\left(-\dot{x}^\alpha \dot{x}^\beta \xi_{\alpha\beta}\right)^{1/2}}. \tag{60}$$

From (60), one can verify that

$$\mathcal{H} \equiv P_\mu P_\nu \xi^{\mu\nu} + m_0^2 = 0, \tag{61}$$

where $\xi^{\mu\nu} = \text{diag}(-1,-1,1,1)$ is the inverse flat metric of $\xi_{\mu\nu}$. Moreover, if we define the canonical Hamiltonian

$$\mathcal{H}_{\mathbf{c}} \equiv \dot{x}^\mu P_\mu - \mathcal{L}_1, \tag{62}$$

one sees that (60) also implies that

$$\mathcal{H}_{\mathbf{c}} \equiv 0. \tag{63}$$

According to the Dirac constraint Hamiltonian system formalism, one can write the total Hamiltonian as

$$\mathcal{H}_T = \mathcal{H}_{\mathbf{c}} + \lambda \mathcal{H}, \tag{64}$$

where λ is a Lagrange multiplier. By using the constraint (61), as well as (63) and (64), one can write the first-order Lagrangian

$$\mathcal{L}_2 = \dot{x}^\mu P_\mu - \frac{\lambda}{2}(P_\mu P_\nu \xi^{\mu\nu} + m_0^2). \tag{65}$$

At the quantum level one requires to apply the constraint (61) to the physical sates Φ in the form

$$[\hat{P}_\mu \hat{P}_\nu \xi^{\mu\nu} + m_0^2]\Phi = 0, \tag{66}$$

where $\hat{P}_\mu$ is an operator associated with P_μ.

By starting with (66), our goal now is to construct a Dirac-type equation in $(2+2)$-dimensions. Let us first write (66) in the form

$$[-\hat{P}_1 \hat{P}_1 + \hat{P}_a \hat{P}_b \eta^{ab} + m_0^2]\Phi = 0. \tag{67}$$

Here, the flat metric η^{ab} is given by $\eta^{ab} = \text{diag}(-1,1,1)$, and the indices $a, b, ...$ take values in the set $\{2,3,4\}$. Consider matrices ϱ^a such that

$$\varrho^a \varrho^b + \varrho^b \varrho^a = 2\eta^{ab}. \tag{68}$$

Using (68) one sees that (67) can be written as

$$[(-\hat{P}_1 + \varrho^a \hat{P}_a)(\hat{P}_1 + \varrho^b \hat{P}_b) + m_0^2]\Phi = 0. \tag{69}$$

Now, we define two spinors

$$\Phi_L \equiv \Phi \tag{70}$$

and

$$\Phi_R \equiv -\frac{1}{m_0}(\hat{P}_1 + \varrho^b \hat{P}_b)\Phi_L. \tag{71}$$

Explicitly (71) leads to

$$(\hat{P}_1 + \varrho^b \hat{P}_b)\Phi_L + m_0 \Phi_R = 0, \tag{72}$$

while (69), (70) and (71) give

$$(\hat{P}_1 - \varrho^a \hat{P}_a)\Phi_R + m_0 \Phi_L = 0. \tag{73}$$

These last two equations can be expressed in a matrix form

$$\left(\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \hat{P}_1 + \begin{bmatrix} 0 & \varrho^a \\ -\varrho^a & 0 \end{bmatrix} \hat{P}_a + \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} m_0 \right) \begin{pmatrix} \Phi_R \\ \Phi_L \end{pmatrix} = 0, \tag{74}$$

where $I = \text{diag}(1,1)$ is the identity matrix in two dimensions. One can of course write (74) in the more compact form

$$(\Gamma^\mu \hat{P}_\mu + m_0)\Psi = 0. \tag{75}$$

Here, we used the following definitions

$$\Psi \equiv \begin{pmatrix} \Phi_R \\ \Phi_L \end{pmatrix}, \tag{76}$$

$$\Gamma^1 \equiv \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{77}$$

and

$$\Gamma^a \equiv \begin{bmatrix} 0 & \varrho^a \\ -\varrho^a & 0 \end{bmatrix}. \tag{78}$$

By promoting $\hat{P}_\mu \to i\partial_\mu$, one recognize in (75) the Dirac type equation in $(2+2)$-dimensions.

We shall show that (75) is deeply linked to the $SL(2,\mathbb{R})$-group. First, observe that an explicit representation of the matrices ϱ_1 and ϱ_a in (78) is

$$\varrho_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varrho_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\varrho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \varrho_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{79}$$

Notice first that the determinant of each of the matrices (79) is different from 0. This suggests to relate such matrices with the general group $GL(2,\mathbb{R})$. Indeed, the matrices in (79) can be considered as a basis for a general matrix M in the following manner:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \varrho_1 a + \varrho_2 b + \varrho_3 c + \varrho_4 d, \quad (80)$$

where $a, b, c, d \in \mathbb{R}$, given by

$$a = \tfrac{1}{2}(A + D), \quad b = \tfrac{1}{2}(-B + C),$$
$$c = \tfrac{1}{2}(A - D), \quad d = \tfrac{1}{2}(B + C), \quad (81)$$

Explicitly, (80) can be read

$$M = \begin{pmatrix} a + c & -b + d \\ b + d & a - c \end{pmatrix}. \quad (82)$$

Without loss of generality, one may assume that $\det(M) \neq 0$, in such a way that M is contained in the Lie group $GL(2,\mathbb{R})$. If one also impose the condition that $\det(M) = 1$, the matrix M belongs to the Lie group $SL(2,\mathbb{R})$.

It is worthwhile to mention that, by writing ϱ_a in tensorial notation

$$\varepsilon_{ij} = \varrho_2, \quad \eta_{ij} = \varrho_3, \quad \lambda_{ij} = \varrho_4, \quad (83)$$

one can construct a gravity model in 2 dimensions (see Ref. 16 for details).

Rewriting (72) and (73) respectively as follows

$$(\varrho_1 \hat{P}_1 + \varrho_2 \hat{P}_2 + \varrho_3 \hat{P}_3 + \varrho_4 \hat{P}_4)\Phi_L + m_0 \Phi_R = 0, \quad (84)$$

and

$$(\varrho_1 \hat{P}_1 - \varrho_2 \hat{P}_2 - \varrho_3 \hat{P}_3 - \varrho_4 \hat{P}_4)\Phi_R + m_0 \Phi_L = 0, \quad (85)$$

one sees that both (84) and (85) have the matrix form (80). This means that these two equations can be indentified with the Lie group $SL(2,\mathbb{R})$. Indeed, taking into account (80), we see that (84) and (85) can be rewritten as

$$\begin{bmatrix} \hat{P}_1 + \hat{P}_3 & -\hat{P}_2 + \hat{P}_4 \\ \hat{P}_2 + \hat{P}_4 & \hat{P}_1 - \hat{P}_3 \end{bmatrix} \Phi_L + m_0 \Phi_R = 0. \quad (86)$$

and

$$\begin{bmatrix} \hat{P}_1 - \hat{P}_3 & \hat{P}_2 - \hat{P}_4 \\ -\hat{P}_2 - \hat{P}_4 & \hat{P}_1 + \hat{P}_3 \end{bmatrix} \Phi_R + m_0 \Phi_L = 0, \quad (87)$$

respectively. One observes that (86) and (87) are matrix-like moments similar to the general matrix (80). Similarly, one can identify the moments matrices contained in the expressions (86) and (87) with the symmetry group $SL(2,\mathbb{R})$. Let us introduce a new momenta matrix

$$\hat{\mathcal{P}}^{\pm} = \frac{1}{m_0} \begin{bmatrix} \hat{P}_1 \pm \hat{P}_3 & \pm(-\hat{P}_2 + \hat{P}_4) \\ \pm(\hat{P}_2 + \hat{P}_4) & \hat{P}_1 \mp \hat{P}_3 \end{bmatrix}. \quad (88)$$

Consequently, the equations (86) and (87) become

$$\hat{\mathcal{P}}^{+}\Phi_L + \Phi_R = 0 \quad (89)$$

and

$$\hat{\mathcal{P}}^{-}\Phi_R + \Phi_L = 0. \quad (90)$$

Note that taking into account the constraint (86) we have

$$\det \hat{\mathcal{P}}^{\pm}\Phi_{R,L} = \Phi_{R,L}. \quad (91)$$

Symbolically, we can consider

$$\det \hat{\mathcal{P}}^{\pm} = I \quad (92)$$

But this means that both $\hat{\mathcal{P}}^{+}$ and $\hat{\mathcal{P}}^{-}$ are elements of $SL(2,\mathbb{R})$-group and therefore the Dirac type Eq. (74) or (79) has a structure associated with the group $SL(2,\mathbb{R})^{+} \times SL(2,\mathbb{R})^{-}$. In fact, this may be understood considering the isomorphism $SO(2,2) \sim SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$.

As it is known, the Dirac equation describes massive particles with $(1/2)$-spin. When the mass m_0 is the mass of the electron, the Dirac equation correctly determines the quantum theory of the electron. On the other hand, the Dirac type equation (74) in $(2+2)$-dimensions also describes massive particles with $(1/2)$-spin,. However, there is a significant distinction for this signature: while in the case of Dirac equation in $(1+3)$-dimensions Ψ can be choosen as a Majorana or Weyl spinor (but not both at the same time), one can choose Ψ as a Majorana-Weyl spinor in $(2+2)$-dimensions.

6. Final Comments

We have proved in some detail that $SL(2,R)$-symmetry and Lorentz symmetry $SO(t,s)$ imply together that the signatures $1+1$ and $2+2$ are exceptional. One may be motivated to relate this result with different physical scenarios. Of course, the signature $1+1$ can be related to string theory. But what about the $2+2$ signature? We already know that this signature arises in a number of physical scenarious, including in a background for $N=2$ strings [17-18] (see also Refs. 19-21), Yang-Mills in Atiyah Singer background [22] (see also Refs. 23 for the importance of the $2+2$ signature in mathematics), Majorana-Weyl spinor [24-25] and more recently in loop quantum gravity in terms of oriented matroid theory [26] (see also Refs. 27-29). But one wonders whether the $2+2$ signature can be linked to quantum gravity itself in $1+3$ dimensions. One possibility to answer this question is to search for a mechanism which can transform self-dual canonical gravity in $2+2$ dimensions into self-dual canonical gravity in $1+3$. This is equivalent to change one time dimension by one space dimension and *vice versa*. Surprisingly this kind of transformation has already be considered in the context of the sigma model (see Ref. 30 and references therein). In fact, it was shown in Ref. 27 that similar mechanism can be implemented at the level of quantum self-dual canonical gravity $2+2$ dimensions.

Acknowledgments

This work was partially supported by PROFAPI-UAS 2009.

1. J.M. Maldacena and H. Ooguri, *J. Math. Phys.* **42** (2001) 2929; hep-th/0001053.
2. E. Witten, *Phys. Rev. D* **44** (1991) 314.
3. O.F. Hernandez, "*An Understanding of SU*(1,1) / U(1) *conformal field theory via bosonization*" , Presented at 4th Mexican School of Particles and Fields, Dec 2-12, 1990, (Oaxtepec, Mexico. Published in Mexican School 1990), 429-436.
4. S. Hwang, *Nucl. Phys. B* **354** (1991) 100 .
5. I. Bars, *Class. Quant. Grav.* **18** (2001) 3113 ; hep-th/0008164.
6. I. Bars, C. Deliduman and O. Andreev, *Phys. Rev. D* **58** (1998) 066004; hep-th/9803188.
7. I. Bars, *Int. J. Mod. Phys. A* **25** (2010) 5235; arXiv:1004.0688 [hep-th].
8. J.A. Nieto, *Nuovo Cim. B* **120** (2005) 135; hep-th/0410003.
9. M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, New Jersey, 1992).
10. J. Govaerts, *Hamiltonian Quantisation and Constrained Dynamics* (Leuven University Press, Leuven, 1991).
11. A. Hanson, T. Regge and C. Teitelboim, *Constrained Hamiltonian Systems* (Accademia Nazionale dei Lincei, Roma, 1976).
12. V.M. Villanueva, J.A. Nieto, L. Ruiz and J. Silvas, *J. Phys. A* **38** (2005) 7183; hep-th/0503093.
13. J.M. Romero and A. Zamora, *Phys. Rev. D* **70** (2004) 105006; hep-th/0408193.
14. P.A.M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1964).
15. S.V. Ketov, H. Nishino and S. J. Gates Jr., *Nucl. Phys. B* **393** (1993) 149; hep-th/9207042.
16. J.A. Nieto and E.A. Leon, *Braz. J. Phys.* **40** (2010) 383; arXiv:0905.3543 [hep-th].
17. H. Ooguri and C. Vafa, *Nucl. Phys. B* **367** (1991) 83.
18. H. Ooguri and C. Vafa, *Nucl. Phys. B* **361** (1991) 469.
19. E. Sezgin, *Is there a stringy description of selfdual supergravity in (2+2)-dimensions?*, Published in "*Trieste High energy physics and cosmology*" (1995) 360-369; hep-th/9602099.
20. Z. Khviengia, H. Lu, C.N. Pope, E. Sezgin, X.J. Wang and K.W. Xu, *Nucl. Phys. B* **444** (1995) 468; hep-th/9504121.
21. S.V. Ketov, Class. Quantum Grav. **10** (1993) 1689; hep-th/9302091.
22. M.A. De Andrade, O.M. Del Cima and L.P. Colatto, *Phys. Lett. B* **370** (1996) 59; hep-th/9506146.
23. M.F. Atiyah. and R.S. Ward, *Commun. Math. Phys.* **55** (1977) 117.
24. P.G.O. Freund, *Introduction to Supersymmetry* (Cambridge University Press, Melbourne, 1986).
25. S.V. Ketov, H. Nishino and S.J. Gates Jr., *Phys. Lett. B* **307** (1993) 323; hep-th/9203081.
26. J.A. Nieto, *Rev. Mex. Fis.* **57** (2011) 400; arXiv:1003.4750 [hep-th].
27. J. A. Nieto, *Int. J. Geom. Meth. Mod. Phys.* **09** (2012) 1250069; arXiv:1107.0718 [gr-qc].
28. J.A. Nieto, *Adv. Theor. Math. Phys.* **10** (2006) 747; hep-th/0506106.
29. J.A. Nieto, *Adv. Theor. Math. Phys.* **8** (2004) 177; hep-th/0310071.
30. C.M. Hull, *JHEP* **9811** (1998) 017; hep-th/9807127.