Coordinate systems adapted to constants of motion

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Received 7 January 2013; accepted 10 June 2013

We present some examples of mechanical systems such that given n constants of motion in involution (where n is the number of degrees of freedom), we can identify a coordinate system in which the Hamilton–Jacobi equation is separable (or R-separable), with the separation constants being the values of the given constants of motion. Analogous results for the Schrödinger equation are also given.

Keywords: Hamilton–Jacobi equation; constants of motion; separation of variables; R-separability; Schrödinger equation

Presentamos algunos ejemplos de sistemas mecánicos tales que dadas n constantes de movimiento en involución (donde n es el número de grados de libertad), podemos identificar un sistema de coordenadas en el cual la ecuación de Hamilton–Jacobi es separable (o R-separable), con las constantes de separación siendo los valores de las constantes de movimiento dadas. Se dan resultados análogos para la ecuación de Schrödinger.

Descriptores: Ecuación de Hamilton–Jacobi; constantes de movimiento; separación de variables; R-separabilidad; ecuación de Schrödinger

PACS: 45.20.Jj; 02.30.Jr; 03.65.-w

1. Introduction

As is well known, in the framework of classical mechanics, given a Hamiltonian function for a system with n degrees of freedom, a complete solution of the corresponding Hamilton–Jacobi (HJ) equation yields $2n$ constants of motion or, equivalently, the solution of the equations of motion (see, *e.g.*, Refs. 1 and 2). The complete solutions of the HJ equation are usually obtained by means of separation of variables [1], but the successful application of this method depends on an appropriate choice of the coordinates. When the HJ equation can be solved by separation of variables, the n separation constants arising in the solution are the values of n constants of motion, which need not be related to "obvious" symmetries of the Hamiltonian. For instance, it turns out that the HJ equation for the Kepler problem in two dimensions can be solved by separation of variables in parabolic coordinates, and the separation constants are the (conserved) values of the Hamiltonian and of one component of the Laplace–Runge–Lenz vector (see, *e.g.*, Ref. 2, p. 169).

Making use of the Liouville theorem, in an *arbitrary* coordinate system, we can find a complete solution of the HJ equation, which need not be separable, if we have n constants of motion in involution [3–5]. In this paper we give some examples where the complete solution of the HJ equation obtained in this manner allows us to identify the coordinate system in the configuration space in which the HJ equation is separable or R-separable, and the separation constants are the values of the n constants of motion we started with. A solution of the HJ equation is R-separable if it is the sum of functions of one variable and some function that may depend on all the coordinates but not on the parameters contained in the solution. The concept of R-separability is more common (though not widely known) in the case of second-order linear partial differential equations (see, *e.g.*, Refs. 6 and 7).

In Sec. 2, following Ref. 5, we give a summary of the Liouville theorem and in Sec. 3 we present various examples where we start with n constants of motion for a given Hamiltonian and we find a coordinate system where the HJ equation admits separable or R-separable solutions; we also analyze the separability of the corresponding Schrödinger equation.

2. The Liouville theorem

Given a Hamiltonian function $H(q^i, p_i, t)$ for a system with n degrees of freedom, we assume known n functionally independent constants of motion

$$Q^i = Q^i(q^j, p_j, t), \qquad i = 1, 2, \ldots, n, \tag{1}$$

which may depend explicitly on the time. (It may be remarked that in the few modern books dealing with the Liouville theorem, the attention is usually restricted to constants of motion that do not depend explicitly on the time, see, *e.g.*, Refs. 8 and 9.) Furthermore, we shall assume that Eqs. (1) can be inverted so that the p_i can be expressed in terms of Q^j, q^j, and t:

$$p_i = F_i(q^j, t, Q^j). \tag{2}$$

Substituting these expressions into the Hamiltonian we obtain a function

$$\widetilde{H}(q^i, t, Q^i) \equiv H(q^i, F_i(q^j, t, Q^j), t) \tag{3}$$

and, treating the Q^i as parameters, there exists a function S (that depends parametrically on the Q^i) such that

$$F_i \mathrm{d}q^i - \widetilde{H} \mathrm{d}t = \mathrm{d}S, \tag{4}$$

if and only if the Q^i are in involution, that is

$$\{Q^i, Q^j\} = 0, \tag{5}$$

where $\{\ ,\ \}$ denotes the Poisson bracket. (In the language of differentiable manifolds, F_i and $\widetilde{H}$ are the pullbacks of p_i and H under the inclusion map of the submanifold defined by $Q^i = \text{const.}$ into the extended phase space.) The function S defined by Eq. (4) is a complete solution of the HJ equation, which need not be separable or R-separable in the coordinates q^i (see Ref. 5 and the examples given below).

Once we have the explicit expression of S, proceeding in the usual manner, we can obtain the n additional constants of motion, $P_1, P_2, \ldots, P_n$, by means of

$$P_i = -\frac{\partial S}{\partial Q^i}.$$

3. Coordinate systems associated with solutions of the HJ equation

In this section, we give several explicit examples where, with the aid of the Liouville theorem, we find coordinate systems in which the HJ equation can be solved by separation of variables.

3.1. Particle in a uniform field

A simple but illustrative example is given by the Hamiltonian

$$H = \frac{1}{2m}({p_x}^2 + {p_y}^2) + mgy, \tag{6}$$

for which all constants of motion are readily obtained (here m and g are constants). In fact, the HJ equation is separable in the coordinates (x, y) of the configuration space, and the separation constants are the values of H and p_x, which are constants of motion as a consequence of the fact that t and x do not appear in the Hamiltonian.

Another constant of motion (which is not related to "obvious" symmetries of the Hamiltonian) is

$$\frac{1}{m} p_x p_y + mgx, \tag{7}$$

therefore, H and $(1/m)p_x p_y + mgx$ are in involution and can be taken as Q^1 and Q^2, respectively. A straightforward computation leads to the expressions

$$p_x + p_y = \pm\sqrt{2m(Q^1 + Q^2) - 2m^2 g(x+y)},$$
$$p_x - p_y = \pm\sqrt{2m(Q^1 - Q^2) + 2m^2 g(x-y)}. \tag{8}$$

Hence, by writing Eq. (4) in the form (here, by abuse of notation, we write p_x and p_y in place of F_1 and F_2)

$$\frac{1}{2}(p_x+p_y)\,\mathrm{d}(x+y) + \frac{1}{2}(p_x-p_y)\,\mathrm{d}(x-y) - Q^1\mathrm{d}t = \mathrm{d}S, \tag{9}$$

and taking into account that $p_x \pm p_y$ is a function of $x \pm y$ only, we see that the function S is the sum of three one-variable functions that depend on $x+y$, $x-y$, and t (with Q^1 and Q^2 being treated as parameters). In other words, the HJ equation admits separable solutions in the coordinates (u, v) defined by

$$u \equiv x + y, \qquad v \equiv x - y, \tag{10}$$

and the separation constants are the values of H and $(1/m)p_x p_y + mgx$.

Indeed, in the coordinates (10), the Hamiltonian (6) takes the form

$$H = \frac{1}{m}({p_u}^2 + {p_v}^2) + mg\frac{u-v}{2} \tag{11}$$

and the HJ equation is

$$\frac{1}{m}\left[\left(\frac{\partial S}{\partial u}\right)^2 + \left(\frac{\partial S}{\partial v}\right)^2\right] + mg\frac{u-v}{2} + \frac{\partial S}{\partial t} = 0,$$

which admits separable solutions of the form

$$S = f(u) + h(v) - Et, \tag{12}$$

where E is a separation constant (the value of H). The functions f and h must obey

$$\frac{1}{m}\left(\frac{\mathrm{d}f}{\mathrm{d}u}\right)^2 + \frac{mgu}{2} - \frac{E}{2} = \frac{A}{2},$$
$$\frac{1}{m}\left(\frac{\mathrm{d}h}{\mathrm{d}v}\right)^2 - \frac{mgv}{2} - \frac{E}{2} = -\frac{A}{2}, \tag{13}$$

where A is a second separation constant (the factor $1/2$ is introduced for later convenience).

The meaning of the parameter A is obtained by subtracting the two equations (13) (in order to eliminate E),

$$\begin{aligned} A &= \frac{1}{m}\left[\left(\frac{\mathrm{d}f}{\mathrm{d}u}\right)^2 - \left(\frac{\mathrm{d}h}{\mathrm{d}v}\right)^2\right] + mg\frac{u+v}{2} \\ &= \frac{1}{m}({p_u}^2 - {p_v}^2) + mg\frac{u+v}{2} \\ &= \frac{1}{m} p_x p_y + mgx. \end{aligned}$$

Thus, the two separation constants, E and A, are the values of the constants of motion H and $(1/m)p_x p_y + mgx$, which led in the first place to the coordinates (u, v).

From Eqs. (12) and (13) we get

$$\begin{aligned} S &= \frac{1}{2}\int \sqrt{2m(E+A) - 2m^2 gu}\,\mathrm{d}u \\ &+ \frac{1}{2}\int \sqrt{2m(E-A) + 2m^2 gv}\,\mathrm{d}v - Et, \end{aligned}$$

which coincides with the principal function S obtained from Eqs. (8) and (9). The second half of constants of motion, $P_i = -\partial S/\partial Q^i$, is given by

$$P_1 = \frac{p_y}{mg} + t, \qquad P_2 = \frac{p_x}{mg}. \tag{14}$$

The Schrödinger equation

$$-\frac{\hbar^2}{m}\left(\frac{\partial^2\psi}{\partial u^2} + \frac{\partial^2\psi}{\partial v^2}\right) + mg\frac{u-v}{2}\psi = \mathrm{i}\hbar\frac{\partial\psi}{\partial t} \tag{15}$$

[*cf.* Eq. (11)] also admits separable solutions of the form $\psi = U(u)V(v)\exp(-\mathrm{i}Et/\hbar)$, where E is a separation constant, and the one-variable functions U and V obey the equations

$$-\frac{\hbar^2}{m}\frac{\mathrm{d}^2U}{\mathrm{d}u^2} + \frac{mgu}{2}U - \frac{E}{2}U = \frac{A}{2}U,$$
$$-\frac{\hbar^2}{m}\frac{\mathrm{d}^2V}{\mathrm{d}v^2} - \frac{mgv}{2}V - \frac{E}{2}V = -\frac{A}{2}V, \qquad (16)$$

where A is another separation constant. The solutions of these last equations can be expressed in terms of Airy functions or Bessel functions of order $1/3$. One can readily verify that, as a consequence of Eqs. (16), the wavefunctions $\psi = U(u)V(v)\exp(-\mathrm{i}Et/\hbar)$ are eigenfunctions of the Hamiltonian (with eigenvalue E) and of the operator $(1/m)({p_u}^2 - {p_v}^2) + mg\frac{u+v}{2} = (1/m)p_xp_y + mgx$ (with eigenvalue A).

3.2. An R-separable solution

Another pair of constants of motion in involution for the Hamiltonian (6) is given by the functions

$$Q^1 \equiv p_x, \qquad Q^2 \equiv p_y + mgt \qquad (17)$$

[see Eqs. (14)] (note that Q^2 depends explicitly on the time). The left-hand side of Eq. (4) is now given by

$$Q^1\mathrm{d}x + (Q^2 - mgt)dy - \frac{1}{2m}\left[(Q^1)^2 + (Q^2 - mgt)^2\right]dt - mgydt = \mathrm{d}\left[-mgyt + Q^1x + Q^2y - \frac{(Q^1)^2t}{2m} + \frac{(Q^2 - mgt)^3}{6m^2g}\right] \qquad (18)$$

thus showing that S is the sum of the function $-mgyt$ (that does not depend on the parameters Q^i) and three one-variable functions that depend on x, y, and t. That is, the HJ equation in the coordinates (x, y) admits R-separable solutions of the form $S = -mgyt + f(x) + h(y) + \phi(t)$, and the separation constants are the values of p_x and $p_y + mgt$. As pointed out above, the HJ equation in the coordinates (x, y) also admits *separable* solutions.

Correspondingly, the Schrödinger equation

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) + mgy\psi = \mathrm{i}\hbar\frac{\partial\psi}{\partial t} \qquad (19)$$

admits (multiplicative) R-separable solutions of the form

$$\psi = \exp\left(-\frac{\mathrm{i}}{\hbar}mgyt\right)X(x)Y(y)T(t), \qquad (20)$$

[*cf.* Eq. (18)]. Substituting Eq. (20) into Eq. (19) we find

$$\psi = \exp\frac{\mathrm{i}}{\hbar}\left[-mgyt + Q^1x + Q^2y - \frac{(Q^1)^2t}{2m} + \frac{(Q^2 - mgt)^3}{6m^2g}\right] \qquad (21)$$

[*cf.* Eq. (18)] where Q^1 and Q^2 are separation constants. The wavefunction (21) is a common eigenfunction of the operators p_x and $p_y + mgt$ with eigenvalues Q^1 and Q^2, respectively. As in the case of the HJ equation, the Schrödinger equation (19) also admits separable solutions (which involve Airy functions). It may be noticed that, among other things, the wavefunctions (21) do not involve special functions.

3.3. Particle in a central field

The usual Hamiltonian for a particle in a central field, in two dimensions, can be written in the form

$$H = \frac{1}{2m}({p_x}^2 + {p_y}^2) + V(\sqrt{x^2 + y^2}), \qquad (22)$$

where V is a function of one variable and (x, y) are Cartesian coordinates. In addition to the Hamiltonian itself, the angular momentum about the origin, $L_z = xp_y - yp_x$, is conserved. Choosing H and L_z as the two constants of motion in involution, Q^1 and Q^2, respectively, we find

$$p_x = \frac{-Q^2y \pm x\sqrt{2m(x^2 + y^2)(Q^1 - V) - (Q^2)^2}}{x^2 + y^2},$$
$$p_y = \frac{Q^2x \pm y\sqrt{2m(x^2 + y^2)(Q^1 - V) - (Q^2)^2}}{x^2 + y^2}$$

and, therefore, the left-hand side of Eq. (4) is

$$Q^2\frac{(-y\mathrm{d}x + x\mathrm{d}y)}{x^2 + y^2} \pm \sqrt{2m(x^2 + y^2)(Q^1 - V) - (Q^2)^2}\,\frac{(x\mathrm{d}x + y\mathrm{d}y)}{x^2 + y^2} - Q^1\mathrm{d}t = Q^2\,\mathrm{d}\arctan\frac{y}{x} \pm \sqrt{2m(Q^1 - V) - \frac{(Q^2)^2}{x^2 + y^2}}\,\mathrm{d}\left(\sqrt{x^2 + y^2}\right) - Q^1\mathrm{d}t,$$

which is the total differential of the sum of functions of $\arctan(y/x)$, $\sqrt{x^2 + y^2}$, and t, thus showing that the HJ equation is separable in polar coordinates and the separation constants are the values of H and L_z.

3.4. Charged particle in a uniform magnetic field

Another example where the HJ equation (and the Schrödinger equation) admits R-separable solutions corresponds to a charged particle in a uniform magnetic field, with the vector potential chosen as

$$\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}.$$

In Cartesian coordinates, with the z-axis pointing along the magnetic field, the Hamiltonian is

$$H = \frac{1}{2m}\left[\left(p_x + \frac{eB}{2c}y\right)^2 + \left(p_y - \frac{eB}{2c}x\right)^2 + p_z^2\right], \qquad (23)$$

where e is the charge of the particle. Three constants of motion in involution are given by

$$Q^1 = H, \qquad Q^2 = p_x - \frac{eB}{2c}y, \qquad Q^3 = p_z.$$

Hence

$$p_x = Q^2 + \frac{eB}{2c}y,$$
$$p_y = \frac{eB}{2c}x \pm \sqrt{2mQ^1 - \left(Q^2 + \frac{eBy}{c}\right)^2 - (Q^3)^2},$$
$$p_z = Q^3,$$

and Eq. (4) takes the form

$$\left(Q^2 + \frac{eB}{2c}y\right)\mathrm{d}x + \left(\frac{eB}{2c}x \pm \sqrt{2mQ^1 - \left(Q^2 + \frac{eBy}{c}\right)^2 - (Q^3)^2}\right)\mathrm{d}y + Q^3\mathrm{d}z - Q^1\mathrm{d}t = \mathrm{d}S.$$

Thus,

$$S = \frac{eB}{2c}xy + Q^2x \pm \int \sqrt{2mQ^1 - \left(Q^2 + \frac{eBy}{c}\right)^2 - (Q^3)^2}\,\mathrm{d}y + Q^3z - Q^1t, \tag{24}$$

is an R-separable solution of the HJ equation; the first term mixes the coordinates x and y but the remaining four are functions of one variable.

The Schrödinger equation

$$\frac{1}{2m}\left[\left(\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x} + \frac{eB}{2c}y\right)^2 + \left(\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial y} - \frac{eB}{2c}x\right)^2 + \left(\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial z}\right)^2\right]\psi = \mathrm{i}\hbar\frac{\partial\psi}{\partial t}$$

admits R-separable solutions of the form

$$\psi = \exp\left(\frac{\mathrm{i}}{\hbar}\frac{eB}{2c}xy\right)X(x)Y(y)Z(z)T(t).$$

In fact,

$$\psi = \exp\frac{\mathrm{i}}{\hbar}\left(\frac{eB}{2c}xy + Q^2x + Q^3z - Q^1t\right)Y(y),$$

where Q^1, Q^2, Q^3 are constants, and Y satisfies the separated equation

$$\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} + \frac{1}{\hbar^2}\left[2mQ^1 - \left(Q^2 + \frac{eBy}{c}\right)^2 - (Q^3)^2\right]Y = 0$$

[*cf.* Eq. (24)].

In this case, the method not only yields a solution of the HJ equation in an arbitrary coordinate system, but also in an arbitrary gauge, since the constants of motion are gauge-independent.

4. Concluding remarks

Apart from the fact that we do not know under what conditions the method followed here leads to separable or R-separable solutions in some coordinate system, it should be clear that the identification of the coordinates may be very difficult in some cases.

It may be noticed that in the examples presented in this paper the coordinate systems found with the aid of Liouville's theorem are all orthogonal, but there is no reason to believe that this will happen in all cases. An unexpected result is that the HJ equation (as well as the Schrödinger equation), written in a specific coordinate system, admits simultaneously separable and R-separable solutions (Sec. 3.2).

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