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Black holes from Myers-Perry solution

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From the five dimensional Myers-Perry solution and consider that de metric MP corresponding to the Kaluza-Klein ansatz (zero mode), we obtained $4D$ solution with non-minimally coupled scalar and electromagnetic fields, characterized by three parameters, r_0, a, b, related to the mass, angular momentum and electromagnetic field, respectively and proposing that the $4D$ solution is a solution type black hole. Then for $a \neq 0$, $b = 0$ the electromagnetic field vanishes and the black hole is stationary. For $a = 0$, $b \neq 0$ the solution is static with electric field. If $a \neq 0$, $b \neq 0$ the solution is stationary with electric field and, due to the rotation, a magnetic field appears. The scalar field that arises from the dimensional reduction is present in all cases. At infinity the solution is asymptotically flat and the trace of the scalar field get lost, turning out that this solution is in agreement with the no hair conjecture.

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1. Introduction

Kaluza's theory (1921), complemented by Klein's interpretation (1929), known as Kaluza-Klein theory (KK) is the proposal to unify the theories of general relativity and electromagnetic theory in a five dimensional vacuum space-time. The Einstein's equations (EE) for a vacuum five dimensional space-time are equivalent to four dimensional EE with matter consisting of scalar and electromagnetic fields. A complete analysis of the KK theory can be consulted in [1], where among other results, $4D$ space-times are interpreted as perfect fluids and also geodesic trajectories are studied.

Previous work on this line includes the thermodynamic properties and classification of stationary, spherically symmetric, asymptotically flat solutions of the Kaluza-Klein theory corresponding to regular black holes in four dimensions [2]. It has been shown that the regular black holes that are constructed from the Kaluza-Klein reduction are interpreted as rotating bound states of D0 and D6 branes [3]. Moreover, structures similar to a Kaluza-Klein bubble arise when five dimensional deSitter space-time with a deficit angle is considered [4]. In [5] charged axisymmetric stationary black holes for a wide class of scalar tensor theories with minimally coupled scalar field were generated from the known general relativity electrovacuum solutions. For these solutions no regular horizons coexist with scalar field as well as they reinforce the Bekenstein's statement that there is no asymptotically flat, stationary and stable black holes in General Relativity endowed with a scalar field.

In [6] it is demonstrated that a solution for a rotating black hole with electric and scalar charges can be generated by a boost transformation of the fifth coordinate from the Kerr solution.

The Myers-Perry black hole [7] is the N-dimensional generalization of the Kerr geometry; in five dimensions it is a regular black hole with two angular momenta. Previously, in [8] it was addressed a reduction from five to four dimensions of the Myers-Perry solution ; there the region corresponding to the ergosphere, possessing two Killing vectors, was interpreted as colliding gravitational waves. In this work by applying the idea Kaluza-Klein to the five dimensional Myers-Perry solution we obtained a $4D$ solution with scalar and electromagnetic fields. The obtained solution is characterized by three free parameters interpreted as mass, angular momentum and electromagnetic field. Particular cases are static or stationary, with or without electromagnetic field, while the scalar field that arises from the dimensional reduction is always present, but at infinity its track is lost.

The 4D solution is consider a solution type Black holes, then the derived black hole is regular, except in the case that the two parameters related to rotation and electric field are both zero. The fact that the solution is derived from a regular $5D$ black hole does not guarantee that the $4D$ solution be singularity-free. Actually, the lift to $5D$ of singular $4D$ space-times has been used as a mechanism to release of singularities [9].

Another aspect of interest to be extracted from the derived solution is if the scalar field that arises from the dimensional reduction(zero mode), can be detected at infinity, being then a counter example of the no hair conjecture. In all cases the energy density associated to the scalar field vanishes at infinity. However the Riemann tensor associated to the black hole with scalar field vanishes more rapidly compared to the corresponding Schwarzschild's black hole, in a sort of smoothing of the curvature produced by the scalar field.

The paper is organized as follows:in Sec. 2 it is presented the $5D$ Myers-Perry solution; in Sec. 3 the obtained $4D$ space-times, their fields and asymptotic behavior are analyzed; the horizons of the $4D$ solution are studied and the new interpretation of the Myers-Perry's parameters is discussed. Section 4 presents interesting particular spaces-times for the different values of the parameters; in some cases the thermodynamics is discussed. Final remarks are given in the last section.

2. Myers-Perry solution

The Myers-Perry (MP) solution [7] is a solution of vacuum Einstein equations in an arbitrary dimension. In particular we address the five dimensional ($5D$) case representing a regular rotating black hole (two rotations) that is a generalization of the Kerr solution. This solution possesses three Killing vectors and one Killing tensor [10]. In the Boyer-Lindquist coordinates $(r, t, \theta, \phi, \psi)$ [10] the MP line element is;

$$ds^2 = \frac{\rho^2 r^2}{\Delta} dr^2 + \rho^2 d\theta^2 - dt^2 + (r^2 + a^2)\sin^2\theta d\phi^2 + (r^2 + b^2)\cos^2\theta d\psi^2 + \frac{r_0^2}{\rho^2}[dt + a\sin^2\theta d\phi + b\cos^2\theta d\psi]^2, \quad (1)$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta, \quad \Delta = (r^2 + a^2)(r^2 + b^2) - r_0^2 r^2. \quad (2)$$

the angles ϕ and ψ take values from the interval $[0, 2\pi]$, and θ takes values in $[0, \pi/2]$; a and b are the two angular momenta, r_0 is related to the black hole mass. The black hole horizon (double root) is located at ;

$$r_\pm^2 = \frac{1}{2}\left[r_0^2 - a^2 - b^2 \pm \sqrt{(r_0^2 - a^2 - b^2)^2 - 4a^2b^2}\right] \quad (3)$$

and represents a S^3 sphere. The MP solution is regular except if $a = b = 0$ that corresponds to the Tangherlini´s solution [11] and only in this case it has a singularity at $r = 0$.

3. Myers-Perry in $4D$

It is well known that the $5D$ Kaluza-Klein theory in vacuum is equivalent to a $4D$ space-time equipped with scalar and electromagnetic fields [1]. There are several ways to reduce from five to four dimensions, in here we adopt the Klein's compactification approach, *i.e.* to consider that the fifth coordinate has a circular topology at a small enough scale; this is also known as the cylinder condition.

The relation between the $5D$ Kaluza Klein metric $\hat{g}_{AB}$ and the $4D$ space time ($g_{\alpha\beta}$) is given by

$$\hat{g}_{AB} = \begin{pmatrix} g_{\alpha\beta} + \kappa^2\widetilde{\phi}^2 A_\alpha A_\beta & \kappa\widetilde{\phi}^2 A_\alpha \\ \kappa\widetilde{\phi}^2 A_\beta & \widetilde{\phi}^2 \end{pmatrix}, \quad (4)$$

where we have scaled the electromagnetic potential A_α by a constant κ, and $\widetilde{\phi}$ is the scalar field. $[A, B]$ run over $\{0, 1, 2, 3, 4\}$ and $[\alpha, \beta]$ over $\{1, 2, 3, 4\}$. The obtained $g_{\alpha\beta}$ is a solution to the action [1]

$$S = -\int d^4x\sqrt{-g}\widetilde{\phi}\left(\frac{R}{16\pi G} + \frac{\widetilde{\phi}^2}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{2g^{\alpha\beta}}{3\kappa^2}\frac{\partial_\alpha\widetilde{\phi}\partial_\beta\widetilde{\phi}}{\widetilde{\phi}^2}\right). \quad (5)$$

The previous action is written in the string frame (coupling between the scalar field and R); it is well known that is related to the action in the Einstein frame via a conformal transformation of the metric tensor.

We establish a comparison between the $5D$ Myers-Perry solution to interpret it in $4D$ (zero mode and coordinate r is not the radius of S^1). There are three possibilities to select the fifth coordinate (Killing directions: ϕ, t and ψ) but only two are spacelike, ϕ and ψ. Given the symmetry between ψ and ϕ and between a and b in a phase of $\pi/2$, it is physically equivalent to choose ϕ or ψ as the compact coordinate. Choosing ψ as the fifth coordinate, the scalar field and electromagnetic potential are given by:

$$\widetilde{\phi}^2 = g_{\psi\psi} = \cos^2\theta\left(r^2 + b^2 + \frac{r_0^2}{\rho^2}b^2\cos^2\theta\right), \quad (6)$$

$$\kappa\widetilde{\phi}^2 A_t = g_{t\psi} = b\cos^2\theta\frac{r_0^2}{\rho^2}, \quad (7)$$

$$\kappa\widetilde{\phi}^2 A_\phi = g_{\psi\phi} = \frac{r_0^2}{\rho^2}ab\sin^2\theta\cos^2\theta, \quad (8)$$

notice that $A_\phi = a\sin^2\theta A_t$, as in Kerr-Newman black hole, and that the coupling between the scalar and electromagnetic field is through the parameter b. The line element in $4D$ has the form:

$$ds_{4D}^2 = \left(\frac{r_0^2}{\rho^2} - 1 - \kappa^2\widetilde{\phi}^2 A_t^2\right)dt^2 + \frac{\rho^2 r^2}{\Delta}dr^2 + \left[(r^2 + a^2)\sin^2\theta + \frac{r_0^2}{\rho^2}a^2\sin^4\theta - \kappa^2\widetilde{\phi}^2 A_\phi^2\right]d\phi^2 + \rho^2 d\theta^2 + \left(a\sin^2\theta\frac{r_0^2}{\rho^2} - \kappa^2\widetilde{\phi}^2 A_t A_\phi\right)dtd\phi \quad (9)$$

this line element represents a stationary space-time with scalar field ($\widetilde{\phi}$) and electromagnetic potential $A_\mu = (A_t, 0, A_\phi, 0)$. The corresponding components of the Maxwell tensor $F_{ij} = \partial_i A_j - \partial_j A_i$ are given by,

$$F_{tr} = E_r = \partial_t A_r - \partial_r A_t = -\partial_r A_t = \frac{br_0^2\cos^4\theta}{k}\frac{(b^2 + r^2 + \rho^2)}{\rho^4\widetilde{\phi}^4} \quad (10)$$

$$F_{t\theta} = E_\theta = \partial_t A_\theta - \partial_\theta A_t = -\partial_\theta A_t = -\frac{r_0^2 b\cos^4\theta\sin 2\theta}{k}\frac{[(a^2 - b^2)(b^2 + r^2) + b^2r_0^2]}{\rho^4\widetilde{\phi}^4} \quad (11)$$

$$F_{\phi r} = B_r = \partial_\phi A_r - \partial_r A_\phi = -\partial_r A_\phi = a\sin^2\theta E_r \quad (12)$$

$$F_{\theta\phi} = B_\theta = \partial_\theta A_\phi - \partial_\phi A_\theta = \partial_\theta A_\phi = \frac{abr_0^2\sin 2\theta\cos^4\theta}{k}\frac{[(a^2 + r^2)(b^2 + r^4) + b^2r_0^2]}{\rho^4\widetilde{\phi}^4} \quad (13)$$

and

$$F_{t\phi} = 0 \qquad F_{r\theta} = 0 \tag{14}$$

From these expressions we see that it is a stationary solution with a rotation parameterized by a: if $a = 0$ the solution becomes diagonal and therefore static. Particularly interesting is the fact that the stationarity is not derived from boosting a static solution. The scalar field is present no matter the value of the parameters, as an effect of the dimensional reduction. This circumstance implies that the obtained solutions are not subject to the unicity theorems valid in $4D$ for the Einstein-Maxwell solutions.

The electromagnetic field derives from the $5D$ angular momentum b, since when $b = 0$ the electromagnetic field vanishes. Moreover, the magnetic field components $F_{\phi r}$ and $F_{\theta\phi}$ are associated to the angular momentum a since the solution with $a = 0$ and $b \neq 0$ does only possess electric field (components F_{tr} and $F_{t\theta}$).

3.1. Event Horizons

Since that the $4D$ solution comes from a solution that represents a black hole in $5D$, we search for the possible horizons considering that the $4D$ solution (9) is a solution that represent a black hole. We obtained that the $4D$ space-time (9) has one horizon derived from $g_{rr} \to \infty$ ($\Delta = 0$) that corresponds to the horizon in $5D$ (3), but in this case is a S^2 sphere with the same restriction $r_0 \geq a + b$.

We determine the event horizon solving for $g_{tt} = 0$; the solution is the double root:

$$r_{\pm}^2 = \frac{1}{2}\left[B \pm \sqrt{B^2 + 4Q}\right] \tag{15}$$

where

$$\begin{aligned} B &= r_0^2 - a^2\cos^2\theta - b^2\sin^2\theta - b^2 \\ Q &= -b^2a^2\cos^2\theta - b^4\sin^2\theta + r_0^2b^2\sin^2\theta \end{aligned} \tag{16}$$

When ξ_t^α is the time like Killing vector and ξ_ϕ^α is the rotational Killing vector, they both satisfy Killing's equation $\xi_{\alpha;\beta} + \xi_{\beta;\alpha}$, then the norm of the time like Killing vector is $\xi^\alpha\xi_\alpha = g_{tt}$.

If the norm of Killing vector goes to zero, we obtain the Killing horizon which is the information described in (15) (ergosphere).

From Eq. (15) only r_+^2 is physically acceptable since $r_-^2 < 0$. In Fig. (1) we show the two horizons, one corresponds to a spherically symmetric ($\Delta = 0$) and the other is an ergosphere whose shape depend on the values of the parameters a and b:

The previous statement can be posed in terms of trapped surfaces, whose existence is determined from the invariant K, [13]: the marginally trapped surfaces are defined when $K = 0$ and coincide in many instances with the classical horizons. On the other hand the space-time presents singularities if K diverges.

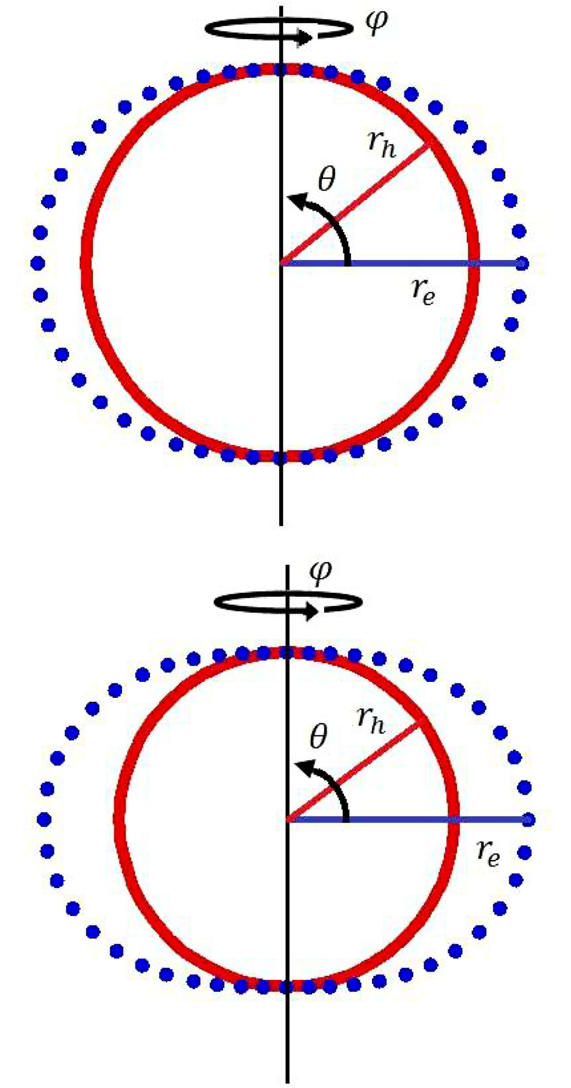

FIGURE 1. Numerical result for the horizon r_h and possible ergosphere r_e. The first graphic is for $a = b = 3$ and $r_0 = 8$. The second one corresponds to $a = 5$, $b = 0$ and $r_0 = 9$

Considering fixed coordinates $x^a = \{t, r\}$, the invariant K is given by;

$$K_{t,r} = \frac{\Delta}{\rho^6\cos^4\theta}\frac{M^2}{N^2}, \tag{17}$$

$$\begin{aligned} N &= [r_0^2b^2\cos^2\theta + \rho^2(r^2+b^2)] \\ &\quad \times [(r^2+a^2)(r_0^2b^2\cos^2\theta + \rho^2(r^2+b^2)) \\ &\quad + a^2r_0^2\rho^2(r^2+b^2)\sin^2\theta], \\ M &= (\rho^2+r^2+a^2)[r_0^2b^2\cos^2\theta + \rho^2(r^2+b^2)]^2 \\ &\quad + r_0^4a^2b^2(\rho^2+r^2+b^2)\cos^2\theta\sin^2\theta, \end{aligned} \tag{18}$$

K is zero only if $\Delta = 0$, and in this case we say that a horizon is present (horizon r_h); otherwise, Eq. (15) determines the ergosphere r_e. K does not diverge except in the case $a = b = 0$, in which $K \propto 1/r^2$ and the $4D$ solution (black hole) is singular at $r = 0$.

3.2. Interpretation of parameters

In five dimensions the parameters a and b are related to the two angular momenta of the MP black hole and r_0^2 is related to its mass. After we apply the KK reduction, the interpretation of the MP parameters in 4D may be obtained from the asymptotic behavior of the metric components in Weyl coordinates $(\tilde{r}, z)$. The transformation that relates the Boyer-Lindquist (r, θ) and Weyl $(\tilde{r}, z)$ coordinates is given by [14]:

$$\begin{aligned} \tilde{r} &= \frac{r}{2}\sqrt{\Delta}\sin 2\theta, \\ z &= \frac{r^2}{2}\cos 2\theta\left(1 - \frac{r_0^2 - a^2 - b^2}{2r^2}\right), \\ \Delta &= r^2\left(1 + \frac{a^2}{r^2}\right)\left(1 + \frac{b^2}{r^2}\right) - r_0^2, \end{aligned} \tag{19}$$

In these coordinates the following limits at infinity apply $\sqrt{\tilde{r}^2 + z^2} \to \infty$ with $z/\sqrt{\tilde{r}^2 + z^2}$ =finite. The asymptotic behavior of the metric functions in $4D$ Minkowski-spacetime is (cf. Eqs. (4.8) in [14]):

$$\begin{aligned} G_{11} &= -1 + \frac{2M}{\sqrt{\tilde{r}^2 + z^2}} + O[(\tilde{r}^2 + z^2)]^{-1} \\ G_{12} &= -\frac{2Jr^2}{(\tilde{r}^2 + z^2)^{3/2}} + O[(\tilde{r}^2 + z^2)]^{-1} \end{aligned} \tag{20}$$

And comparing Eqs. (20) with the expansions from Eq. (9), the mass and angular momentum in $4D$ are:

$$M_{4D} = \frac{2M_{5D}}{3\pi} = \frac{r_0^2}{4} \qquad J_{4D} = \frac{J_{5D}}{2\pi} = \frac{ar_0^2}{8} \tag{21}$$

In this case the parameter r_0 is related to the mass and a to the angular momentum. On the other side the parameter b is related to electromagnetic field.

3.3. Asymptotic Behavior

The asymptotic behavior of the $4D$ solution when $r \to \infty$ is:

$$ds_{4D}^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2, \tag{22}$$

that is a Minkowski space in spherical coordinates, then the space time in $4D$ is asymptotically flat.

In the limit $r \to \infty$ apparently the scalar field $\widetilde{\phi} \to \sqrt{r^2 + b^2}\cos\theta$ diverges, however, the energy density T_0^0 that corresponds to the scalar field vanishes asymptotically, as we shall show in the next subsection. Moreover, on the axial axis $r \to 0$ the scalar field is constant.

In the limit $r \to 0$, the electromagnetic field is constant; while in the limit $r \to \infty$ the field behaves like

$$\begin{aligned} E_r &\to \frac{2br_0^2}{\kappa r^6} \sim \frac{1}{r^6}, \\ E_\theta &\to \frac{2\sin\theta\cos\theta b r_0^2(b^2 - a^2)}{\kappa r^6} \sim \frac{1}{r^6}. \end{aligned} \tag{23}$$

$$\begin{aligned} B_r &\to \frac{2br_0^2 a\sin^2\theta}{\kappa r^6} \sim \frac{1}{r^6}, \\ B_\theta &= \frac{abr_0^2\sin 2\theta}{\kappa r^4} \sim \frac{1}{r^4}. \end{aligned} \tag{24}$$

The magnetic field arises from the rotation, since if $a = 0$, the magnetic field vanishes. As for the electric field, its asymptote is signing some kind of electrical multipole. If one compares with the electric field components for a given multipole, q_{lm}, whose dependence on r goes like $1/(r^{l+2})$; m being related to the azimuthal angle [12]. On this basis we can think of a electric multipole of order $l = 4$.

3.4. Energy-momentum tensor of the scalar field

To calculate the total energy-momentum tensor of the complete scalar and electromagnetic fields is very cumbersome. Since we have shown that at infinity the electromagnetic field vanishes, then it is important to determine what the behavior of the scalar field is at infinity. To this end we calculate the scalar component of $T_{\mu\nu}$. If the electromagnetic parameter vanishes, $b = 0$, the solution (9) represents a stationary ($a \neq 0$) or static ($a = 0$) black hole with scalar field; the information about the scalar matter can be extracted from (see [1])

$$T_{\alpha\beta} = \frac{1}{8\pi G}\frac{\nabla_\beta(\partial_\alpha\widetilde{\phi})}{\widetilde{\phi}}, \tag{25}$$

with the scalar field ($\widetilde{\phi}$) given by Eq. (6), the component corresponding to the energy density is

$$T_t^t = \frac{r_0^2(r^2 + 2a^2 - a^2\cos^2\theta)}{(r^2 + a^2\cos^2\theta)^3}. \tag{26}$$

In the limit of large distances, $r \to \infty$, the energy density associated to the scalar field vanishes as $T_t^t \propto r_0^2/r^4$; then the scalar field is associated to the source, vanishing at large distances, this in agreement with the asymptotic flatness of the metric (22).

The no scalar hair conjecture excludes the availability of any knowledge of a scalar field from the far away exterior geometry of a black hole even when a scalar field is present in the space-time along with gravity. Our results are then in agreement with the no-hair conjecture.

4. Particular cases

The $4D$ space-time (9) has three free parameters a, b and r_0, making it possible to obtain different space-times depending on the different values of these parameters. We shall address separately these cases as follows:

4.1. Case $r_0 \neq 0$, $b = 0$ and $a \neq 0$

From Eqs. (7) and (8) it is clear that if $b = 0$ and $a \neq 0$, both A_t and A_ϕ vanish (stationary Brans-Dicke case); the scalar field reduces to $\widetilde{\phi}^2 = r^2\cos^2\theta$. The line element (9) takes the form:

$$ds^2_{4D} = -\left(1 - \frac{r_0^2}{r^2 + a^2\cos^2\theta}\right)dt^2 + \left(\frac{r^2 + a^2\cos^2\theta}{r^2 + a^2 - r_0^2}\right)dr^2 + (r^2 + a^2\cos^2\theta)d\theta^2 + \frac{a\sin^2\theta_0^2}{r^2 + a^2\cos^2\theta}dtd\phi + \left(r^2 + a^2 + \frac{r_0^2 a^2\sin^2\theta}{r^2 + a^2\cos^2\theta}\right)\sin^2\theta d\phi^2 \tag{27}$$

Moreover the line element (27) is non diagonal ($g_{t\phi} \neq 0$), being then a stationary solution with scalar field. It has a spherical horizon $r_+^2 = r_0^2 - a^2$ hidden by the ergosphere $r_+^2 = r_0^2 - a^2\cos^2\theta$. The case when $r_0 = a$ the line element (27) is singular at $r = 0$, singularity that lies behind the horizon $r_+^2 = a^2\sin^2\theta$

4.2. Case $r_0 \neq 0$, $a = 0$ and $b \neq 0$

In the case $a = 0$, $b \neq 0$, one of the electromagnetic potential components is zero, $A_\phi = 0$, but we still have the electric field associated to A_t, and since there is no rotation, $B_i = 0$. The line element (9) represents a static space-time ($g_{t\phi} = 0$), with a spherical horizon with radios $r_+^2 = r_0^2 - b^2$, whose size is reduced from r_0 as an effect of the electromagnetic field. The line element is given by:

$$ds^2 = \left(\frac{r_0^2}{\rho^2} - 1 - \kappa^2\widetilde{\phi}^2 A_t^2\right)dt^2 + r^2\sin^2\theta d\phi^2 + (r^2 + b^2\sin^2\theta)\left[d\theta^2 + \frac{dr^2}{r^2 + b^2 - r_0^2}\right] \tag{28}$$

When $r_0 = b$ the line element (28) presents a naked singularity at $r = 0$.

4.3. Case $r_0 \neq 0$, $a = 0$ and $b = 0$

In this case the electromagnetic field vanishes and we have a static space-time with a scalar field. The metric is given by

$$ds^2_{4D} = -\left(1 - \frac{r_0^2}{r^2}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{r_0^2}{r^2}\right)} + r^2\sin\theta d\psi^2 + r^2 d\theta^2, \tag{29}$$

The solution (29) presents a spherical horizon at $r^2 = r_0^2$ and has a singularity at $r = 0$; it is very similar to the Gibbons-Maeda black hole [15] but without axion charge.

In this case the expressions for the Riemann tensor components can be handle, some of them are,

$$R^{trtr}_{\rm sc} = -\frac{3r_0^2}{r^4}, \quad R^{t\theta t\theta}_{\rm sc} = -\frac{r_0^2}{r^4(r_0^2 - r^2)}, \quad R^{\phi\theta\phi\theta}_{\rm sc} = \frac{r_0^2}{r^8\sin^2\theta}, \quad R^{r\theta r\theta}_{\rm sc} = \frac{r_0^2(r_0^2 - r^2)}{r^8}, \tag{30}$$

And one can compare them with the corresponding to a Schwarzschild solution with mass m,

$$R^{trtr}_{\rm Schw} = -\frac{2m}{r^3}, \quad R^{t\theta t\theta}_{\rm Schw} = \frac{m}{r^4(r - 2m)}, \quad R^{\phi\theta\phi\theta}_{\rm Schw} = \frac{2m}{r^7\sin^2\theta}, \quad R^{r\theta r\theta}_{\rm Schw} = -\frac{m(r - 2m)}{r^6}, \tag{31}$$

From the comparison we see that for the black hole with scalar field, its corresponding Riemann tensor vanishes, for instance $R^{\phi\theta\phi\theta}$, as r^{-8} while the Schwarzschild's goes as r^{-7}, meaning that at infinity Schwarzschild's gravitational field persists farther than the corresponding to the black hole with scalar field. Roughly speaking,

$$R^{abcd}_{\rm scalar} \propto \frac{R^{abcd}_{\rm Schw}}{r}, \tag{32}$$

It indicates that the scalar field has a smoothing effect on the curvature. In principle such effect should be detectable with very fine precision experiments, for instance, in geodesic deviation near a black hole of known mass, the measured deviation should be stronger in absence of scalar field.

In this case is also easy to check the energy conditions satisfied by the scalar field. The necessary condition for the existence of a black hole with spherical symmetry $T_t^t \leq \frac{1}{r_h^2}$ is satisfied and does not impose additional conditions on r_0; the weak energy condition $T_r^r - T_t^t \geq 0$ is satisfied as well.

The line element (29) corresponds to a static space-time with spherical symmetry and scalar field and the mass is given by [16]:

$$M(r) = \frac{r_h}{2} - \frac{1}{2}\int_{r_h}^{r} T_t^t(r)r^2 dr = \frac{r_0^2}{2r} \tag{33}$$

where r_h is the radio of the horizon that in this case is $r_h = r_0$. Testing the condition for tidal forces at the horizon $2M' - r_h M'' < 1$, it is satisfied. The regularity of geometry at the horizon ensures that tidal gravitational forces are bounded there. Test particles following geodesics feel nothing particular as they cross the horizon.

4.4. Case $r_0 \neq 0$, $a = b$

In this case the space-time has a spherical horizon at $r_\pm^2 = \frac{1}{2}(r_0^2 - 2a^2 \pm \sqrt{r_0^4 - 4a^2r_0^2})$ wrapped by the ergoshere $r_\pm^2 = \frac{1}{2}(r_0^2 - 2a^2 \pm \sqrt{r_0^4 - 4a^2r_0^2\cos^2\theta})$. The electromagnetic and scalar fields do not vanish.

4.5. Case $r_0 = 0$, $a \neq 0$, $b \neq 0$

When the parameter related to the mass r_0 is zero, the solution is static without electromagnetic field,

$$ds^2_{4D} = -dt^2 + \frac{\rho^2}{(r^2 + a^2)(r^2 + b^2)} r^2 dr^2 + (r^2 + a^2)\sin^2\theta d\phi^2 + \rho^2 d\theta^2 \quad (34)$$

This space-time is regular and does not posses horizon; the scalar field is given by $\tilde{\phi} = (r^2 + b^2)\cos^2\theta$; there is no energy density associated to $\tilde{\phi}$, being $T^t_t = 0$.

5. Black Hole Thermodynamics

Thermodynamic properties of black holes $4D$ in vacuum are well known, and is interesting to determine the thermodynamic quantities of the stationary solution (no electromagnetic fields, $b = 0$) that represents a black hole (27) with scalar field. The surface gravity [16] is given by:

$$\widehat{\kappa} = \lim_{r \to r_h} \frac{1}{2} \frac{\partial g_{tt}}{\partial r} = 2\pi T, \quad (35)$$

where r_h is the radio of horizon and T is the black hole temperature. The specific in terms of the energy density (only scalar field is present):

$$C_q = -2\pi r_h^2 \frac{1 + r_h^2 T^t_t(r_h)}{1 - r_h^2 T^t_t(r_h) - r_h^3 \frac{dT^t_t}{dr}(r_h)} \quad (36)$$

For the stationary black hole the surface gravity and the specific heat are given, respectively, by:

$$\widehat{\kappa} = \lim_{r \to r_h} \frac{1}{2} \frac{\partial g_{tt}}{\partial r} = \lim_{r \to r_h} \frac{r_0^2 r}{r^2 + a^2 \cos^2\theta} = \frac{r_0^2 \sqrt{r_0^2 - a^2}}{(r_0^2 - a^2 \sin^2\theta)^2} \quad (37)$$

$$C_q = -2\pi(r_0^2 - a^2) \frac{M(r_0^2 - a^2 \sin^2\theta)}{N} \quad (38)$$

where

$$M = (r_0^2 - a^2 \sin^2\theta)^3 + r_0^2 (r_0^2 - a^2)(r_0^2 + a^2 \sin^2\theta)$$

$$N = (r_0^2 - a^2 \sin^2\theta)^4 - r_0^2 (r_0^2 - a^2)(r_0^4 - a^4 \sin^4\theta) + 4r_0^2 (r_0^2 - a^2)^2 (r_0^2 + 2a^2 \sin^2\theta) \quad (39)$$

in this case the temperature and specific heat depend on the coordinate (θ). The denominator N does not vanish for any value and therefore we can not conjecture about a phase transition.

On the other hand when we consider the static black hole ($a = 0$), the behavior is very similar to Schwarzschild's, being the surface gravity, $\widehat{k} = \frac{1}{r_0}$; and the temperature $T = \frac{\widehat{\kappa}}{2\pi} = \frac{1}{2\pi r_0}$. The specific heat is given by $C_q = -\pi r_0^2$.

6. Conclusions

A $4D$ space-time a scalar and electromagnetic fields is obtained through the Kaluza-Klein reduction from a $5D$ Myers-Perry black hole; the angular momentum and mass of the MP solution are related in the $4D$ solution to mass and angular momentum; the second angular momentum of the MP solution becomes associated to the electromagnetic field. The event horizon is regular and in the stationary case there is an ergosphere. The $4D$ space-time is asymptotically flat and regular. There are several interesting particular cases: it may be static or stationary and the electromagnetic field may vanish or not depending of the chosen values of the parameters; the scalar field is present in all cases, as a result of the dimensional reduction.

Near the horizon the energy density of the scalar field is nonzero. Moreover we check that in the spherically symmetric case the weak energy condition holds and the tidal gravitational forces are bounded in the vicinity of horizon. The no-hair conjecture holds in this three-parameter solution: at infinity there is no trace of the scalar field because the energy density associated to the scalar field vanishes at large distances.

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