Quasinormal frequencies of the Dirac field in a D-dimensional Lifshitz black hole

A. López-Ortega

Departamento de Física, Escuela Superior de Física y Matemáticas,
Instituto Politécnico Nacional, Unidad Profesional Adolfo López Mateos, Edificio 9, México, 07738 D. F., México,
email: alopezo@esfm.ipn.mx

In a D-dimensional Lifshitz black hole we calculate exactly the quasinormal frequencies of a test Dirac field in the massless and zero angular eigenvalue limits. These results are an extension of the previous calculations in which the quasinormal frequencies of the Dirac field are determined, but in four dimensions. We discuss the four-dimensional limit of our expressions for the quasinormal frequencies and compare with the previous results. We also determine whether the Dirac field has unstable modes in the D-dimensional Lifshitz spacetime.

Keywords: Quasinormal modes; Dirac field; Lifshitz spacetimes; classical stability.

PACS: 04.40.-b; 04.30.Nk; 04.70.Bw; 04.50.Gh

1. Introduction

We know condensed matter systems for which the space and time at the critical points (Lifshitz fixed points) show the anisotropic scale invariance

$$x_i \to \lambda^2 x_i, \qquad t \to \lambda^{2\hat{z}} t, \tag{1}$$

where $\hat{z} > 1$ is the critical exponent. To extend the AdS-CFT correspondence to systems with Lifshitz points, in recent times a detailed analysis is carried out of the spacetimes that at large r asymptote to the so called Lifshitz metric [1,2]

$$ds^2 = \frac{r^{2\hat{z}}}{l^{2\hat{z}}} dt^2 - \frac{l^2}{r^2} dr^2 - r^2 d\Omega^2, \tag{2}$$

where $d\Omega^2$ denotes the line element of the $(D-2)$-dimensional plane $\mathbb{R}^{D-2}$ and l is a positive constant. In this metric the parameter $\hat{z}$ coincides with the critical exponent of the formulas (1).

Owing to the quasinormal modes (QNM) are useful to determine relevant physical properties of the black holes [3-6], recently the quasinormal frequencies (QNF) of several asymptotic Lifshitz black holes have been computed [7-16]. In this work we study the damped oscillations of the Dirac field propagating in the D-dimensional ($D \geq 4$) asymptotic Lifshitz static black hole with $\hat{z} = 2$ [2,7]

$$ds^2 = \frac{r^4}{l^4} \left(1 - \frac{r_+^2}{r^2} \right) dt^2 - \frac{l^2 dr^2}{r^2 - r_+^2} - r^2 d\Omega^2, \tag{3}$$

where r_+ locates the event horizon. The D-dimensional Lifshitz black hole (3) is a solution of a Lagrangian with scalar and vector fields [2] or with higher curvature terms [7].

To explore the classical stability of this D-dimensional Lifshitz black hole in Ref. 7 Giacomini, *et al.* calculate exactly the QNF of the massive Klein-Gordon field. They find that its spectrum of QNF is discrete and its QNM are stable. Furthermore the QNF depend on the spacetime dimension D (see the expressions (33) and (34) of Ref. 7). Their results are

extended in Refs. 15 and 16. Catalán, *et al.* [15] calculate the QNF of the Dirac field propagating in the four-dimensional Lifshitz black hole (3), that is, when $d\Omega^2$ is the line element of the two-dimensional plane $\mathbb{R}^2$. In the D-dimensional Lifshitz black hole (3) the QNF of the electromagnetic field are calculated exactly in Ref. 16. In this reference it is found that for $D = 5, 6, 7$, and for the scalar type electromagnetic field we need to impose a slightly different boundary condition as $r \to \infty$ to get a discrete and stable spectrum of QNF, since the usually imposed Dirichlet boundary condition lead us to a continuous spectrum of QNF with possible unstable QNM. For other values of the spacetime dimension the spectrum of QNF for the scalar type electromagnetic field is discrete and stable when we impose the Dirichlet boundary condition as $r \to \infty$. For the vector type electromagnetic field the usual Dirichlet boundary condition lead us to a discrete and stable spectrum of QNF. As for the Klein-Gordon field [7] the QNF of the electromagnetic field depend on the spacetime dimension. Hence in asymptotic Lifshitz black holes for the electromagnetic and Klein-Gordon fields their spectra of QNF depend on the spacetime dimension.

In this work our main objective is to show that the analytical results of Ref. 15 in four dimensions (those on the exact calculation of the QNF for the massless Dirac field and for the massive Dirac field with angular eigenvalue equal to zero) can be extended to the D-dimensional Lifshitz black hole (3) ($D \geq 4$), that is, when $d\Omega^2$ is the line element of the $(D-2)$-dimensional plane $\mathbb{R}^{D-2}$. These calculations allow us to study additional properties about the behavior of the Lifshitz black holes under small perturbations and using these results we examine whether the spectrum of QNF for the Dirac field depend on the spacetime dimension as for the Klein-Gordon and electromagnetic fields. We also study the classical stability of the Dirac field in the D-dimensional Lifshitz spacetime (2).

In a curved spacetime it is well known that in some physical phenomena the fermion fields behave in a different way

than the boson fields. For example, the fermion fields do not suffer superradiance when they are scattered by rotating black holes [17-20]. Thus we believe that is relevant to study the QNM of the Dirac field in asymptotically Lifshitz spacetimes. See Refs. 21 to 24 to find other examples where the spectrum of QNF for the Dirac field is calculated exactly in other higher dimensional spacetimes.

Thus in what follows for the massless Dirac field and for the massive Dirac field with angular eigenvalue equal to zero we calculate their QNF when they propagate in the D-dimensional Lifshitz black hole (3) and determine whether the D-dimensional Lifshitz spacetime (2) is stable against Dirac perturbations. We note that the method exploited in this work to solve the Dirac equation in the D-dimensional Lifshitz black hole (3) is different from the procedure used in Ref. 15 for the four-dimensional case, since the procedure of the previous reference is adapted to four-dimensional spacetimes.

We organize this paper as follows. Based on Refs. 25 to 28 in Sec. 2 we give the main results of the method that simplifies the Dirac equation to a pair of coupled partial equations when the D-dimensional background is maximally symmetric. In Sec. 3 we calculate exactly the QNF of the Dirac field in the D-dimensional Lifshitz black hole (3). We compute exactly the QNF for the massive Dirac field with angular eigenvalue equal to zero and for the massless Dirac field with $\kappa \neq 0$ since for these two limiting cases we can solve exactly the radial equations. In Sec. 4 we determine whether the Dirac field has unstable modes in the D-dimensional Lifshitz spacetime (2) with $\hat{z} = 2$. Finally we discuss some relevant facts in Sec. 5.

2. Dirac equation in D-dimensional maximally symmetric spacetimes

If F, G, and H are functions of r and here $\mathrm{d}\Omega^2$ is the line element of a $(D-2)$-dimensional maximally symmetric space, then for a D-dimensional spacetime ($D \geq 4$) of the form

$$\mathrm{d}s^2 = F(r)^2\mathrm{d}t^2 - G(r)^2\mathrm{d}r^2 - H(r)^2\mathrm{d}\Omega^2, \qquad (4)$$

it is known that the Dirac equation

$$i\overline{\nabla}\psi = m\psi, \qquad (5)$$

simplifies to the coupled system of partial differential equations [25-28] (see for example the formulas (30) in Ref. 28)

$$\partial_t\psi_1 + \frac{F}{G}\partial_r\psi_1 = -\left(i\kappa\frac{F}{H} + imF\right)\psi_2,$$

$$\partial_t\psi_2 - \frac{F}{G}\partial_r\psi_2 = \left(i\kappa\frac{F}{H} - imF\right)\psi_1, \qquad (6)$$

where ψ_1 and ψ_2 are the components of a two-dimensional spinor that depends on the coordinates (t, r), and κ are the eigenvalues of the Dirac operator on the $(D-2)$-dimensional

maximally symmetric space with line element $\mathrm{d}\Omega^2$, usually called the base manifold.

For the D-dimensional Lifshitz black hole (3) and for the D-dimensional Lifshitz spacetime (2) the symbol $\mathrm{d}\Omega^2$ denotes the line element of the $(D-2)$-dimensional plane $\mathbb{R}^{D-2}$, which is a maximally symmetric space, therefore we can use the coupled system of partial differential equations (6) to study the behavior of the Dirac field in these two backgrounds. Furthermore the eigenvalues κ are equal to $\kappa = i\eta$ with $\eta \in \mathbb{R}$ [29]. Notice that $\kappa = 0$ is an allowed eigenvalue of the Dirac operator on the base manifolds of the Lifshitz spacetime (2) and of the Lifshitz black hole (3).

3. Quasinormal modes of the Dirac field

Here we calculate exactly the QNF of the Dirac field propagating in the D-dimensional Lifshitz black hole (3). First we notice that in the D-dimensional Lifshitz black hole (3) the partial differential equations (6) simplify to

$$z(z^2-1)\frac{\mathrm{d}R_1}{\mathrm{d}z} - i\tilde{\omega}R_1 = -z(z^2-1)^{1/2}\left(\frac{i\hat{\kappa}}{z} + i\tilde{m}\right)R_2,$$

$$z(z^2-1)\frac{\mathrm{d}R_2}{\mathrm{d}z} + i\tilde{\omega}R_2 = -z(z^2-1)^{1/2}\left(\frac{i\hat{\kappa}}{z} - i\tilde{m}\right)R_1, \quad (7)$$

when the components ψ_1 and ψ_2 have the harmonic time dependence

$$\psi_j = \mathrm{e}^{-i\omega t}R_j(r), \qquad (8)$$

with $j = 1, 2$. In Eqs. (7) we define the quantities $z = r/r_+$, $\hat{\kappa} = (\kappa l)/r_+$, $\tilde{m} = ml$, and $\tilde{\omega} = (\omega l^3)/r_+^2$. In what follows, taking as a basis Eqs. (7) we calculate exactly the QNF of the massless Dirac field with $\kappa \neq 0$ in Subsec. 3.1, and then we determine the QNF of the massive Dirac field with $\kappa = 0$ in Subsec. 3.2. We study these two limiting cases since we have not been able to simplify Eqs. (7) when $m \neq 0$ and $\kappa \neq 0$ simultaneously.

3.1. Massless Dirac field

For a classical field propagating in the Lifshitz black hole (3) we define its QNM as the oscillations that satisfy the boundary conditions [7-16]

  i) They are purely ingoing near the horizon.

 ii) They go to zero as $r \to \infty$.

For the D-dimensional Lifshitz black hole (3) in the massless limit the system of differential equations (7) simplifies to

$$z(z^2-1)\frac{\mathrm{d}R_1}{\mathrm{d}z} - i\tilde{\omega}R_1 = -(z^2-1)^{1/2}i\hat{\kappa}R_2,$$

$$z(z^2-1)\frac{\mathrm{d}R_2}{\mathrm{d}z} + i\tilde{\omega}R_2 = -(z^2-1)^{1/2}i\hat{\kappa}R_1. \qquad (9)$$

From these we obtain the decoupled equations for the radial functions R_1 and R_2

$$\frac{\mathrm{d}^2 R_j}{\mathrm{d}z^2} + \left(\frac{1}{z} + \frac{z}{z^2 - 1}\right)\frac{\mathrm{d}R_j}{\mathrm{d}z}$$
$$+ \left(\frac{\tilde{\omega}^2 + i\epsilon\tilde{\omega}}{(z^2-1)^2} + \frac{\hat{\kappa}^2 - \tilde{\omega}^2}{z^2(z^2-1)}\right)R_j = 0, \quad (10)$$

with $\epsilon = 1$ ($\epsilon = -1$) for R_1 (R_2). Here we study in detail the radial function R_1 and we notice that similar results are valid for the radial function R_2.

Making the change of variable

$$u = \frac{z^2 - 1}{z^2}, \quad (11)$$

and taking the function R_1 as

$$R_1 = u^A(1 - u)^{B+1/4}f_1, \quad (12)$$

where the constants A and B are solutions of the algebraic equations

$$A^2 - \frac{A}{2} + \frac{\tilde{\omega}^2 + i\tilde{\omega}}{4} = 0, \qquad B^2 - \frac{1}{16} = 0, \quad (13)$$

we find that the function f_1 is a solution of the hypergeometric differential equation [30-32]

$$u(1-u)\frac{\mathrm{d}^2 f_1}{\mathrm{d}u^2} + (c - (a+b+1)u)\frac{\mathrm{d}f_1}{\mathrm{d}u} - abf_1 = 0, \quad (14)$$

with the parameters a, b, and c equal to

$$a = A + B + \frac{1}{4} + \frac{\sqrt{\hat{\kappa}^2 - \tilde{\omega}^2}}{2},$$
$$b = A + B + \frac{1}{4} - \frac{\sqrt{\hat{\kappa}^2 - \tilde{\omega}^2}}{2},$$
$$c = 2A + \frac{1}{2}. \quad (15)$$

In what follows we take

$$A = \frac{1}{2} - i\frac{\tilde{\omega}}{2}, \qquad B = \frac{1}{4}. \quad (16)$$

We expect to find the same physical results for the other values of the parameters A and B.

If we assume that the quantity c is not an integer to discard the solutions involving logarithmic terms [31,32], then we get that the radial function R_1 is given by

$$R_1 = u^{1/2-i\tilde{\omega}/2}(1-u)^{1/2}\left(C_1 \, {}_2F_1(a,b;c;u)\right.$$
$$\left. + C_2 \, u^{1-c} {}_2F_1(a-c+1,b-c+1;2-c;u)\right), \quad (17)$$

where ${}_2F_1(a,b;c;u)$ denotes the hypergeometric function and C_1, C_2 are constants [30-32]. We point out that in the coordinate u the horizon of the Lifshitz black hole (3) is located at $u = 0$ and this coordinate satisfies $u \to 1$ as $r \to \infty$.

Thus near the horizon ($u = 0$) the previous radial function behaves as

$$R_1 \approx C_1 u^{1/2-i\tilde{\omega}/2} + C_2 u^{i\tilde{\omega}/2} \approx C_1 \exp(r_+^2 r_*/l^3)$$
$$\times \exp(-i\omega r_*) + C_2 \exp(i\omega r_*), \quad (18)$$

where r_* denotes the tortoise coordinate of the Lifshitz black hole (3) and it is equal to

$$r_* = \frac{l^3}{2r_+^2}\ln(u) = \frac{l^3}{2r_+^2}\ln\left(\frac{z^2-1}{z^2}\right), \quad (19)$$

that is, $r_* \in (-\infty, 0)$ for $r \geq r_+$.

Considering that we have a time dependence of the form $\exp(-i\omega t)$ (see the formula (8)), near the horizon the first term of the formula (18) is an ingoing wave and the second term represents an outgoing wave. Hence to have a purely ingoing wave near the horizon we must take $C_2 = 0$. Thus the radial function that satisfies the boundary condition of the QNM near the horizon is

$$R_1 = C_1 u^{1/2-i\tilde{\omega}/2}(1-u)^{1/2} {}_2F_1(a,b;c;u)$$
$$= C_1 u^{1/2-i\tilde{\omega}/2}\left[(1-u)^{1/2}\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right.$$
$$\times {}_2F_1(a,b;a+b-c+1;1-u) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}$$
$$\left.\times {}_2F_1(c-a,c-b;c+1-a-b;1-u)\right], \quad (20)$$

where in the second line of the previous expression we use the Kummer formula for the hypergeometric function ${}_2F_1(a,b;c;u)$ that for $c-a-b$ different from an integer establishes [30-32] (see for example the formula (4) of Sec. 4.8 in Ref. 31)

$${}_2F_1(a,b;c;u) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$
$$\times {}_2F_1(a,b;a+b+1-c;1-u) \quad (21)$$
$$+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-u)^{c-a-b}$$
$$\times {}_2F_1(c-a,c-b;c+1-a-b;1-u).$$

In our case we can use the Kummer formula since $c-a-b = -1/2$, which differs from an integer.

Therefore from the expression (20) for the function R_1 we obtain that as $u \to 1$

$$\lim_{u\to 1} R_1 = C_1 \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}, \quad (22)$$

since the first term goes to zero in this limit and to satisfy the boundary condition ii) of the QNM as $r \to \infty$ we must impose

$$a = -n, \qquad \text{or} \qquad b = -n, \qquad n = 0, 1, 2, \ldots, \quad (23)$$

from which we find that in the D-dimensional Lifshitz black hole (3) the QNF of the massless Dirac field are equal to

$$\omega = -\frac{ir_+^2}{l^3}\left(n+1+\frac{\eta^2 l^2}{r_+^2(4n+4)}\right), \qquad (24)$$

with $\eta \in \mathbb{R}$ [29]. Using a similar method we get that for the component ψ_2 its QNF are equal to

$$\omega = -\frac{ir_+^2}{l^3}\left(n+\frac{1}{2}+\frac{\eta^2 l^2}{r_+^2(4n+2)}\right). \qquad (25)$$

For $D = 4$ and making the identification $r_+^2 = l^2/2$, as in Ref. 15, we find that the QNF (24) and (25) of the massless Dirac field become

$$\omega = -\frac{i}{l}\left(\frac{n}{2}+\frac{1}{2}+\frac{\eta^2}{2(2n+2)}\right),$$

$$\omega = -\frac{i}{l}\left(\frac{n}{2}+\frac{1}{4}+\frac{\eta^2}{2(2n+1)}\right), \qquad (26)$$

that are equal to the QNF of the Dirac field given in Ref. 15. Moreover we note that for the QNF (24) and (25) the quantity c of the formulas (15) is not an integer, as we assumed.

3.2. Massive Dirac field with $\kappa = 0$

Another limit for which we can solve exactly the radial equations is for the massive Dirac field with angular eigenvalue equal to zero. Therefore in what follows we calculate exactly the QNF of the massive Dirac field with $\kappa = 0$ propagating in the D-dimensional Lifshitz black hole (3). In contrast to Catalán, *et al.* [15] that consider negative values of the mass m, we study positive values of m. For $\kappa = 0$ the system of differential equations (7) simplifies to

$$z(z^2-1)\frac{dR_1}{dz}-i\tilde{\omega}R_1 = -z(z^2-1)^{1/2}i\tilde{m}R_2,$$

$$z(z^2-1)\frac{dR_2}{dz}+i\tilde{\omega}R_2 = z(z^2-1)^{1/2}i\tilde{m}R_1, \qquad (27)$$

from which we obtain the following decoupled equations for the radial functions R_1 and R_2

$$\frac{d^2R_j}{dz^2}+\frac{z}{z^2-1}\frac{dR_j}{dz}+\left(\frac{\tilde{\omega}^2-i\epsilon\tilde{\omega}}{z^2}\right.$$

$$\left.+\frac{\tilde{\omega}^2+i\epsilon\tilde{\omega}}{(z^2-1)^2}-\frac{\tilde{\omega}^2-i\epsilon\tilde{\omega}+\tilde{m}^2}{z^2-1}\right)R_j = 0, \qquad (28)$$

where ϵ takes the same values for R_1 and R_2 that in the previous subsection.

Here we study in detail the radial function R_1 (similar results are valid for the function R_2). Making the change of variable (11) and taking R_1 as

$$R_1 = u^A(1-u)^B f_1, \qquad (29)$$

with the quantities A and B being solutions of the algebraic equations

$$A^2-\frac{A}{2}+\frac{\tilde{\omega}^2+i\tilde{\omega}}{4} = 0, \qquad B^2-\frac{\tilde{m}^2}{4} = 0, \qquad (30)$$

we find that the function f_1 is a solution of the hypergeometric differential equation (14) with parameters a, b, and c equal to [30-32]

$$a = A+B+\frac{1}{4}+\frac{\sqrt{-\tilde{\omega}^2+i\tilde{\omega}+1/4}}{2},$$

$$b = A+B+\frac{1}{4}-\frac{\sqrt{-\tilde{\omega}^2+i\tilde{\omega}+1/4}}{2},$$

$$c = 2A+\frac{1}{2}. \qquad (31)$$

In what follows we take

$$A = \frac{1}{2}-i\frac{\tilde{\omega}}{2}, \qquad B = \frac{\tilde{m}}{2}. \qquad (32)$$

We expect to get similar results for the other values of the constants A and B.

If the parameter c is not an integer, then the radial function R_1 is given by [31,32]

$$R_1 = u^{1/2-i\tilde{\omega}/2}(1-u)^{\tilde{m}/2}\left(C_1 \, _2F_1(a,b;c;u)\right.$$

$$\left.+ C_2 \, u^{1-c} \, _2F_1(a-c+1,b-c+1;2-c;u)\right), \qquad (33)$$

with C_1 and C_2 constants. Near the horizon of the Lifshitz black hole (3) ($u = 0$) we find that the previous radial function behaves as

$$R_1 \approx C_1 \exp(r_+^2 r_*/l^3)$$

$$\times \exp(-i\omega r_*) + C_2 \exp(i\omega r_*). \qquad (34)$$

Since we assume a time dependence of the form $\exp(-i\omega t)$ we see that in the previous approximation for the radial function R_1 the first term is an ingoing wave, whereas the second term is an outgoing wave. Hence to get a purely ingoing wave near the horizon of the Lifshitz black hole (3) we must impose $C_2 = 0$.

Therefore the radial function satisfying the boundary condition i) of the QNM is equal to

$$R_1 = C_1 u^{1/2-i\tilde{\omega}/2}(1-u)^{\tilde{m}/2} \, _2F_1(a,b;c;u)$$

$$= C_1 u^{1/2-i\tilde{\omega}/2}\left[(1-u)^{\tilde{m}/2}\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right.$$

$$\times \, _2F_1(a,b;a+b+1-c;1-u)$$

$$+(1-u)^{-\tilde{m}/2}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}$$

$$\left.\times \, _2F_1(c-a,c-b;c+1-a-b;1-u)\right], \qquad (35)$$

where in the second line of the previous equation we use Kummer's formula for the hypergeometric function (21),

since the quantity $c - a - b = -\tilde{m}$ is not an integer [30-32]. From the last expression for the radial function R_1, in the limit $u \to 1$ we note that the first term goes to zero, whereas the second diverges in this limit. Therefore to satisfy the boundary condition ii) of the QNM we must impose

$$a = -n, \qquad \text{or} \qquad b = -n, \qquad n = 0, 1, 2, 3, \ldots \quad (36)$$

Using the values for the parameters a and b of the formulas (31) we obtain that in the D-dimensional Lifshitz black hole (3) the QNF of the massive Dirac field with $\kappa = 0$ are equal to

$$\omega = -\frac{ir_+^2}{l^3}\left(n + \frac{1}{2} + \frac{\tilde{m}}{2}\right). \quad (37)$$

For the radial function R_2 a similar method also gives the previous QNF. If we make $r_+^2 = l^2/2$, as previously, for $D = 4$ the QNF (37) coincide with those calculated in Ref. 15, that is, the formula (37) produces the values reported in the previous reference, except that our expression (37) gives the additional QNF $-(i/l)(1/4 + ml/4)$ for the component ψ_2 of Ref. 15. Furthermore, for the QNF (37) we get that the parameter c of the formulas (31) is not an integer, as we assumed.

4. Modes of the Dirac field in the Lifshitz spacetime

To extend the results of the previous section here we determine the modes of the Dirac field in the D-dimensional Lifshitz spacetime (2). In a similar way to the electromagnetic field [16], in the Lifshitz spacetime (2) the modes of the Dirac field must satisfy the boundary conditions:

1) The modes go to zero as $r \to \infty$.

2) The modes are regular at $r = 0$.

Our objective is to determine for the Dirac field the existence of unstable modes that satisfy the previous boundary conditions, that is, for the Dirac field propagating in the D-dimensional Lifshitz spacetime (2) we are looking for modes whose amplitudes increase with the time and that fulfill the boundary conditions 1) and 2).

For the Dirac field propagating in the D-dimensional Lifshitz spacetime (2) the coupled system of differential equations (6) simplify to

$$y^{\hat{z}+1}\frac{dR_1}{dy} - i\hat{\omega}R_1 = -y^{\hat{z}}\left(\frac{i\kappa}{y} + i\hat{m}\right)R_2,$$

$$y^{\hat{z}+1}\frac{dR_2}{dy} + i\hat{\omega}R_2 = -y^{\hat{z}}\left(\frac{i\kappa}{y} - i\hat{m}\right)R_1, \quad (38)$$

when we take the components ψ_1 and ψ_2 as in the formula (8) and we define the quantities $y = r/l$, $\hat{\omega} = \omega l$, and $\hat{m} = ml$. Since we have not been able to simplify in an appropriate form the system of differential equations (38) when $m \neq 0$ and $\kappa \neq 0$ simultaneously, in a similar way to the Lifshitz

black hole (3), in what follows we calculate the modes of the massless Dirac field with $\kappa \neq 0$ and of the massive Dirac field with $\kappa = 0$.

4.1. Massless Dirac field ($\kappa \neq 0$)

For the Dirac field propagating in the D-dimensional Lifshitz spacetime (2), in the massless limit we get that the coupled system of differential equations (38) simplifies to

$$y\frac{dR_1}{dy} - \frac{i\hat{\omega}}{y^{\hat{z}}}R_1 = -\frac{i\kappa}{y}R_2,$$

$$y\frac{dR_2}{dy} + \frac{i\hat{\omega}}{y^{\hat{z}}}R_2 = -\frac{i\kappa}{y}R_1, \quad (39)$$

from which we obtain for the functions R_1 and R_2 the decoupled equations

$$\frac{d^2R_j}{dy^2} + \frac{2}{y}\frac{dR_j}{dy} + \left(\frac{\hat{\omega}^2}{y^{2\hat{z}+2}}\right.$$
$$\left. + \frac{i\hat{\omega}(z-1)\epsilon}{y^{\hat{z}+2}} + \frac{\kappa^2}{y^4}\right)R_j = 0, \quad (40)$$

with $j = 1, 2$, and $\epsilon = 1$ for R_1 whereas $\epsilon = -1$ for R_2, as previously. In a similar way to the D-dimensional Lifshitz black hole (3) we restrict to the case $\hat{z} = 2$, since for this value of the critical exponent we can solve exactly the radial equations (40). Thus for $\hat{z} = 2$ we find that the differential equations (40) take the form

$$\frac{d^2R_j}{dx^2} + (\kappa^2 + \epsilon i\hat{\omega})R_j + \hat{\omega}^2 x^2 R_j = 0, \quad (41)$$

where we define the coordinate x by

$$x = \frac{1}{y}. \quad (42)$$

Making the ansatz

$$R_j = e^{i\hat{\omega}x^2/2}\hat{R}_j, \quad (43)$$

we obtain that the functions $\hat{R}_j$ must be a solution of the differential equation

$$v\frac{d^2\hat{R}_j}{dv^2} + \left(\frac{1}{2} - v\right)\frac{d\hat{R}_j}{dv} - \frac{i\hat{\omega}(\epsilon+1) - \eta^2}{4i\hat{\omega}}\hat{R}_j = 0, \quad (44)$$

where we use that $\kappa = i\eta$, as previously [29], and we define the variable v by

$$v = -i\hat{\omega}x^2. \quad (45)$$

We notice that the differential equations (44) for the functions $\hat{R}_j$ are confluent hypergeometric differential equations [31,32]

$$v\frac{d^2f}{dv^2} + (b - v)\frac{df}{dv} - af = 0, \quad (46)$$

with parameters

$$a_j = -\frac{\eta^2}{4i\hat{\omega}} + \frac{\epsilon+1}{4}, \qquad b_j = \frac{1}{2}. \quad (47)$$

If C_{1j} and C_{2j} are constants, the solutions of the differential equations (41) are [32]

$$R_j = C_{1j}e^{-v/2}U(a_j, b_j; v)$$
$$+ C_{2j}e^{v/2}U(b_j - a_j, b_j; e^{-i\pi}v), \qquad (48)$$

where $U(a, b; v)$ denotes the Tricomi solution of the confluent hypergeometric differential equation (46) [32]. Since we are looking for unstable modes, in what follows we assume that the imaginary parts of the frequencies fulfill $\mathbb{Im}(\omega) > 0$, and hence from the formula (45) we get that $\mathbb{Re}(v) > 0$. Notice that we choose a time dependence $\exp(-i\omega t)$ (see the formulas (8)) and for $\mathbb{Im}(\omega) > 0$ the amplitude of the Dirac field increases with the time, that is, for $\mathbb{Im}(\omega) > 0$ the modes are unstable.

For $v \to \infty$ ($r \to 0$) we find that the functions R_j of the formula (48) behave as

$$R_j \approx \frac{C_{1j}}{e^{v/2}v^{a_j}} + C_{2j}e^{v/2}v^{a_j - b_j}, \qquad (49)$$

where we use that the Tricomi solution satisfies [32]

$$U(a, b; v) \approx v^{-a}, \qquad (50)$$

as $v \to \infty$. Owing to $\mathbb{Re}(v) > 0$ we find that the first term in the formula (49) goes to zero as $v \to \infty$, whereas the second term diverges in this limit. Hence to fulfill the boundary condition 2) we must take $C_{2j} = 0$, and the radial functions R_j simplify to

$$R_j = C_{1j}e^{-v/2}U(a_j, b_j; v). \qquad (51)$$

It is convenient to recall that when the quantity b is different from an integer the Tricomi solution $U(a, b; v)$ satisfies [32]

$$U(a, b; v) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} {}_1F_1(a, b; v)$$
$$+ \frac{\Gamma(b-1)}{\Gamma(a)}v^{1-b} {}_1F_1(a-b+1, 2-b; v), \qquad (52)$$

where ${}_1F_1(a, b; v)$ is the confluent hypergeometric function [31,32]. Using this property of the Tricomi solution we write the radial functions (51) as

$$R_j = C_{1j}\left[\frac{\Gamma(1/2)e^{-v/2}}{\Gamma(a_j+1/2)}{}_1F_1(a_j, b_j; v)\right.$$
$$\left. + \frac{\Gamma(-1/2)}{\Gamma(a_j)}e^{-v/2}v^{1/2}{}_1F_1(a_j - b_j + 1, 3/2; v)\right]. \qquad (53)$$

In the limit $v \to 0$ ($r \to \infty$) we get that the previous functions behave in the form

$$\lim_{v \to 0} R_j \approx \frac{\Gamma(1/2)}{\Gamma(a_j+1/2)}. \qquad (54)$$

Hence to satisfy the boundary condition 1) we must impose

$$a_j + \frac{1}{2} = -n, \qquad n = 0, 1, 2, 3, \ldots, \qquad (55)$$

from which we obtain that the frequencies of the modes that fulfill the boundary conditions 1) and 2) are equal to

$$\hat{\omega}_j = -i\frac{\eta^2}{4}\frac{1}{n + (\epsilon+1)/4 + 1/2}. \qquad (56)$$

Considering that for $\epsilon = 1$ and $\epsilon = -1$ it is true that

$$\frac{\epsilon+1}{4} + \frac{1}{2} > 0, \qquad (57)$$

we obtain that for the frequencies (56) their imaginary parts satisfy $\mathbb{Im}(\hat{\omega}_j) < 0$, and we notice that this fact contradicts our assumption that the frequencies of the modes fulfill $\mathbb{Im}(\omega) > 0$. Therefore we do not find unstable modes satisfying the boundary conditions 1) and 2) and we infer that the modes of the massless Dirac field with $\kappa \neq 0$ are stable in the D-dimensional Lifshitz spacetime (2).

4.2. Massive Dirac field with $\kappa = 0$

As for the D-dimensional Lifshitz black hole (3), in the D-dimensional Lifshitz spacetime (2) with $\hat{z} = 2$ we can solve exactly the radial equations (38) of the massive Dirac field in the limit when the angular eigenvalue goes to zero. It is convenient to recall that $\kappa = 0$ is an allowed eigenvalue for the Dirac operator on the base manifold of the Lifshitz spacetime (2) [29]. In this limit, from Eqs. (38) we obtain the following decoupled equations for the functions R_1 and R_2

$$\frac{\mathrm{d}^2R_j}{\mathrm{d}y^2} + \frac{1}{y}\frac{\mathrm{d}R_j}{\mathrm{d}y} + \left(\frac{\hat{\omega}^2}{y^6} + \frac{2i\hat{\omega}\epsilon}{y^4} - \frac{\hat{m}^2}{y^2}\right)R_j = 0, \quad (58)$$

where the quantities j and ϵ take the same values that in the previous sections. Making the change of variable (42) and taking the functions R_j as in the formula (43), in this case we find that the functions $\hat{R}_j$ of this expression must be solutions of the differential equations

$$\frac{\mathrm{d}^2\hat{R}_j}{\mathrm{d}v^2} + \left(\frac{1}{v} - 1\right)\frac{\mathrm{d}\hat{R}_j}{\mathrm{d}v} - \left(\frac{(\epsilon+1)}{2v} + \frac{\hat{m}^2}{4v^2}\right)\hat{R}_j = 0, \quad (59)$$

where we use the variable v of the formula (45).

Proposing that the functions $\hat{R}_j$ take the form

$$\hat{R}_j = v^{A_j}\check{R}_j, \qquad (60)$$

with the quantities A_j being solutions of

$$A_j^2 - \frac{\hat{m}^2}{4} = 0, \qquad (61)$$

and substituting the expression (60) into Eq. (59), we find that the functions $\check{R}_j$ must solve the differential equations[i]

$$v\frac{\mathrm{d}^2\check{R}_j}{\mathrm{d}v^2} + (2A_j + 1 - v)\frac{\mathrm{d}\check{R}_j}{\mathrm{d}v}$$
$$- \left(\frac{1+\epsilon}{2} + A_j\right)\check{R}_j = 0. \qquad (62)$$

As in the previous subsection these are confluent hypergeometric differential equations (46) with parameters

$$a_j = A_j + \frac{1+\epsilon}{2}, \qquad b_j = 2A_j + 1. \tag{63}$$

In what follows we take $A_j = \hat{m}/2$ and hence the previous quantities are equal to

$$a_j = \frac{\hat{m} + (1+\epsilon)}{2}, \qquad b_j = 1 + \hat{m}. \tag{64}$$

Thus the radial functions R_j are

$$R_j = \mathrm{e}^{-v/2} v^{\hat{m}/2} \bigg(C_{1j} U(a_j, b_j; v) $$
$$+ C_{2j} \mathrm{e}^v U(b_j - a_j, b_j; \mathrm{e}^{-i\pi} v) \bigg), \tag{65}$$

where C_{1j} and C_{2j} are constants, as previously. In what follows we assume that the imaginary parts of the frequencies satisfy $\mathbb{Im}(\omega) > 0$, as in the previous subsection, and using the property (50) of the Tricomi solution, we find that the second term of the radial functions (65) diverges as $v \to \infty$. Therefore to get a well behaved solution as $v \to \infty$ we take $C_{2j} = 0$ and the radial functions that satisfy the boundary condition 2) are equal to

$$R_j = C_{1j} \mathrm{e}^{-v/2} v^{\hat{m}/2} U(a_j, b_j; v). \tag{66}$$

Taking into account the property (52) of the Tricomi solution we obtain that the previous radial functions transform into

$$R_j = C_{1j} \mathrm{e}^{-v/2} v^{\hat{m}/2} \left[\frac{\Gamma(1-b_j)}{\Gamma(a_j - b_j + 1)} {}_1F_1(a_j, b_j; v) \right.$$
$$\left. + \frac{\Gamma(b_j - 1)}{\Gamma(a_j)} v^{1-b_j} {}_1F_1(a_j - b_j + 1, 2 - b_j; v) \right], \tag{67}$$

that in the limit $v \to 0$ behave as

$$R_j \approx \frac{\Gamma(1-b_j)}{\Gamma(a_j - b_j + 1)} \frac{v^{\hat{m}/2}}{\mathrm{e}^{v/2}} + \frac{\Gamma(b_j - 1)}{\Gamma(a_j)} \frac{1}{\mathrm{e}^{v/2} v^{\hat{m}/2}}. \tag{68}$$

From this expression we notice that in the limit $v \to 0$ the first term goes to zero, whereas the second term diverges. Hence to fulfill the boundary condition 1) we must impose

$$a_j = -n, \qquad n = 0, 1, 2, 3, \ldots, \tag{69}$$

form which we obtain

$$\frac{\hat{m}}{2} + \frac{1+\epsilon}{2} = -n. \tag{70}$$

Since in the previous formulas for $\epsilon = 1$ and $\epsilon = -1$ the left hand side is positive we can not satisfy the conditions (70) and hence for the massive Dirac field with angular eigenvalue equal to zero we do not find modes with $\mathbb{Im}(\omega) > 0$ that fulfill the boundary conditions 1) and 2). Therefore for this field we do not obtain unstable modes satisfying the boundary conditions 1) and 2).

5. Discussion

From the expressions (24) and (25) for the QNF of the massless Dirac field and from the formula (37) for the QNF of the massive Dirac field with $\kappa = 0$ we find that their imaginary parts satisfy $\mathbb{Im}(\omega) < 0$ and considering that we take a time dependence of the form $\exp(-i\omega t)$ we get that in the D-dimensional Lifshitz black hole (3) the QNM of the massless Dirac field with $\kappa \neq 0$ and of the massive Dirac field with $\kappa = 0$ are stable since their amplitudes decay as the time increases. Thus when we impose the Dirichlet boundary condition ii) as $r \to \infty$, for the Dirac field we find a discrete spectrum of QNF for all $D \geq 4$ and its QNM are stable, as for the Klein-Gordon field [7]. Notice that for the electromagnetic field of scalar type propagating in the D-dimensional Lifshitz black hole (3) we need to impose a slightly different boundary condition as $r \to \infty$ when $D = 5, 6, 7$ [16].

For the Dirac field propagating in the D-dimensional Lifshitz black hole (3) from the expressions (24), (25), and (37) we find that its QNF do not depend explicitly on the spacetime dimension, and hence the QNM of the Dirac field behave in a different way than those of the electromagnetic and Klein-Gordon fields.

In a similar way to the electromagnetic field [16], we also show that in the D-dimensional Lifshitz spacetime (2) with $\hat{z} = 2$ the massless Dirac field with $\kappa \neq 0$ and the massive Dirac field with $\kappa = 0$ do not have unstable modes that satisfy the boundary conditions 1) and 2) of Sec. 4. Thus we expect that the D-dimensional Lifshitz spacetime (2) with $\hat{z} = 2$ is linearly stable against Dirac perturbations.

Doubtless, for the D-dimensional Lifshitz black hole (3) and the D-dimensional Lifshitz spacetime (2) the generalization of our results for the Dirac field with $m \neq 0$ and $\kappa \neq 0$ simultaneously deserves further research.

We recall that in the two-dimensional Witten black hole the QNF of the massive Dirac field are equal to [33] (see the formulas (61) and (62) in Ref. 33)

$$\omega_1 = -i \left(\frac{n}{2} + \frac{1}{2} - \frac{m_w^2}{(2n+2)} \right),$$
$$\omega_1 = -i \left(\frac{n}{2} + \frac{1}{4} - \frac{m_w^2}{(2n+1)} \right),$$
$$\omega_2 = -i \left(\frac{n}{2} + \frac{1}{4} - \frac{m_w^2}{(2n+1)} \right),$$
$$\omega_2 = -i \left(\frac{n}{2} - \frac{m_w^2}{2n} \right), \tag{71}$$

where m_w is the mass of the Dirac field in the two-dimensional black hole and $n = 1, 2, 3, \ldots$, in the second set of QNF ω_2. If in the values of the previous QNF ω_1 we make the identification $m_w = i\eta/\sqrt{2}$ and we also divide by l the whole expressions of the QNF ω_1, then we obtain the QNF (26) of the massless Dirac field in the Lifshitz black hole (3). Something similar happens with the QNF ω_2 of the

two-dimensional Witten black hole. We do not know an explication for this fact, but it points out to some connection between these two backgrounds.

In Ref. 34 Emparan and Tanabe show that for a large class of black holes, in the limit $D \to \infty$ their QNF behave as

$$\omega_D = \left[\frac{D}{2} + K - \left(\frac{e^{i\pi}}{2} \left(\frac{D}{2} + K \right) \right)^{1/3} \alpha_p \right] \frac{1}{r_+}, \quad (72)$$

where K is the angular momentum number, r_+ is the radius of the event horizon, and the quantities $-\alpha_p$ are the zeroes of the Airy function. Emparan and Tanabe proposes that the behavior for the QNF given in the formula (72) would be true for other black holes, but as shown in Ref. 16 for the electromagnetic and Klein-Gordon fields propagating in the D-dimensional Lifshitz black hole (3) the expression (72) does not describe the behavior of their QNF in the limit $D \to \infty$. From our expressions (24), (25), and (37) for the QNF of the Dirac field and taking into account that they do not depend

on the spacetime dimension D, we infer that for the Dirac field propagating in the Lifshitz black hole (3) the formula (72) does not produce the behavior of its QNF in the limit $D \to \infty$. For example, the QNF (72) are complex numbers with real part different from zero, but in the D-dimensional Lifshitz black hole the QNF (24), (25), and (37) of the Dirac field are purely imaginary. Furthermore the QNF of the Dirac field in the D-dimensional Lifshitz black hole (3) depend on the radius of the horizon as r_+^2, whereas in the formula (72) the dependence on the radius of the horizon is in the form $1/r_+$. These facts suggest that in the limit $D \to \infty$ the behavior of the QNF is not universal.

6. Acknowledgments

i. It is convenient to note that in Eqs. (62) the frequency $\hat{\omega}$ does not appear explicitly and it is contained only in the variable v (see the definition (45)).

1. S. Kachru, X. Liu and M. Mulligan, *Phys. Rev. D* **78** (2008) 106005 [arXiv:0808.1725 [hep-th]].

2. K. Balasubramanian and J. McGreevy, *Phys. Rev. D* **80** (2009) 104039. [arXiv:0909.0263 [hep-th]].

3. K. D. Kokkotas and B. G. Schmidt, *Living Rev. Rel.* **2** (1999) 2. [arXiv:gr-qc/9909058].

4. H. P. Nollert, *Class. Quant. Grav.* **16** (1999) R159.

5. E. Berti, V. Cardoso and A. O. Starinets, *Class. Quant. Grav.* **26** (2009) 163001. [arXiv:0905.2975 [gr-qc]].

6. R. A. Konoplya and A. Zhidenko, *Rev. Mod. Phys.* **83** (2011) 793. [arXiv:1102.4014 [gr-qc]].

7. A. Giacomini, G. Giribet, M. Leston, J. Oliva and S. Ray, *Phys. Rev. D* **85** (2012) 124001. [arXiv: 1203.0582 [hep-th]].

8. Y. S. Myung and T. Moon, *Phys. Rev. D* **86** (2012) 024006. [arXiv:1204.2116 [hep-th]].

9. P. A. Gonzalez, J. Saavedra and Y. Vasquez, *Int. J. Mod. Phys. D* **21** (2012) 1250054. [arXiv:1201.4521 [gr-qc]].

10. B. Cuadros-Melgar, J. de Oliveira and C. E. Pellicer, *Phys. Rev. D* **85** (2012) 024014 [arXiv:1110.4856 [hep-th]].

11. P. A. Gonzalez, F. Moncada and Y. Vasquez, *Eur. Phys. J. C* **72** (2012) 2255. [arXiv:1205.0582 [gr-qc]].

12. R. Becar, P. A. Gonzalez and Y. Vasquez, *Int. J. Mod. Phys. D* **22** (2013) 1350007. [arXiv:1210.7561 [gr-qc]].

13. S. Lepe, J. Lorca, F. Pena and Y. Vasquez, *Phys. Rev. D* **86** (2012) 066008. [arXiv:1205.4460 [hep-th]].

14. M. Catalan, E. Cisternas, P. A. Gonzalez and Y. Vasquez, arXiv:1404.3172 [gr-qc].

15. M. Catalan, E. Cisternas, P. A. Gonzalez and Y. Vasquez, *Eur. Phys. J. C* **74** (2014) 2813. [arXiv:1312.6451 [gr-qc]].

16. A. López-Ortega, *Gen. Rel. Grav.* **46** (2014) 1756. [arXiv:1406.0126 [gr-qc]].

17. W. G. Unruh, *Phys. Rev. D* **10** (1974) 3194.

18. W. G. Unruh, *Phys. Rev. Lett.* **31** (1973) 1265.

19. M. Martellini and A. Treves, *Phys. Rev. D* **15** (1977) 3060.

20. B. R. Iyer and A. Kumar, *Phys. Rev. D* **18** (1978) 4799.

21. A. Lopez-Ortega, *Int. J. Mod. Phys. D* **9** (2009) 1441. [arXiv:0905.0073 [gr-qc]].

22. A. López-Ortega, *Gen. Rel. Grav.* **39** (2007) 1011. [arXiv:0704.2468 [gr-qc]].

23. R. Becar, P. A. Gonzalez and Y. Vasquez, *Phys. Rev. D* **89** (2014) 023001. [arXiv:1306.5974 [gr-qc]].

24. A. Lopez-Ortega, *Rev. Mex. Fis.* **56** (2010) 44. [arXiv:1006.4906 [gr-qc]].

25. G. W. Gibbons and A. R. Steif, *Phys. Lett. B* **314** (1993) 13. [arXiv:gr-qc/9305018].

26. S. R. Das, G. W. Gibbons and S. D. Mathur, *Phys. Rev. Lett.* **78** (1997) 417. [arXiv:hep-th/9609052].

27. I. I. Cotaescu, *Mod. Phys. Lett. A* **13** (1998) 2991. [arXiv:gr-qc/9808030].

28. A. Lopez-Ortega, *Lat. Am. J. Phys. Educ.* **3** (2009) 578. [arXiv:0906.2754 [gr-qc]].

29. N. Ginoux, *The Dirac Spectrum* (Lecture Notes in Mathematics Vol. 1976, Springer, Dordrecht, 2009).

30. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions, Graphs, and Mathematical Table* (Dover Publications, New York, 1965).

31. Z. X. Wang and D. R. Guo, *Special Functions* (World Scientific Publishing, Singapore, 1989).

32. F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST Handbook of Mathematical Functions*, (Cambridge University Press, New York, 2010).

33. A. Lopez-Ortega and I. Vega-Acevedo, *Gen. Rel. Grav.* **43** (2011) 2631. [arXiv:1105.2802 [gr-qc]].

34. R. Emparan and K. Tanabe, *Phys. Rev. D* **89** (2014) 064028. [arXiv:1401.1957 [hep-th]].