# A physical interpretation of fractional calculus in observables terms: analysis of the fractional time constant and the transitory response

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This work presents the analysis of the fractional time constant and the transitory response (delay, rise, and settling times) of a RC circuit as a physical interpretation of fractional calculus in observables terms, the definition of Caputo fractional derivative is applied. The physical interpretation of these observables allows a clearer understanding of the concept of fractional derivative.

*Keywords:* Fractional calculus; fractional time constant; fractional differential equations; transitory response.

Este trabajo presenta el análisis de la constante de tiempo transitoria y de la respuesta en frecuencia (tiempo de retraso, elevación y asentamiento) de un circuito RC como una interpretación física del cálculo fraccionario en términos de estos observables, la definición de derivada fraccionaria de Caputo es aplicada. La interpretación física de estos observables permite tener un entendimiento claro del concepto de derivada fraccionaria.

*Descriptores:* Calculo fraccionario; constante de tiempo fraccionaria; ecuaciones diferenciales fraccionarias; respuesta transitoria.

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## 1. Introduction

Fractional calculus (FC), involving derivatives an integrals of non-integer order, is the natural generalization of the classical calculus, which during recent years became a powerful and widely used tool for better modeling and control of processes in many areas of science and engineering [1-11]. Many physical phenomena have “intrinsic” fractional order description and so FC is necessary in order to explain them [12]. In many applications FC provide more accurate models of the physical systems than ordinary calculus do. Since, its success in description of anomalous diffusion [13-16] non-integer order calculus both in one and multidimensional space, it has become an important tool in many areas of physics, mechanics, chemistry, engineering, finances and bioengineering [17-21]. Fundamental physical considerations in favor of the use of models based on derivatives of non-integer order are given in [22,23]. The Lagrangian and Hamilton formulation of dynamics and electromagnetic field in view of fractional calculus has been reported in [24-29]. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes [30]. This is the main advantage of FC in comparison with the classical integer-order models, in which such effects are in fact neglected. Another large field which requires the use of FC is the theory of fractals [31-32]. The development of the theory of fractals has opened further perspective for the theory of fractional derivatives, especially in modeling dynamical processes in self-similar and porous structures. Fractional-order models have been already used for modeling of electrical circuits (such as domino ladders, tree structures, etc.) and elements (coils, memristor, etc.). The review of such models can be found in [33-35].

Applications of fractional calculus in electromagnetic theory are given in [36-41]. In Ref. [42] the time evolution of the fractional electromagnetic waves using the time fractional Maxwell’s equations is presented. Unlike the work of the authors mentioned above, in which the pass from an ordinary derivative to a fractional one is direct, here first we analyze the ordinary derivative operator and try to bring it to the fractional form in a consistent manner [43]. In Ref. [44] an alternative fractional construction for the electromagnetic waves in terms of the fractional derivative of the Caputo type is presented.

The next section it is described of RC circuit and the fractional time constant and the transitory response.

## 2. Fractional Calculus

The definitions of the fractional order derivative are not unique and there exist several definitions, including: Grünwald-Letnikov, Riemann-Liouville, Weyl, Riesz and the Caputo representation for fractional order derivative. In the Caputo case, the derivative of a constant is zero and we can define, properly, the initial conditions for the fractional differential equations which can be handled by using an analogy with the classical integer case. For these reasons, in this paper we prefer to use the Caputo fractional derivative. The Caputo fractional derivative (CFD) for a function of time, $f(t)$, is defined as follows [6]

$$ {}_{0}^{C}D_t^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(\eta)}{(t-\eta)^{\gamma-n+1}} d\eta, \qquad (1)$$

where $n = 1, 2, \ldots \in N$ and $n - 1 < \gamma \leq n$. We consider the case $n = 1$, *i.e.*, in the integrand there is only a first derivative. In this case, $0 < \gamma \leq 1$, is the order of the fractional derivative.

The Caputo derivative operator satisfies the following properties

$$ {}_{0}^{C}D_t^{\gamma}[f(t)+g(t)] = {}_{0}^{C}D_t^{\gamma} f(t) + {}_{0}^{C}D_t^{\gamma} g(t), $$
$$ {}_{0}^{C}D_t^{\gamma} c = 0, \text{ where } c \text{ is constant.} \qquad (2)$$

The Caputo definition of the fractional derivative is very useful in the time domain studies, because the initial conditions for the fractional order differential equations with the Caputo derivatives can be given in the same manner as for the ordinary differential equations with a known physical interpretation.

Laplace transform to CFD gives [6]

$$ L[{}_{0}^{C}D_t^{\gamma} f(t)] = S^{\gamma} F(S) - \sum_{k=0}^{m-1} S^{\gamma-k-1} f^{(k)}(0). \qquad (3)$$

The inverse Laplace transform requires the introduction of the Mittag-Leffler function. The Mittag-Leffler function is defined by the series expansion as

$$ E_a(t) = \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(am+1)}, \qquad (a > 0), \qquad (4)$$

When $a = 1$, from (4) we obtain

$$ E_1(t) = \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(m+1)} = \sum_{m=0}^{\infty} \frac{t^m}{m!} = e^t. \qquad (5)$$

Therefore, the Mittag-Leffler function can be seen as a generalization of the exponential function.

## 3. Application Example

Ohm's law states that the current flowing through a conductor between two given points is directly proportional to the potential difference and inversely proportional to the resistance between them. The mathematical formula can be written as follows

$$ v(t) = Ri(t), \qquad (6)$$

where $i(t)$ is the current flowing through the conductor measured in ampers (A), $v(t)$ is the potential difference measured between two points of the conductor in units of volts $V$ and $R$ is the resistance of the conductor measured in ohms. The current is a flow of electric charge through a conductive medium. In electric circuits this charge is often carried by moving electrons in a wire. The change in the charge $q$ with respect to time $t$ is,

$$ i(t) = \frac{dq}{dt}. \qquad (7)$$

Taking this into account, Ohm's law can be written as a function of the charge $q(t)$

$$ v(t) = R\frac{dq}{dt}. \qquad (8)$$

The idea is to rewrite Ohm's law in terms of a fractional (non-integer) derivative. For this purpose we introduce a fractional time derivative operator as follows

$$ \frac{d^{\gamma}}{dt^{\gamma}}, \qquad 0 < \gamma \leq 1, \qquad (9)$$

where $\gamma$ is an arbitrary parameter very close to 1, which represents the order of the derivative and in the case $\gamma = 1$ it becomes an ordinary (integer) derivative operator. However, the ordinary time operator has dimensions of inverse seconds $\mathrm{s}^{-1}$. Then the expression (9),

$$ \left[\frac{d^{\gamma}}{dt^{\gamma}}\right] = \frac{1}{\mathrm{s}^{\gamma}}, \qquad 0 < \gamma \leq 1, \qquad (10)$$

is not an ordinary time derivative, because of the dimension, $\mathrm{s}^{-\gamma}$.

To be consistent with dimensionality, we introduce a new parameter, $\sigma$, as follows

$$ \left[\frac{1}{\sigma^{1-\gamma}}\frac{d^{\gamma}}{dt^{\gamma}}\right] = \frac{1}{\mathrm{s}}, \qquad 0 < \gamma \leq 1, \qquad (11)$$

such that when $\gamma = 1$ the expression (11) becomes an ordinary derivative. This is true if the parameter $\sigma$ has dimensions of seconds, $[\sigma] = \mathrm{s}$. Therefore, we can change the ordinary time derivative operator by the fractional as follows

$$ \frac{d}{dt} \rightarrow \frac{1}{\sigma^{1-\gamma}}\frac{d^{\gamma}}{dt^{\gamma}}, \qquad n-1 < \gamma \leq n, \qquad (12)$$

where $n$ is integer. These two expressions represent time derivatives, since their dimensions are inverse seconds. The parameter $\sigma$ characterizes the fractional structures (components that show an intermediate behavior between a system conservative (capacitor) and dissipative (resistor)), of the fractional time operator [43]. Using the expression (12), Ohm's law (8) becomes a fractional Ohm's law

$$ v(t) = \frac{R}{\sigma^{1-\gamma}}\frac{d^{\gamma}q}{dt^{\gamma}}, \qquad 0 < \gamma \leq 1, \qquad (13)$$

when $\gamma = 1$, from the expression(13) we have (8).

The RC circuit is represented in Fig. 1. Applying Kirchhoff's law, we have

$$ R\frac{dq}{dt} + \frac{1}{C}q(t) = v(t), \qquad (14)$$

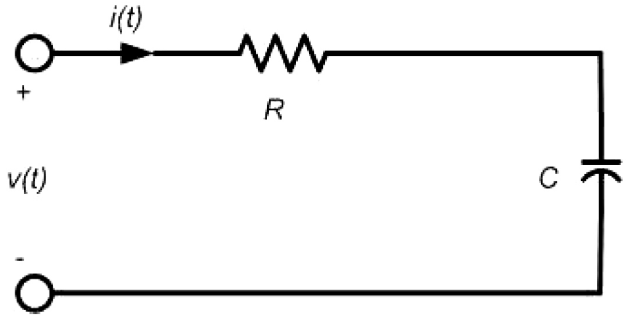


FIGURE 1. RC Circuit.

where $R$ (resistance), $C$ (capacitance) and $v(t)$ is the voltage source. The general solution of (14) is well known and has the form

$$q(t) = ce^{-t/\tau} + e^{-t/\tau}\int e^{t/\tau}v(t)dt, \tag{15}$$

where $\tau = RC$ is the time constant measured in seconds.

Using the expression (13), the fractional differential equation for the RC circuit has the form

$$\frac{d^\gamma q}{dt^\gamma} + \frac{1}{\tau_\gamma}q(t) = \frac{C}{\tau_\gamma}v(t), \tag{16}$$

where

$$\tau_\gamma = \frac{RC}{\sigma^{1-\gamma}}, \tag{17}$$

it can be called fractional time constant due to its dimensionality $s^\gamma$. When $\gamma = 1$, from (17) we have the well known time constant $\tau = RC$.

Assuming that $v(0) = 0$ and for any time $t$, $v(t) = V_0 u(t)$, where $V_0$ is a constant source of voltage and $u(t)$ is the step function. Applying the Laplace transform in (16) with zero initial condition (steady state)

$$S^\gamma Q(S) + \frac{1}{\tau_\gamma}Q(S) = \frac{CV_0}{\tau_\gamma S}. \tag{18}$$

Solving for $Q(S)$, we obtain

$$Q(S) = \frac{CV_0}{\tau_\gamma S\left(S^\gamma + \frac{1}{\tau_\gamma}\right)}. \tag{19}$$

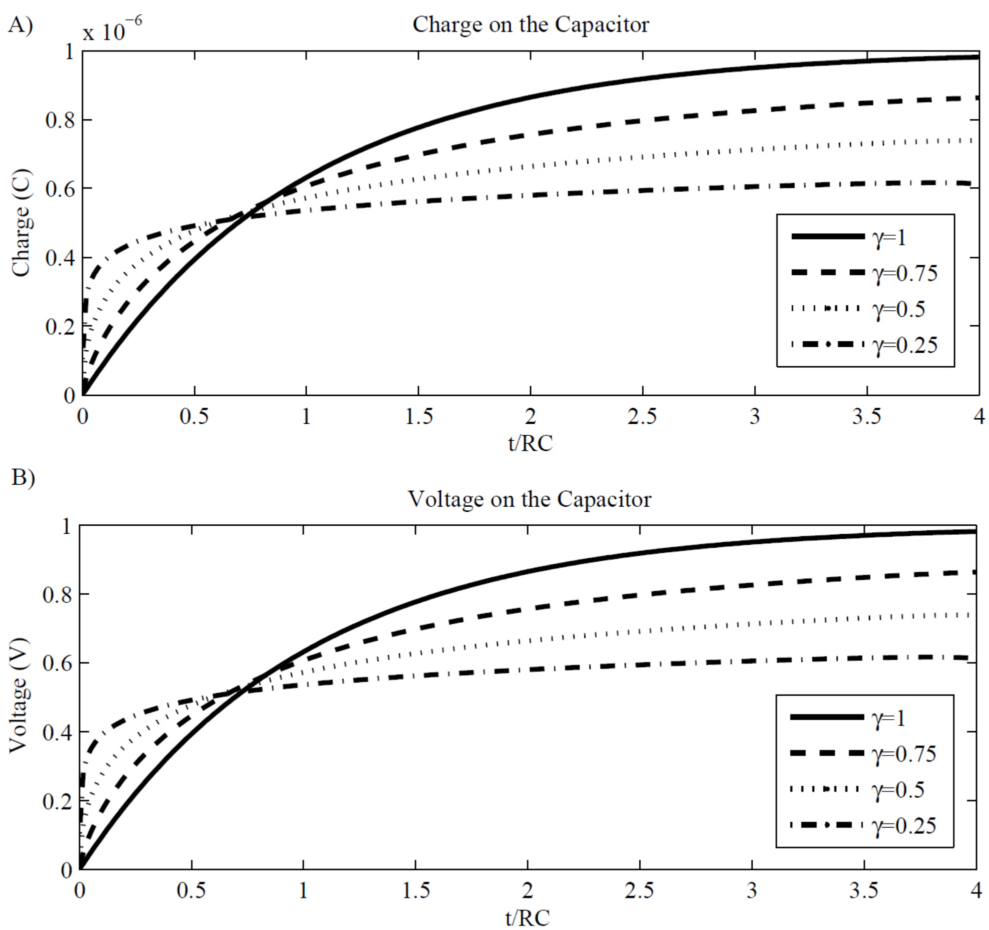


FIGURE 2. Charge on the capacitor, in Figure A), exponents: $\gamma = 0.25$, $\gamma = 0.5$, $\gamma = 0.75$ and $\gamma = 1$, in Figure B), voltage on the capacitor, exponents: $\gamma = 0.25$, $\gamma = 0.5$, $\gamma = 0.75$ and $\gamma = 1$.

Applying the inverse Laplace transform [12] in (19) we obtain the behavior of the charge with respect to time $t$.

$$q(t) = CV_0\Big\{1 - E_\gamma\Big(-\frac{1}{\tau_\gamma}t^\gamma\Big)\Big\}$$

$$= CV_0\Big\{1 - E_\gamma\Big(-\frac{\sigma^{1-\gamma}}{RC}t^\gamma\Big)\Big\}, \tag{20}$$

$$q(t) = CV_0\Big\{1 - E_\gamma\Big(-\frac{1}{\tau_\gamma}t^\gamma\Big)\Big\}, \tag{21}$$

where $E_\gamma(t)$ is the Mittag-Leffler function.

The parameter $\gamma$, which represents the order of the fractional differential Eq. (16), can be related to the parameter $\sigma$, which characterizes the presence of fractional structures in the system. In our case the relationship is given by the expression

$$\gamma = \frac{\sigma}{RC}. \tag{22}$$

Then, the magnitude

$$\delta = 1 - \gamma, \tag{23}$$

characterizes the existence of fractional structures in the system. This can be seen as follows: if $\gamma = 1$, from (22) we have $\sigma = RC$ and thus $\delta = 0$ in (23), which means that in the system there is not any fractional structures, that is, it is a regular $RC$ circuit. However, in the interval, $0 < \gamma < 1$, or the equivalent, $0 < \sigma < RC$, the magnitude $\delta$ increases and tends to unity because are increasingly fractional structures in the system.

Substituting the expression (22) in (21) we have

$$q(t) = CV_0\Big[1 - E_\gamma(-\gamma^{1-\gamma}\tilde{t}^\gamma)\Big], \tag{24}$$

where $\hat{t} = t/RC$ is a dimensionless parameter. From (24), we have the voltage in the capacitor

$$v(t) = V_0\Big[1 - E_\gamma(-\gamma^{1-\gamma}\tilde{t}^\gamma)\Big]. \tag{25}$$

The current is, $i(t) = dq(t)/dt$, then from (24) we obtain

$$i(t) = \frac{V_0}{R}\frac{d}{d\hat{t}}\Big[1 - E_\gamma(-\gamma^{1-\gamma}\tilde{t}^\gamma)\Big]. \tag{26}$$

Given the values, $R = 1M\Omega$, $C = 1\mu F$, we simulate the Eqs. (24), (25) and (26), obtaining the Fig. 2 show the behavior of the charge and voltage (in the same Figure). Figure 2 B) shows the voltage on the capacitor for the following fractional exponents $\gamma = 0.25$, $\gamma = 0.5$, $\gamma = 0.75$ and $\gamma = 1$.

## 4. Analysis of the Fractional Time Constant

The time constant is the time required for one capacitor to charge to $63.2$ of the total charge (maximum voltage) after a direct current source is connected to an RC circuit. The capacitor does not reach its maximum load (and voltage) in a time constant. If a new constant lag time hill be charged capacitor is now $86.5$ of the total load. This situation is similar, when the capacitor is discharged. When the CD source voltage is removed an RC circuit has a constant time after the voltage on the capacitor has gone from $100$ to $36.8$ (it has lost $63.2$ of its original value). Table I shows the value (in percent) of these two cases.

TABLE I. Values of Charge and Discharge vs., Time Constant.

| Time Constant | % Load or Growth | % Discharge or Decrease |
|---|---|---|
| 1 | 63.2 | 36.8 |
| 2 | 86.5 | 13.5 |
| 3 | 95.0 | 5.0 |
| 4 | 98.2 | 1.8 |
| 5 | 99.3 | 0.7 |

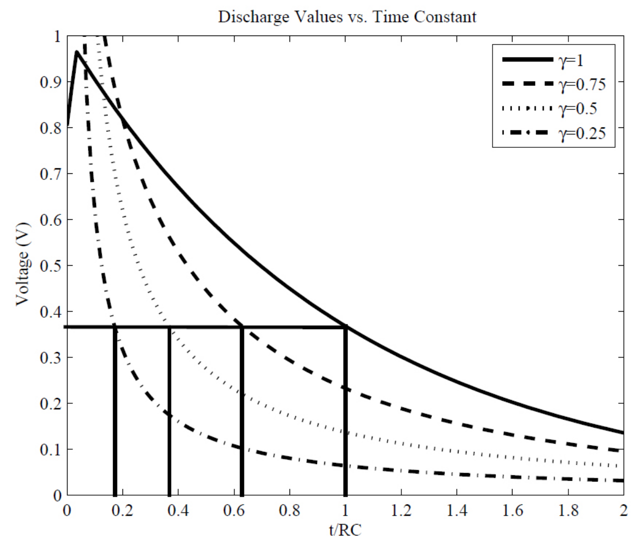


FIGURE 3. Discharge values of voltage in the RC circuit, exponent $\gamma = 1$, $\tau = 0.368$ located in $t = 1$ second, fractional exponents: $\gamma = 0.75$, $\tau = 0.368$ located in $t = 0.628$ seconds, $\gamma = 0.5$, $\tau = 0.368$ located in $t = 0.369$ seconds and $\gamma = 0.25$, $\tau = 0.368$ located in $t = 0.177$ seconds.

The discharge values are show in the Fig. 3 for the fractional exponents $\gamma = 1$, $\gamma = 0.75$, $\gamma = 0.5$ y $\gamma = 0.25$, respectively.

In assessing the fractional exponent shows that the time constant tends to move forward in time as this exponent $\gamma$

TABLE II. Discharge Values vs., Time Constant.

| $\gamma$ | Time(s) | Voltage (V) |
|---|---|---|
| 1 | 1 | 0.368 |
| 0.75 | 0.628 | 0.368 |
| 0.5 | 0.369 | 0.368 |
| 0.25 | 0.177 | 0.368 |

change from $\gamma = 1$, $\gamma = 0.75$, $\gamma = 0.5$ to $\gamma = 0.25$, respectively. That is, capacitor discharge occurs in less time than it would take the entire order of exponent. This phenomenon indicates the existence of another capacitive element, different from the ideal capacitor in the RC circuit shown in Fig. 1, showing fractional structures (components that show an intermediate behavior between a system conservative (capacitor) and dissipative (resistor)). The Table II shows the discharge values vs., time constant.

## 5. Transient Response

Then define three design specifications of the transient response [47], in the Table III is that in evaluating the output for each value of fractional exponent. Delay time $\bar{t}_d$ which is the time takes the output to reach $10\%$ of its final value. The rise time $\bar{t}_r$, is the time it takes the output to go from $10\%$ to $90\%$ of its final value and the settling time $\bar{t}_{ss}$ is defined as the time required for the response to $2\%$ around its final value and remain in that value.

The Table III describes the behavior of the delay time, rise time and settling time, respectively, for different values of $\gamma$. It is observed that for all $\gamma\epsilon(0.1]$ stored charge in the RC circuit is directly proportional to the potential difference across the capacitor, it follows that the transient behavior of the system can be analyzed using, equally, the graph of Figure 2 A) or Figure 2B), in this case is selected in Fig. 2 B) for the analysis of the transient.

TABLE III. Fractional Exponent vs., Output.

| $\gamma$ | $\bar{t}_d$ | $\bar{t}_r$ | $\bar{t}_{ss}$ |
|---|---|---|---|
| 1 | 3.16227 | 0.10000 | 1.77827 |
| 0.75 | 31.62277 | 0.06309 | 2.63026 |
| 0.5 | 100000 | 0.01000 | 39.81071 |
| 0.25 | 3162277.66016 | 0.00010 | 1995.26231 |

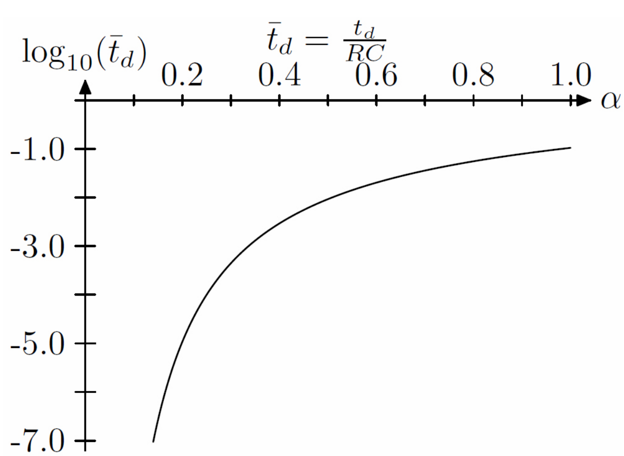


FIGURE 4. Plot of the delay time versus fractional order derivative, in the graph $\alpha = \gamma$.

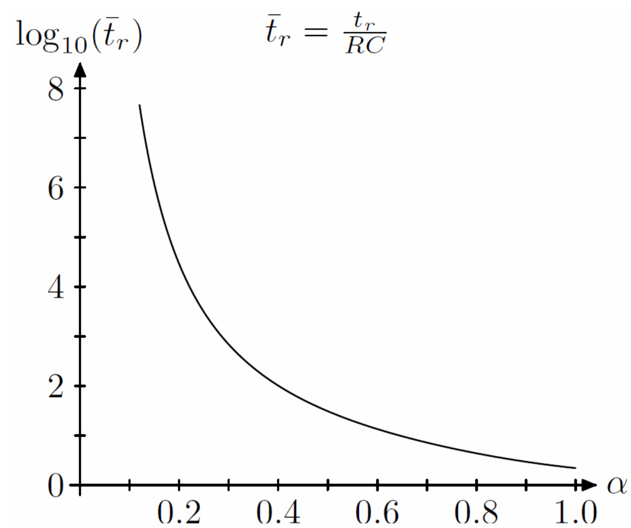


FIGURE 5. Plot of the rise time versus fractional order derivative, in the graph $\alpha = \gamma$.

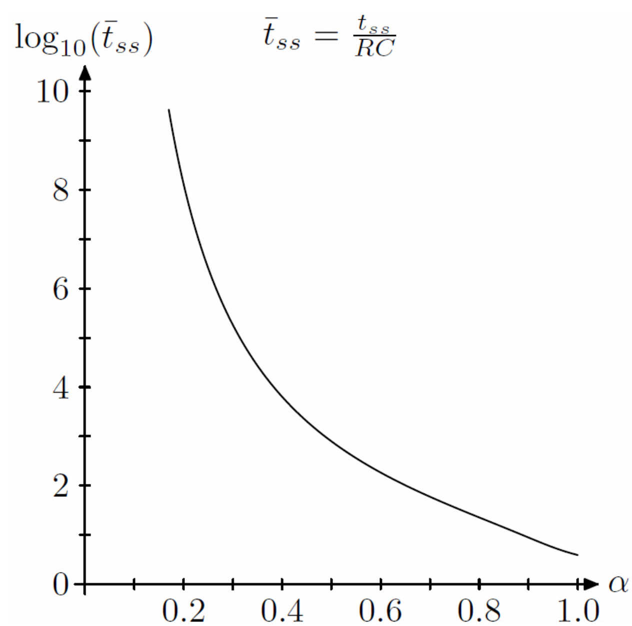


FIGURE 6. Plot of the settling time versus fractional order derivative, in the graph $\alpha = \gamma$.

For the time delay $\bar{t}_d$ can see that as the order of the fractional derivative $\gamma$, the time delay decreases with the decreases of the order of derivative, likewise, the delay sensitivity of the order of derivative increases with decreasing the order of the derivative. No values are plotted $\bar{t}_d$ for smaller values $\gamma$ because these are very small. Apparently, it has an exponential decreases settling time for values under the order of the derivative, see Fig. 4.

For the rise time $\bar{t}_r$ as the order of the derivative varies from $0.25$ to $1.0$. As can be seen, the rise time increases as the order of the derivative decreases, becoming both more sensitive. Is evident the effect that the order of the derivative can have on clock systems and semiconductors circuits, see Fig. 5.

For the settling time $\bar{t}_{ss}$ can be seen that as the order of the fractional derivative $\gamma$ tends to zero the settling time tends to infinity, that is, the settling time decreases with increasing the order of the fractional derivative, likewise, the settling time sensitivity regarding the order of the derivative also decreases. Apparently, there is an exponential growth settling time for smaller values of the derivative order, see Fig. 6.

## 6. Conclusion

Fractional calculus is a very useful tool in describing the evolution of systems with memory, which typically are dissipative and to complex systems. In this work, by use of the concept of time constant and transitory response we discuss tow important consequences of application of fractional operators in physics.

In assessing the fractional exponent shows that the time constant tends to move forward in time as this exponent $\gamma$ change from $\gamma = 1$, $\gamma = 0.75$, $\gamma = 0.5$ to $\gamma = 0.25$, respectively. That is, capacitor discharge occurs in less time than it would take the entire order of exponent. This phenomenon indicates the existence of another capacitive element, different from the ideal capacitor in the RC circuit shown in Fig. 1, showing fractional structures (components that show an intermediate behavior between a system conservative (capacitor) and dissipative (resistor)).

Respect to transient response we conclude that the settling time decreases with increasing the order of the fractional derivative, likewise, the settling time sensitivity regarding the order of the derivative also decreases. The rise time increases as the order of the derivative decreases, becoming both more sensitive. The time delay decreases with the decreases of the order of derivative, likewise, the delay sensitivity of the order of derivative increases with decreasing order of the derivative.

Is evident the effect that the order of the derivative can have on clock systems and semiconductors circuits, which must be small rise times damage to electronic circuits and large rise times produce large errors in clock circuits. On the other hand, in the steady state behavior is observed the reduction in the bandwidth having as consequence a lower data transmission capacity.

We emphasize that fractional differentiation with respect to time can be interpreted as an existence of memory effects which correspond to intrinsic dissipation in our system.

We hope that this way of dealing with fractional electrical circuit can be found applications in the power electronics, communication theory, control theory, also in the modeling of cells seen as an electrical RC circuit.

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1. K.B. Oldham and J. Spanier, *The Fractional Calculus*. (Academic Press, New York, 1974).
2. K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*. (John Wiley, NY, 1993).
3. M. Duarte Ortiguera, *Fractional Calculus for Scientists and Engineers* (Springer, NY, 2011).
4. D. Cafagna, *Fractional Calculus: A mathematical Tool From Past for Present Engineers*. (IEEE Industrial Electronics Magazine, Summer 2007).
5. S.G. Samko, A.A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*. (Langhorne, PA: Gordon and Breach Science Publishers, 1993).
6. I. Podlubny, *Fractional Differential Equations* (Academic Press, New York, 1999).
7. D. Baleanu, *Fractional Calculus Models and Numerical Methods* (World Scientific Publishing Company, 2012).
8. C.A. Monje, Y.Q. Chen, B.M. Vinagre, D. Xue, and V. Feliu, *Fractional-Order Systems and Controls*, (Series: Advances in Industrial Control, Springer, 2010).
9. R. Caponetto, G. Dongola, L. Fortuna, and I. Petrás, *Fractional Order Systems: Modeling and Control Applications*, (Singapore: World Scientific, 2010).
10. D. Baleanu, Z.B. Günvenc, and J.A. Tenreiro Machado, (Eds). *New Trends in Nanotechnology and Fractional Calculus Applications*. (Springer, 2010).
11. D. Baleanu, K. Diethelm, E. Scalas, and J.J. Trujillo, *Fractional Calculus Models and Numerical Methods*. (Series on Complexity, Nonlinearity and Chaos. World Scientific, 2012).
12. D. Baleanu, A. K. Golmankhaneh, A. K. Golmankhaneh, and R. R. Nigmatullin, *Nonlinear Dynamics* **60** (2010) 81-86.
13. W. Wyss, *J. Math. Phys.* **27** (1986) 2782-2785.
14. R. Hilfer, *J. Phys. Chem.* **104** (2000) 3914-3917.
15. R. Metzler, and J. Klafter, *Phys. Rep.* **339**, (2000) 1-77.
16. R. Metzler, and J. Klafter, *J. Phys.* **A37** (2004) R161-R208.
17. O.P. Agrawal, J.A. Tenreiro-Machado, and I. Sabatier, (Eds.) *Fractional Derivatives and Their Applications: Nonlinear Dynamics*, **38**, (Berlin: Springer-Verlag, 2004).
18. R. Hilfer, (Ed.) *Applications of Fractional Calculus in Physics* (Singapore: World Scientific, 2000).
19. S. Das, *Functional Fractional Calculus for System Identification and Controls* (Springer, NY, 2008).
20. B.J. West, M. Bologna, and P. Grigolini, *Physics of Fractional Operators*, (Berlin: Springer-Verlag, 2003).
21. R.L. Magin, *Fractional Calculus in Bioengineering* (Roddin: Begell House Publisher, 2006).

22. M. Caputo and F. Mainardi, *Pure and App. Geo.* **91** (1971) 134-147.

23. S. Westerlund, *Causality*, (University of Kalmar, Rep. No. 940426, 1994).

24. F. Riewe, *Physical Review E* **53** (1996) 1890-1899.

25. M.A.E. Herzallah, I. Muslih Sami, Dumitru Baleanu, and M. Rabei Eqab, *Nonlinear Dynamics* **66** (2011) 549-555.

26. A. Khalili Golmankhaneh, M. Yengejeh Ali, and D. Baleanu, *International Journal of Theoretical Physics* **51** (2012) 2909-2916.

27. Alireza K. Golmankhaneh, and Lap Lambert, *Investigations in Dynamics: With Focus on Fractional Dynamics* (Academic Publishing, 2012).

28. I. Muslih Sami, M. Saddallah, D. Baleanu, and E. Rabei, *Romanian Journal of Physics* **55** (2010) 659-663.

29. D. Baleanu, I. Muslih Sami, and M. Rabei Eqab, *Nonlinear Dynamics* **53** (2008) 67-74.

30. V. Uchaikin, *Fractional Calculus*. (Edit. Artishok. In Russian, 2008).

31. B. Mandelbrot, *The Fractal Geometry of Nature*, (San Francisco, CA: Freeman, 1982).

32. K. Alireza, V. F. Golmankhaneh, and D. Baleanu, *Romanian Reports in Physics* **65** (2013) 84-93.

33. I. Petrás, I. Podlubny, P. O'Leary, L. Dorcák, B.M. Vinagre, *Analoge Realization of Fractional-Order Controllers*. (Tu Kosice: BERG, Faculty, Slovakia, 2002).

34. I. Petrás. *Fractional-Order Memristor-Based Chua's Circuit* IEEE Transactions on Circuits and Systems-II: Express Briefs, **57**, Issue. 12, 975-979, (2010).

35. I. Petrás, *Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulation*, (London: Springer and Beijing: HEP, 2011).

36. N. Engheta, *Antennas and Propagation Magazine* **39** (1997) 35-46.

37. A. S. Balankin, B. Mena, J. Patiño, and D. Morales, *Physics Letters A* **377** (2013) 738-788.

38. V. E. Tarasov, and J. J. Trujillo, *Annals of Physics* **334** (2013) 1-23.

39. V.E. Tarasov, *Physics of Plasmas* **12** (2005).

40. V.E. Tarasov, *Modern Physics Letters A* **21** (2006) 1587-1600.

41. V.E. Tarasov, *Fractional Dynamics*. (Springer. HEP. 2011).

42. H. Nasrolahpour, *Comm. Nonl. Sci. Num. Simul*. **18** (2013) 2589-2593.

43. J.F. Gómez-Aguilar, J.J. Rosales-García, J.J. Bernal-Alvarado, T. Córdova-Fraga, and R. Guzmán-Cabrera, *Rev. Mex. Fís* **58** (2012) 348-352.

44. J.J. Rosales, M. Guía, J.F. Gómez, and V.I. Tkach, *Discontinuity, Nonlinearity and Complexity* **1** (2012) 325-335.

45. I. Podlubny, *Fract. Calc. App. Anal.* **5** (2002) 367-386.

46. M. Moshre-Torbati and J.K. Hammond, *J. Franklin Inst.* **335B** (1998) 1077-1086.

47. W. Bolton, *Ingeniería de Control*, 2 ed., Alfaomega, (2001).