Heat engines and the Curzon-Ahlborn efficiency

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The so-called Curzon-Ahlborn efficiency is becoming a paradigmatic result with regards to thermodynamic optimization of power cycles. Its wide applicability and sole dependence on the external heat bath temperatures (as the Carnot efficiency does) allows for an easy and fairly comparison with experimental efficiencies of striking validity. Different analytical derivations are presented in order to assess its validity and limitations for a broad variety of thermal cycles and steady state systems based on Finite-Time, Linear-Irreversible and Equilibrium Thermodynamic frameworks. Some conclusions and future perspectives are also outlined.

Keywords: Thermodynamics; heat engines; optimization; Curzon-Ahlborn efficiency.

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1. Introduction

The study of heat devices has been a cornerstone in the development of Thermodynamics and it is usual to find the Second Law in textbooks and monographs based on Clausius and Kelvin-Planck statements for refrigerators and heat engines, which fixes the overall conditions for the heat-work conversion processes in a cyclic system [1]. An step further is the Carnot theorem which plays a central role since it states the quantitative upper bounds for the efficiency of such heat devices. Carnot showed that for a heat device working as a heat engine between two thermal reservoirs at temperatures T_h and T_c ($T_h > T_c$), the maximum possible efficiency is $\eta_C = 1 - \tau$ ($\tau \equiv T_c/T_h < 1$). For a refrigerator device it is well-known that the maximum possible coefficient of performance is given by $\epsilon_C = \tau/(1-\tau)$. These two upper bounds, though of theoretical significance, are a poor guide from a practical point of view when comparing with observed results for real facilities. Two main reasons provoke this disagreement: a) real heat devices are of finite-size and work under finite-time periods. As a consequence, suffer from time- and size-irreversibilities; and b) The different steps of the particular cycles are different to those of the Carnot cycle since usually neither the heat absorption/rejection processes are reversible isothermal processes nor the adiabatic expansion/compression processes are isoentropic in nature.

Addressing above shortages was the main reason behind the work published in 1975 by F. L. Curzon and B. Ahlborn in an oriented pedagogical journal [2]. In their own words: '*We have found it instructive in our classes on thermodynamics to consider another fundamental limitation on efficiency which is caused by the rate at which heat can be exchanged between the working material and the heat reservoirs*'. The analytical result reported for the efficiency at maximum power conditions was $\eta_{CA} = 1 - \sqrt{\tau}$. As the Carnot value, it depends only on the external heat sources temperatures, it is independent of any peculiar characteristic of the heat device, and allows for a useful guide with observed values for a number of real power plants. It is worth noting that the CA-efficiency was also reported early and independently in the nuclear engineering scope by Chambadal [3] and Novikov [4].

The overall goal of this paper is to present some alternative derivations of this result based on three different Thermodynamic branches: Finite-Time based phenomenological models (which assume quite different points of view) [5–8], the well-founded Linear Irreversible Thermodynamics (LIT) framework, and Classical Equilibrium Thermodynamics [1, 9, 10]. From these complementary derivations we analyze the versatility, validity and restrictions of this result, which has evolved into a paradigm in the field of the thermodynamic optimization of energy converters. Some results for the efficiency at maximum power conditions are discussed in connection with the CA-efficiency and a number of future perspectives are finally outlined.

2. Endoreversible model

The original derivation by Curzon-Ahlborn was based on the so-called endoreversible (or exo-irreversible) model. This Finite Time model is built on the following main hypothesis: the working system under consideration obeys a Carnot-like cycle where the isothermal heat addition and heat rejection processes (at temperatures T_h' y T_c', respectively) with the two heat thermal baths at temperatures T_h y T_c follow linear heat transfer laws (see Fig. 1) in such a way that all irreversibilities are associated to the external coupling of the working system with the heat reservoirs, while the adiabatic paths remain as reversible. The validity and applicability of this endoreversible hypothesis has been widely analyzed and

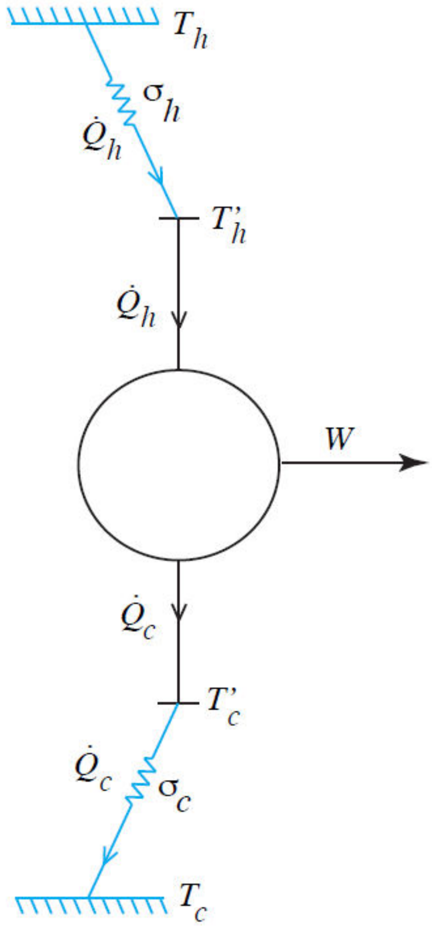

FIGURE 1. Picture of a endoreversible (exo-irreversible) Carnot-line heat engine.

discussed in the specialized literature [11–15]. With this in mind, the heat fluxes can be mathematically described by

$$\dot{Q}_h = \sigma_{\rm h}\left(T_{\rm h} - T'_{\rm h}\right) \equiv \sigma_{\rm h} T_h \left(1 - \frac{1}{a_h}\right) \tag{1}$$

$$\dot{Q}_c = \sigma_{\rm c}\left(T_{\rm c} - T'_{\rm c}\right) \equiv \sigma_{\rm c} T_c \left(a_c - 1\right) \tag{2}$$

where σ_h y σ_c denote, respectively, the thermal conductances associated to the hot and cold heat transfers and $a_h = T_h/T'_h \geq 1$, $a_c = T'_c/T_c \geq 1$.

Because of its internal reversibility, the working fluid fulfills the Clausius equality:

$$\frac{\dot{Q}_h}{T'_h} = \frac{\dot{Q}_c}{T'_c}, \tag{3}$$

and, as consequence, the two internal temperatures T'_h y T'_c (*i.e.*, a_h and a_c) are not independent variable. Substituting Eqs. (1) and (2) in (3) we obtain:

$$a_c = \frac{1}{1 - \sigma_{hc}(a_h - 1)} \tag{4}$$

where $\sigma_{hc} = \sigma_h/\sigma_c$. From the above equations, the two most significant magnitudes in the energy conversion process, the power $\dot{W} = \dot{Q}_h - \dot{Q}_c$ and the efficiency $\eta = \dot{W}/\dot{Q}_h$ read as:

$$\dot{W} = \sigma_h T_h \frac{(a_h - 1) - \sigma_{hc}\left(a_h - 1\right)^2 - \tau\left(a_h^2 - a_h\right)}{a_h\left(1 + \sigma_{hc}\right) - \sigma_{hc} a_h^2} \tag{5}$$

$$\eta = 1 - \frac{a_h \tau}{1 - \sigma_{hc}\left(a_h - 1\right)} \tag{6}$$

Equations (5) and (6) show that, under fixed values for the parameters accounting for the hot and cold conductances and the external T_h and T_c temperatures, $\dot{W}$ and η are only dependent on the upper temperature T'_h through the parameter a_h. The maximum power output is obtained under the usual condition $(\partial \dot{W}/\partial a_h)_{a_h = \bar{a}_h} = 0$. Thus, the optimal $\bar{a}_h$ and $\bar{a}_c$ values are given by:

$$\bar{a}_h = \frac{1 + \sigma_{hc}}{\sigma_{hc} + \sqrt{\tau}} \tag{7}$$

$$\bar{a}_c = \frac{\sqrt{\tau} + \sigma_{hc}}{\sqrt{\tau}(\sigma_{hc} + 1)} \tag{8}$$

From these equations we easily obtain the following expressions for the maximum power output $\dot{W}_{\max}$ and for the efficiency at maximum power $\eta_{\max \dot{W}}$:

$$\dot{W}_{\max} = \sigma_h T_h \frac{\left(1 - \sqrt{\tau}\right)^2}{1 + \sigma_{hc}} \tag{9}$$

$$\eta_{\max \dot{W}}(\tau) = \eta(\bar{a}_h, \bar{a}_c, \tau) = 1 - \sqrt{\tau} \equiv \eta_{CA} \tag{10}$$

The main conclusion we can extract from these results is that the efficiency at maximum power is only dependent of hot and cold external temperatures through the parameter τ and then it is independent on any peculiar characteristic of the system. This simple dependence leads to a fair comparison with experimental results (see Table I and Fig. 2) in spite of the simplicity of the endoreversible (or exo-irreversible) model. We also stress that these results have been obtained assuming linear heat transfer equations. Nonlinear heat transfer laws implemented in the endoreversible model do not reproduce the CA-efficiency [16].

3. Linear Irreversible Thermodynamics

An obvious criticism to the CA-derivation is its lack of generality because of its model dependence. On the other side, the analysis of Curzon and Ahlborn is not bound to devices working near equilibrium and, as a consequence it does not seem easy to reproduce it within the framework of linear irreversible thermodynamics. Any attempt should first substitute simple finite-time models by a macroscopic description of the heat device, and then one must integrate the involved differential equations describing the local behavior of such system to get information about its performance. A valuable step in this way was reported by Van den Broeck [17] showing that

TABLE I. Comparison among observed efficiencies $\eta_{\rm exp}$, with the theoretical η_C and Curzon-Ahlborn, $\eta_{\rm CA}$, values. Data taken from [20].

Power Plant	$T_{\rm c}$(K)	$T_{\rm h}$(K)	τ	$\eta_{\rm exp}$	$\eta_{\rm C}$	$\eta_{\rm CA}$
Doel 4 (Nuclear, Belgium)	566	283	0.50	0.35	0.50	0.31
Almaraz II (Nuclear, Spain)	600	290	0.48	0.34	0.52	0.31
Sizewell B (Nuclear, UK)	581	288	0.50	0.36	0.50	0.30
Cofrentes (Nuclear, Spain)	562	289	0.51	0.34	0.49	0.29
Heysham (Nuclear, UK)	727	288	0.40	0.40	0.60	0.37
West Thurrock (Coal, UK)	838	298	0.36	0.36	0.64	0.40
CANDU (Nuclear, Canada)	573	298	0.52	0.30	0.48	0.28
Larderello (Geothermal, Italy)	523	353	0.68	0.16	0.32	0.18
Calder Hall (Nuclear, UK)	583	298	0.51	0.19	0.49	0.29
(Steam/Mercury, USA)	783	298	0.38	0.34	0.62	0.38
(Steam, UK)	698	298	0.43	0.28	0.57	0.35
(Gas Turbine, Switzerland)	963	298	0.31	0.32	0.69	0.44
(Gas Turbine, France)	953	298	0.31	0.34	0.69	0.44

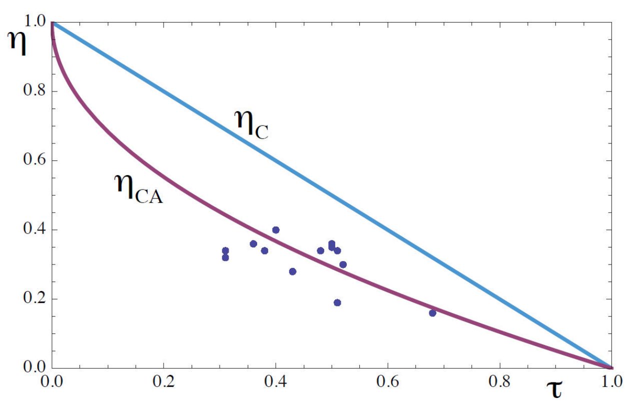

FIGURE 2. Comparison among observed efficiencies (solid points), $\eta_{\rm exp}$, with the theoretical η_C and Curzon-Ahlborn, η_{CA}, values. Data taken from [20].

the CA-efficiency is a result which can be obtained in the well-founded formalism of linear irreversible thermodynamics (LIT) thus supporting their validity and generality.

Irreversible Thermodynamics is based on the assumption of local thermodynamic equilibrium. Accordingly, the local and instantaneous relations between thermodynamic quantities in a system out of equilibrium are the same as for an equilibrium system. The resulting positive entropy production is found to be a sum of products of the so-called thermodynamic fluxes, J_i, and thermodynamic forces, X_j (also called affinities), where each flux depends on all the thermodynamic forces. The relation can be linearized in the limit of vanishing forces by a Taylor expansion, so that each flux linearly depends on each force through some coupling parameter, $L_{ij} = \partial J_i/\partial X_j$, that globally contains the local information of the system. For the particular case of two fluxes and two thermodynamic forces the following relations are well known: $J_1 = L_{11}X_1 + L_{12}X_2$, $J_2 = L_{21}X_1 + L_{22}X_2$. Here the conditions $L_{11} \geq 0$, $L_{22} \geq 0$ and $L_{11}L_{22} - L_{12}L_{21} \geq 0$ imply that entropy production $\dot{\sigma} = J_1X_1 + J_2X_2$ is a non-negative quantity, according to the second law [1,9,10].

In this framework the main steps of the proposal by Van den Broeck [17] are as follow. Consider the cyclic system sketched in Fig. 1, where now the temperature of the external thermal baths are given by $T + \Delta T$ and T, so that the power output of the working system against an external force F (mechanical, electric, or so on) is $\dot{W} = -F\dot{x}$, where $\dot{x}$ is the conjugate variable to F. To get this, the system absorbs a heat flux $|\dot{Q}|$ from the hot thermal bath and delivers a heat flux $|\dot{Q}| - |\dot{W}|$ to the cold thermal bath. The entropy generation is given by

$$\dot{\sigma} = -\frac{|\dot{Q}|}{T+\Delta T} + \frac{|\dot{Q}| - |\dot{W}|}{T} \equiv -\frac{F\dot{x}}{T} + |\dot{Q}|\left(\frac{1}{T} - \frac{1}{T+\Delta T}\right) \tag{11}$$

From this equation it is straightforward to identify the fluxes $J_1 \equiv \dot{x}$ and $J_2 \equiv \dot{Q}$ as well as the the corresponding thermodynamic forces (or affinities) $X_1 \equiv F/T$ and

$$X_2 \equiv \left(\frac{1}{T} - \frac{1}{T+\Delta T}\right) \approx \frac{\Delta T}{T^2}.$$

Taking into account the above elected forces and fluxes, the power output can be written as

$$\dot{W} = -F\dot{x} = -J_1X_1T = -(L_{11}X_1^2 + L_{12}X_1X_2)T \tag{12}$$

which shows a maximum under the condition $(\partial\dot{W}/\partial X_1)_{X_1=\bar{X}_1} = 0$. The resulting optimal force $\bar{X}_1$ s given by

$$\bar{X}_1 = -\frac{L_{12}X_2}{2L_{11}} \tag{13}$$

On the other side, the efficiency of conversion is

$$\eta = \frac{\dot{W}}{\dot{Q}} = -\frac{J_1 X_1 T}{J_2} = -\frac{J_1 X_1 T}{J_2 X_2} X_2 = -\frac{\Delta T}{T}\frac{J_1 X_1}{J_2 X_2} \quad (14)$$

After the linear expressions of J_1 and J_2 in terms of X_1 and X_2, we get that

$$\eta = -\frac{\Delta T}{T} k \frac{L_{11}k + L_{12}}{L_{21}k + L_{22}} \quad (15)$$

with $k \equiv X_1/X_2$. Finally, taking into account the optimal force $\bar{X}_1 = -L_{12}X_2/2L_{11}$ given by Eq. 13 we obtain for the efficiency at maximum power condition the following equation

$$\eta(\bar{X}_1) \equiv \eta_{\max \dot{W}} = \frac{1}{2}\frac{\Delta T}{T}\frac{q^2}{2-q^2} \quad (16)$$

where $q = L_{12}/\sqrt{L_{11}L_{22}}$ is a coupling parameter bounded by $-1 \leq q \leq +1$. In the limit of perfect coupling, $q = 1$, the efficiency is $\eta_{\max \dot{W}} = (1/2)(\Delta T/T)$, which matches at first order the CA-value

$$\eta_{CA} = 1 - \sqrt{\frac{T}{T + \Delta T}} \approx \frac{\Delta T}{2T}.$$

For a more general demonstration not limited to the linear approximation, Van den Broeck [17] uses a cascade construction of infinite cycles, each working between infinitesimal temperature differences, which under perfect coupling conditions allows to recover the exact CA-efficiency. An interesting and striking property of the cascade construction is that the overall performance regimes of the whole system may be different to those showed by each particular device [18, 19].

It should be stressed that this derivation, unlike the original one, does not need any explicit assumption for heat transfers processes or the endoreversible hypothesis, as it emerges as a straightforward consequence of the appropriate selection for the linear fluxes-forces relations from the entropy generation equation.

4. *Low-dissipation* model

A third and complementary derivation for the CA-efficiency, but conceptually very different, was reported more recently by Esposito *et al*. [20]. It does not need the endoreversible assumption, or any specific heat transfer law between the cyclic system and external heat baths couplings. Besides it is independent of the external temperatures values (*i.e.*, beyond the linear-response regime) and incorporates an additional ingredient associated to some symmetry properties which in last instance determines upper and lower bounds for the efficiency of the energy conversion.

Again, let us consider the reversible Carnot cycle in Fig. 1. During the heat absorption process the entropy variation is $\Delta S = Q_h/T_h$ while along the heat rejection step is $-\Delta S = Q_c/T_c$. Globally, the Clausius equality $Q_h/T_h = Q_c/T_c$ applies. The consideration of irreversibilities is taken into account by considering that isothermal heat transfers proceed in finite times, t_h and t_c, during the upper absorption and low rejection processes, respectively. Thus, entropy generation in these processes are Σ_h/t_h and Σ_c/t_c, where Σ_h and Σ_c are coefficients that globally contain all the information on the corresponding irreversibilities.

Because of the linear dependence of the entropy generation on the inverse of time, this model it is called *low-dissipation*. Indeed, under infinite time limits the model recovers reversible behavior without entropy generation. Another valuable characteristic of the model is that does not make any assumption on the temperatures T_h and T_c, and thus is not limited to the linear time domain. The heats involved in the cycle are given as $Q_h = T_h\left(\Delta S - \Sigma_h/t_h\right)$ and $Q_c = T_c\left(-\Delta S - \Sigma_c/t_c\right)$, while the power output is:

$$\begin{aligned} P &= \frac{Q_h + Q_c}{t_h + t_c} \\ &= \frac{(T_h - T_c)\,\Delta S - T_h\Sigma_h/t_h - T_c\Sigma_c/t_c}{t_h + t_c} \end{aligned} \quad (17)$$

The maximum power output is obtained by maximizing this function with respect to the time durations t_h and t_c. This requires to fulfill the following conditions:

$$\begin{aligned} &\left(\frac{\partial P}{\partial t_{\mathrm{c}}}\right)(\bar{t}_{\mathrm{c}}, \bar{t}_{\mathrm{h}}) = 0 \\ &\left(\frac{\partial P}{\partial t_{\mathrm{h}}}\right)(\bar{t}_{\mathrm{c}}, \bar{t}_{\mathrm{h}}) = 0 \\ &\left\{\left(\frac{\partial P}{\partial t_{\mathrm{c}}}\right)\left(\frac{\partial P}{\partial t_{\mathrm{h}}}\right) - \left[\left(\frac{\partial^2 P}{\partial t_{\mathrm{c}}\partial t_{\mathrm{h}}}\right)\right]^2\right\}(\bar{t}_{\mathrm{c}}, \bar{t}_{\mathrm{h}}) < 0 \end{aligned} \quad (18)$$

These conditions give rise to a unique physically acceptable solution:

$$\bar{t}_h = 2\frac{T_h\Sigma_h}{(T_h - T_c)\,\Delta S}\left(1 + \sqrt{\frac{T_c\Sigma_c}{T_h\Sigma_h}}\right) \quad (19)$$

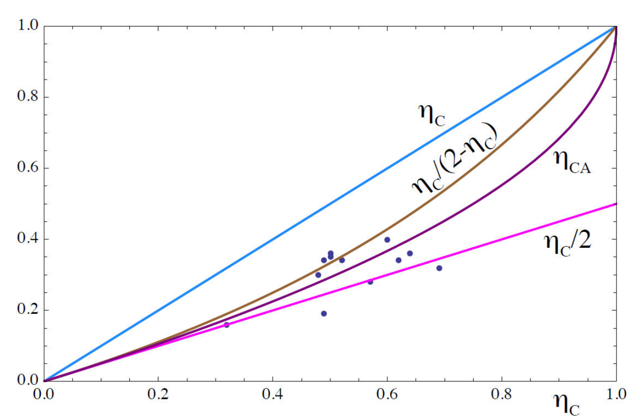

FIGURE 3. Comparison among observed efficiencies (solid points) with η_C, η_{CA}, and the upper $\eta_C = (2 - \eta_C)$ and low $\eta_C = 2$ bounds.

$$\bar{t}_c = 2\frac{T_c\Sigma_c}{(T_h - T_c)\,\Delta S}\left(1 + \sqrt{\frac{T_h\Sigma_h}{T_c\Sigma_c}}\right) \tag{20}$$

The efficiency is given by

$$\eta(t_h, t_c, \tau) = \frac{P}{Q_h} = 1 - \frac{T_c}{T_h}\frac{(\Delta S + \frac{\Sigma_c}{t_c})}{(\Delta S - \frac{\Sigma_h}{t_h})} \tag{21}$$

and taking into account the optimal values given in Eqs. 19 and 20, we finally obtain that the efficiency at maximum power conditions is given as:

$$\eta(\bar{t}_h, \bar{t}_c, \tau) = \eta_C \frac{\left(1 + \sqrt{\frac{T_c\Sigma_c}{T_h\Sigma_h}}\right)}{\left(1 + \sqrt{\frac{T_c\Sigma_c}{T_h\Sigma_h}}\right)^2 + \frac{T_c}{T_h}\left(1 - \frac{\Sigma_c}{\Sigma_h}\right)} \tag{22}$$

When $\Sigma_h = \Sigma_c$ (symmetric dissipation) the exact Curzon-Ahlborn value $1 - \sqrt{T_c/T_h} \equiv 1 - \sqrt{\tau}$ is straightforward recovered while under strong asymmetric limits $\Sigma_h/\Sigma_c \to \infty$ and $\Sigma_h/\Sigma_c \to 0$ the following lower and upper bounds $\eta_C/2 \leq \eta^* \leq \eta_C/(2-\eta_C)$ are obtained. In Fig. 3 can be checked the fairly agreement between efficiencies listed in Table I and the above bounds. One notable difference of this model with the two previous derivations is that now the CA-efficiency needs the optimization with respect two independent variables (t_h and t_c) to be obtained.

5. Conclusions and outlook

Although the original derivation of the the Curzon-Ahlborn efficiency was settled as a pedagogical tool allowing a fairly comparison between theoretical and observed results and it was obtained for a particular model (Carnot-like endoreversible or exo-irreversible cycles with linear transfer laws) it has become a paradigmatic result in studies dealing with thermodynamic optimization of power devices in the frameworks of Finite Time and Linear Irreversible Thermodynamics (and also in Stochastic Thermodynamics [21]).

In the above context, more intriguing is the fact that this result can be also obtained in the realm of (reversible) Classical Equilibrium Thermodynamics: a) Reversible cycles with two adiabatic processes alternating with two others of the same nature (for instance, two isochorics (Otto cycle) or two isobarics (Joule-Brayton cycle)) show an efficiency at maximum work that is exactly the same η_{CA}-value. Nevertheless, for reversible cycles with two adiabatic steps alternating with two others of different nature (for instance Diesel and Atkinson cycles) the efficiency at maximum work conditions is working-fluid dependent, though it is very close, but not exactly the same, to η_{CA} [22, 23]; b) A Carnot-like heat device working at maximum work between two finite thermal baths with constant and equal heat capacities also show an efficiency equal to the Curzon-Ahlborn [1, 24].

Because of the nowadays importance of the energy conversion processes, the great universality of the CA-efficiency in so many different contexts and models deserve future and closer analysis. We list below three sets of recent results that could be useful for future research work in order to provide a clear foundation and understanding of its observed ubiquity:

- The analysis of a great variety of heat engine models (classic, mesoscopic and quantum in nature) has enabled to show that at maximum power conditions, the efficiency coincides or approaches up to second order (depending on some particular symmetry conditions) with the Taylor expansion of the CA-value [25–36]:

$$\eta_{CA} = 1 - \sqrt{\tau} \equiv 1 - \sqrt{1 - \eta_C}$$
$$\approx \frac{\eta_C}{2} + \frac{\eta_C^2}{8} + \frac{6\eta_C^3}{96} + ... \tag{23}$$

  Why so many different models in nature and in different thermodynamic frameworks approaches the above Taylor expansion is still an open question.

- Minimally nonlinear irreversible thermodynamic models have also been proposed for cyclic and steady-state heat devices, in order to account for possible thermal dissipation effects in the interaction between the working system and the external heat reservoirs [37]. These models incorporate to the linear fluxes-forces relations an additional nonlinear dissipation term. As in LIT, the optimization procedure also involves only one degree of freedom, the thermodynamic flux, and the resulting energetic properties obtained under symmetric dissipation conditions allow to recover the optimized low dissipation results when the tight-coupling condition holds. Concerning these models, a different point of view was stated on the basis that dissipations should naturally appear in the LIT-models when the local Onsager relations are extended to a global scale [38, 39]. Another step forward in this direction has been reported by Sheng and Tu [40, 41] by introducing new and valuable concepts as weighted thermal fluxes and weighted reciprocal of temperature.

- A wider connection between (reversible) maximum-work and (finite-time) maximum-power thermal cycles has been recently reported suggesting an extended endorevesibility concept where the meaning of the CA-efficiency has been reconsidered and the role of heat capacities established [42, 43].

We close by stressing another open question concerning with the extension of the CA-efficiency to refrigerator devices. An important shortcoming of endoreversible models is its lack of generality. This problem remains an open question provided that the optimization of the refrigeration power for an endorreversible Carnot refrigerator, with linear finite-time heat transfers, cannot allow to obtain an analogous expression for the efficiency of the refrigerator to Curzon-Ahlborn's value for endoreversible heat engines. So, a number of different optimization criteria has been proposed in order to find a counterpart of the CA-efficiency using mainly low-disspation and linear and non-linear irreversible models [40, 41, 44–47].

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