Complete solutions of the Hamilton–Jacobi equation and the envelope method

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It is shown that the parameters contained in any two complete solutions of the Hamilton–Jacobi equation, corresponding to a given Hamiltonian, are related by means of a time-independent canonical transformation and that, in some cases, a generating function of this transformation is given by the envelope of a family of surfaces defined by the difference of the two complete solutions. Conversely, in those cases, one of the complete solutions is given by the envelope of a family of surfaces defined by the sum of the other complete solution and the generating function of the canonical transformation. Some applications of these results to geometrical optics are also given.

Keywords: Hamilton–Jacobi equation; canonical transformations; envelopes; eikonal equation.

Se muestra que los parámetros contenidos en cualesquier dos soluciones completas de la ecuación de Hamilton–Jacobi, correspondiente a una hamiltoniana dada, están relacionados por medio de una transformación canónica independiente del tiempo y que, en algunos casos, una función generatriz de esta transformación está dada por la envolvente de una familia de superficies definida por la diferencia de las dos soluciones completas. Recíprocamente, en esos casos, una de las soluciones completas está dada por la envolvente de una familia de superficies definida por la suma de la otra solución completa y la función generatriz de la transformación canónica. Se dan también algunas aplicaciones de estos resultados en la óptica geométrica.

Descriptores: Ecuación de Hamilton–Jacobi; transformaciones canónicas; envolventes; ecuación iconal.

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1. Introduction

As is well known, in the framework of classical mechanics, the solution of the Hamilton equations can be obtained from a single complete solution of the Hamilton–Jacobi (HJ) equation (see, *e.g.*, Ref. 1). What is no so widely known is that, as in the case of any first-order partial differential equation (PDE), there is an infinite number of complete solutions of the HJ equation that cannot be obtained from a complete solution substituting the parameters contained in it by functions of other parameters (see, *e.g.*, Refs. 2 and 3).

Lagrange found a method that leads, in principle, to the general solution of a first-order PDE starting from a complete solution (see, *e.g.*, Ref. 4 and the references cited therein). For instance, given a solution $z = f(x, y; a, b)$ of the first-order PDE in two variables

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0, \tag{1}$$

containing two arbitrary parameters, a, b (which, in this context, means that the solution is complete), substituting b by some function of a, $b = \phi(a)$, one obtains the one-parameter family of solutions of (1), $z = f(x, y; a, \phi(a))$. Assuming that the equation

$$\frac{\partial f(x, y; a, \phi(a))}{\partial a} = 0 \tag{2}$$

can be solved for a as a function of x and y, $a = \chi(x, y)$, and eliminating a one obtains a function

$$f(x, y; \chi(x, y), \phi(\chi(x, y))) \tag{3}$$

that is also a solution of (1); this solution is the general solution of Eq. (1) if ϕ is an arbitrary function, or another complete solution of Eq. (1) if ϕ contains two new arbitrary parameters [2, 3].

In the xy-plane, $f(x, y; a, \phi(a)) = 0$ represents a family of curves, parameterized by a, and $f(x, y; \chi(x, y), \phi(\chi(x, y))) = 0$ represents the *envelope* of this family (roughly speaking, the envelope of a family of curves is a curve that touches tangentially each member of the family, see, *e.g.*, Refs. 2 and 3); therefore, this method of finding new solutions of a first-order PDE is equivalent to finding envelopes of families of curves (or surfaces, when there are more variables involved) that represent solutions of the equation.

One example of the use of the concept of envelope is encountered in geometrical optics where the propagation of the light can be described with the aid of wavefronts. According to the Huygens principle, each point of a wavefront is the source of secondary waves, and the new wavefronts are the envelopes of the secondary waves (see, *e.g.*, Refs. 5 and 6). The wavefronts are the level surfaces of the eikonal function, which obeys a first-order PDE (the eikonal equation).

Actually, the concept of envelope is present in many places; for instance, finding the Legendre transform of a

function of several variables amounts to finding the envelope of a family of surfaces.

The aim of this paper is to show that, for a given Hamiltonian, any two complete solutions of the HJ equation are related by means of a time-independent canonical transformation, and that one obtains, in a natural manner, Lagrange's method of envelopes, with similar results for the case of the eikonal equation.

In Sec. 2 we present an elementary constructive proof of the fact that any two complete solutions of the HJ equation are related by a time-independent canonical transformation and that any complete solution of the HJ equation is obtained from any other such solution looking for the envelope of a family of surfaces. In Sec. 3 we apply these results to geometrical optics.

2. Complete solutions of the HJ equation, canonical transformations, and envelopes

For a given Hamiltonian, $H(q_i, p_i, t)$, of a system with n degrees of freedom, the HJ equation is the first-order PDE

$$H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) + \frac{\partial S}{\partial t} = 0. \tag{4}$$

A complete solution of this equation is a function of $2n + 1$ variables, $S(q_i, P_i, t)$, that satisfies Eq. (4) and the condition

$$\det\left(\frac{\partial^2 S}{\partial q_i \partial P_j}\right) \neq 0.$$

The function S generates a canonical transformation

$$Q_i = Q_i(q_j, p_j, t), \qquad P_i = P_i(q_j, p_j, t), \tag{5}$$

such that the new Hamiltonian is equal to zero (see, *e.g.*, Ref. 1). The canonical transformation (5) is determined by

$$p_i = \frac{\partial S}{\partial q_i}, \qquad Q_i = \frac{\partial S}{\partial P_i}, \tag{6}$$

which give p_i and Q_i as functions of q_i, P_i, and t, hence,

$$\mathrm{d}S = p_i(q_j, P_j, t)\,\mathrm{d}q_i$$
$$- H\big(q_i, p_i(q_j, P_j, t), t\big)\,\mathrm{d}t + Q_i(q_j, P_j, t)\,\mathrm{d}P_i \tag{7}$$

(here and henceforth there is sum over repeated indices).

In a similar manner, if $\tilde{S}(q_i, \tilde{P}_i, t)$ is a second complete solution of the HJ equation (4), then

$$\mathrm{d}\tilde{S} = p_i(q_j, \tilde{P}_j, t)\,\mathrm{d}q_i$$
$$- H\big(q_i, p_i(q_j, \tilde{P}_j, t), t\big)\,\mathrm{d}t + \tilde{Q}_i(q_j, \tilde{P}_j, t)\,\mathrm{d}\tilde{P}_i, \tag{8}$$

where $(\tilde{Q}_i, \tilde{P}_i)$ is another set of canonical coordinates, all of which are constants of motion. By combining Eqs. (7) and (8) one finds that

$$\tilde{Q}_i \mathrm{d}\tilde{P}_i - Q_i \mathrm{d}P_i = \mathrm{d}(\tilde{S} - S), \tag{9}$$

provided that P_i and $\tilde{P}_i$ are related in such a way that

$$\frac{\partial(\tilde{S} - S)}{\partial q_i} = 0 \tag{10}$$

[see the first equation in (6)]. The condition

$$\frac{\partial(\tilde{S} - S)}{\partial t} = 0$$

is automatically satisfied as a consequence of Eqs. (10) since, by hypothesis, S and $\tilde{S}$ are solutions of the same HJ equation (see Example 1, below). Equations (10) constitute a system of n equations that, under appropriate conditions, determine the q_i as functions of P_i, $\tilde{P}_i$, and t.

Equation (9) shows that the coordinates (Q_i, P_i) and $(\tilde{Q}_i, \tilde{P}_i)$, associated with S and $\tilde{S}$, respectively, are related by a *time-independent canonical transformation*, and that

$$F \equiv \tilde{S}(q_i, \tilde{P}_i, t) - S(q_i, P_i, t) \tag{11}$$

is a generating function of this transformation *if* $(P_i, \tilde{P}_i)$ are functionally independent. In this latter case, from Eq. (9), we find that

$$\tilde{Q}_i = \frac{\partial F}{\partial \tilde{P}_i}, \qquad Q_i = -\frac{\partial F}{\partial P_i}. \tag{12}$$

Thus, for a given HJ equation, the only difference between its complete solutions is in the sets of canonical coordinates (Q_i, P_i) associated with them. These coordinates are constants of motion that label the solutions of the equations of motion.

Remark 1. Even if $(P_i, \tilde{P}_i)$ are functionally dependent, Eq. (9) implies that F can be expressed as a function of P_i and $\tilde{P}_i$ only, though not in a unique way. As a consequence of Eqs. (10), $F(P_i, \tilde{P}_i) = 0$ is the (equation of the) *envelope* of the family of surfaces $\tilde{S}(q_i, \tilde{P}_i, t) - S(q_i, P_i, t) = 0$, parameterized by the q_i.

Example 1. *Relation between two given complete solutions of the HJ equation*

By means of direct computations one can verify that the functions

$$S(q, P, t) = -mgtq + Pq - \frac{P^2 t}{2m} + \frac{Pgt^2}{2} - \frac{1}{6}mg^2 t^3 \tag{13}$$

and

$$\tilde{S}(q, \tilde{P}, t) = -mgtq + \frac{m}{2t}\left(q + \frac{gt^2}{2} + \tilde{P}\right)^2 - \frac{1}{6}mg^2 t^3 \tag{14}$$

are complete solutions of the HJ equation

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + mgq + \frac{\partial S}{\partial t} = 0, \tag{15}$$

corresponding to a particle of mass m in a uniform gravitational field. Then,

$$\tilde{S} - S = \frac{m}{2t}\left(q + \frac{gt^2}{2} + \tilde{P}\right)^2 - Pq + \frac{P^2 t}{2m} - \frac{Pgt^2}{2} \tag{16}$$

and Eq. (10) yields

$$\frac{m}{t}\left(q + \frac{gt^2}{2} + \tilde{P}\right) - P = 0. \qquad (17)$$

Making use of this last relation in order to eliminate q from Eq. (16) we obtain [see Eq. (11)]

$$F(P, \tilde{P}) = P\tilde{P}. \qquad (18)$$

It may be noticed that t disappeared together with q, and that the canonical transformation generated by the generating function (18) is the "exchange transformation," $\tilde{Q} = P$, $\tilde{P} = -Q$ [see Eqs. (12)].

On the other hand, assuming that $(P_i, \tilde{P}_i)$ are functionally independent, from Eq. (11) we have

$$\tilde{S}(q_i, \tilde{P}_i, t) = S(q_i, P_i, t) + F(P_i, \tilde{P}_i). \qquad (19)$$

In order to express $\tilde{S}$ as a function of q_i, $\tilde{P}_i$, and t only, the parameters P_i appearing on the right-hand side of this last equation are eliminated making use of the relations [see Eqs. (6) and (12)]

$$\frac{\partial(S + F)}{\partial P_i} = 0, \qquad (20)$$

which are analogous to Eqs. (10). In this case, owing to Eqs. (20), $\tilde{S}(q_i, \tilde{P}_i, t) = 0$ is the envelope of the family of surfaces $S(q_i, P_i, t) + F(P_i, \tilde{P}_i) = 0$, parameterized by the P_i.

The remaining case, where $(P_i, \tilde{P}_i)$ are functionally dependent, can be reduced to the first case by replacing one or several parameters P_i, contained in S, by their conjugate variables, Q_i, in order to obtain a set of n parameters that together with the $\tilde{P}_i$ form a functionally independent set. For instance, instead of $S(q_i, P_i, t)$ we can make use of its Legendre transform $S'(q_i, Q_i, t) \equiv S(q_i, P_i, t) - P_i Q_i$, if $(Q_i, \tilde{P}_i)$ are functionally independent. Then, by relabeling the parameters in S', we obtain a function $S'(q_i, P_i, t)$ that will be related to $\tilde{S}(q_i, \tilde{P}_i, t)$ according to (19) and (20).

Thus, we conclude that from a given complete solution of the HJ equation, or a Legendre transform of it, one can obtain any other complete solution by means of an appropriate canonical transformation.

Example 2. *Obtaining a complete solution of the HJ equation from another complete solution*

It is illustrative to consider again the complete solution (13) of the HJ equation (15), with F given by Eq. (18), so that the right-hand side of Eq. (19) is

$$S(q, P, t) + F(P, \tilde{P}) = -mgtq$$
$$+ Pq - \frac{P^2 t}{2m} + \frac{Pgt^2}{2} - \frac{1}{6}mg^2 t^3 + P\tilde{P}. \qquad (21)$$

Then, Eq. (20) yields

$$q - \frac{Pt}{m} + \frac{gt^2}{2} + \tilde{P} = 0,$$

which is equivalent to Eq. (17). Making use of this last relation to eliminate P from the right-hand side of Eq. (21), one recovers the complete solution (14) of Eq. (15).

Remark 2. If the number of parameters $\tilde{P}_i$ contained in F is less than n, the solution $\tilde{S}$ of the HJ equation obtained by means of Eqs. (19) and (20) will not be complete but, nevertheless, it will be a solution. Indeed, we can think that F originally had n parameters $\tilde{P}_i$, and that k of them have taken some fixed values, leaving only $n-k$ arbitrary parameters $\tilde{P}_i$.

3. Geometrical optics

As pointed out in the Introduction, in classical mechanics, the knowledge of a single complete solution of the HJ equation is enough to find the solution of the equations of motion, so that usually there would be no need of additional complete solutions of the HJ equation. By contrast, in geometrical optics it is important to have different (not necessarily complete) solutions of the eikonal equation, which is analogous to the HJ equation.

The eikonal equation is given by

$$(\nabla S)^2 = n^2, \qquad (22)$$

where n is the refractive index of the medium, and S and n are functions of the space coordinates only (assuming that the medium is isotropic) (see, *e.g.*, Refs. 5 and 6). In terms of Cartesian coordinates, (x, y, z), the eikonal equation (22) can be written, *e.g.*, as

$$\pm\left[n^2 - \left(\frac{\partial S}{\partial x}\right)^2 - \left(\frac{\partial S}{\partial y}\right)^2\right]^{1/2} + \frac{\partial S}{\partial z} = 0, \qquad (23)$$

which has the form of the HJ equation (4), with z in place of t. (See also Ref. 7.)

The surfaces $S = $ const. are the wavefronts and the rays of light are the curves orthogonal to the wavefronts. Each solution of the eikonal equation corresponds to a family of wavefronts in such a way that, if a_1 and a_2 are constants (such that $S = a_1$ and $S = a_2$ are nonempty sets), the wavefront $S = a_2$ is obtained from $S = a_1$ by the propagation of the light by a time $(a_2 - a_1)/c$, where c is the speed of light in vacuum.

For instance, if the refractive index is *constant*, one can readily verify that

$$S(x, y, P_1, P_2, z) = n\sqrt{(x - P_1)^2 + (y - P_2)^2 + z^2} \quad (24)$$

is a complete solution of the eikonal equation [with $P_1, P_2 \in (-\infty, \infty)$] and the wavefronts $S = $ const. are spheres [centered at $(P_1, P_2, 0)$]. Choosing the generating function

$$F(P_i, \tilde{P}_i) = P_i \tilde{P}_i \qquad (25)$$

[*cf.* Eq. (18)], which generates the exchange transformation, $\tilde{Q}_i = P_i$, $\tilde{P}_i = -Q_i$, from Eqs. (20) we obtain the conditions

$$-\frac{n(x - P_1)}{\sqrt{(x - P_1)^2 + (y - P_2)^2 + z^2}} + \tilde{P}_1 = 0,$$

$$-\frac{n(y - P_2)}{\sqrt{(x - P_1)^2 + (y - P_2)^2 + z^2}} + \tilde{P}_2 = 0.$$

Making use of these relations to eliminate the P_i from $S + F$ [see Eq. (19)], we obtain a second complete solution of the eikonal equation, namely

$$\tilde{S}(x, y, \tilde{P}_1, \tilde{P}_2, z) = \tilde{P}_1 x + \tilde{P}_2 y + \sqrt{n^2 - \tilde{P}_1{}^2 - \tilde{P}_2{}^2}\, z \quad (26)$$

(with $\tilde{P}_1{}^2 + \tilde{P}_2{}^2 \leqslant n^2$). The wavefronts $\tilde{S} = $ const. are planes [with normal $(\tilde{P}_1, \tilde{P}_2, \sqrt{n^2 - \tilde{P}_1{}^2 - \tilde{P}_2{}^2}\,)$].

Other generating functions lead to other families of wavefronts. According to the results of Sec. 2, with a suitable function F, one can obtain any possible family of wavefronts, starting from *any* complete solution of the eikonal equation [such as (24) or (26)].

Example 3. *Refracted wavefronts produced by a point source*

In this example we shall consider two homogeneous media, of (constant) refractive indices n_0 and n_1, separated by a plane, and we want to find the wavefronts in the second medium produced by a point source in the first medium.

A complete solution of the eikonal equation in the second medium is given by

$$S(x, y, P_1, P_2, z) = P_1 x$$
$$+ P_2 y + \sqrt{n_1{}^2 - P_1{}^2 - P_2{}^2}\, z, \quad (27)$$

its wavefronts are planes and the parameters P_1, P_2 specify the normal to the planes and, therefore, the direction of the light rays. We want to replace the parameters P_1, P_2 by some new parameters, $\tilde{P}_1$, $\tilde{P}_2$, that specify the position of the point source in the first medium. To this end we find the light rays emanating from a point source placed at (x_0, y_0, z_0). Assuming that

$$n(x, y, z) = \begin{cases} n_0, & \text{if } z < 0, \\ n_1, & \text{if } z > 0, \end{cases}$$

from the Hamilton equations applied to the Hamiltonian

$$H = -\sqrt{n^2 - p_x{}^2 - p_y{}^2}$$

[see Eq. (23)], we find that, since H does not depend on x or y, p_x and p_y are constant (which is equivalent to Snell's law) and

$$\frac{dx}{dz} = \frac{\partial H}{\partial p_x} = \frac{p_x}{\sqrt{n^2 - p_x{}^2 - p_y{}^2}},$$

$$\frac{dy}{dz} = \frac{\partial H}{\partial p_y} = \frac{p_y}{\sqrt{n^2 - p_x{}^2 - p_y{}^2}}. \quad (28)$$

These last two equations mean that the light rays are straight lines for $z < 0$, or $z > 0$ (where n is constant). Integrating both sides of the first equation in (28) over z, between z_0 and 0 (with $z_0 < 0$), we obtain

$$x_1 - x_0 = -\frac{z_0 p_x}{\sqrt{n_0{}^2 - p_x{}^2 - p_y{}^2}}, \quad (29)$$
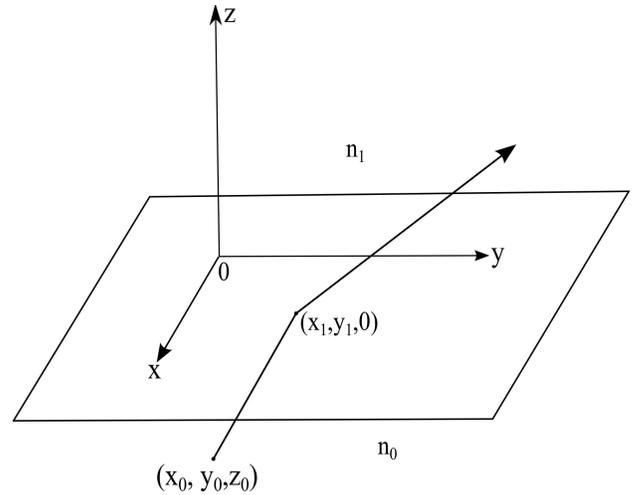
FIGURE 1. A light ray emanating from the point (x_0, y_0, z_0) is refracted at the plane $z = 0$ at $(x_1, y_1, 0)$. The coordinates x_0 and y_0 of the source are taken as the new parameters $\tilde{P}_1$ and $\tilde{P}_2$, and the parameters P_1, P_2 define the direction of the refracted ray.

where x_0 and x_1 are the values of x at $z = z_0$ and $z = 0$, respectively (see Fig. 1) and, in a similar manner,

$$y_1 - y_0 = -\frac{z_0 p_y}{\sqrt{n_0{}^2 - p_x{}^2 - p_y{}^2}}. \quad (30)$$

Being the solution of the Hamilton equations, the relation between the values of the canonical coordinates (x, y, p_x, p_y) at $z = z_0$ and at $z = 0$ must be a canonical transformation (in fact, we shall find its generating function below). Since we are looking for a canonical transformation to replace the parameters P_i that specify the direction of the light rays in the second medium by parameters $\tilde{P}_i$ that specify the position of the point source, the values of the canonical coordinates (x, y, p_x, p_y) at $z = 0$ will be denoted as (Q_1, Q_2, P_1, P_2), and those at $z = z_0$ will be denoted as $(\tilde{P}_1, \tilde{P}_2, -\tilde{Q}_1, -\tilde{Q}_2)$ (the minus signs are necessary to have a canonical transformation); then, from Eqs. (29) and (30), we obtain

$$\tilde{Q}_1 = -P_1, \qquad \tilde{P}_1 = Q_1 + \frac{z_0 P_1}{\sqrt{n_0{}^2 - P_1{}^2 - P_2{}^2}},$$

$$\tilde{Q}_2 = -P_2, \qquad \tilde{P}_2 = Q_2 + \frac{z_0 P_2}{\sqrt{n_0{}^2 - P_1{}^2 - P_2{}^2}}.$$

A straightforward computation shows that $\tilde{Q}_i d\tilde{P}_i - Q_i dP_i$ is the differential of [see Eq. (9)]

$$F(P_i, \tilde{P}_i) = -P_1 \tilde{P}_1 - P_2 \tilde{P}_2 - z_0 \sqrt{n_0{}^2 - P_1{}^2 - P_2{}^2}. \quad (31)$$

(Note that z_0 is treated as a parameter.) The generating function (31) coincides with the *characteristic function* W obtained by calculating optical lengths in Ref. 5 [Eq. (6·23)].

Then, from Eqs. (19), (27), and (31), we obtain

$$\tilde{S} = P_1 x + P_2 y + \sqrt{{n_1}^2 - {P_1}^2 - {P_2}^2}\, z$$
$$- P_1 \tilde{P}_1 - P_2 \tilde{P}_2 - z_0 \sqrt{{n_0}^2 - {P_1}^2 - {P_2}^2}, \qquad (32)$$

with the P_i determined by the conditions [see Eqs. (20)]

$$x - \frac{z P_1}{\sqrt{{n_1}^2 - {P_1}^2 - {P_2}^2}}$$
$$- \tilde{P}_1 + \frac{z_0 P_1}{\sqrt{{n_0}^2 - {P_1}^2 - {P_2}^2}} = 0, \qquad (33)$$

$$y - \frac{z P_2}{\sqrt{{n_1}^2 - {P_1}^2 - {P_2}^2}}$$
$$- \tilde{P}_2 + \frac{z_0 P_2}{\sqrt{{n_0}^2 - {P_1}^2 - {P_2}^2}} = 0. \qquad (34)$$

In this case, instead of giving the refracted wavefronts in the implicit form $\tilde{S} = \text{const.}$, it is simpler to express them in parameterized form: for a given value of $\tilde{S}$, Eqs. (32)–(34) can be employed to find the coordinates x, y, z of the points of the wavefront in terms of the parameters P_1 and P_2 (with the coordinates of the source, $(\tilde{P}_1, \tilde{P}_2, z_0) = (x_0, y_0, z_0)$, fixed).

It should be remarked that the eikonal functions (24), (26), (27), and (32) correspond to homogeneous media (*i.e.*, constant refractive indices), but that the basic results are applicable to any isotropic medium.

4. Concluding remarks

The results of Sec. 2 show that the Lagrange method of envelopes arises in a natural way in the cases considered here, and that the new solutions obtained in this manner have a very special structure, being sums of the form (19). Furthermore, we have shown that the function F, appearing in Eq. (19), is related to a canonical transformation. It may be pointed out that the results derived here are applicable to any first-order PDE that contains the unknown function S only through its derivatives.

As we have shown in Sec. 3, the formalism of Hamiltonian mechanics can be conveniently applied to the geometrical optics, allowing us to derive the relevant relations without having to use other standard tools (such as the Fermat principle, Snell's law, or the concept of optical length).

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