# Investigation of electrical RC circuit within the framework of fractional calculus

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In this paper, charging and discharging processes of different capacitors in electrical RC circuit are considered theoretically and experimentally. The non-local behaviors in these processes, arising from the time fractality, are investigated via fractional calculus. In this context, the time fractional differential equation related to electrical RC circuit is proposed by making use of Caputo fractional derivative. The resulting solution exhibits a feature in between power law and exponential law forms, and is obtained in terms of Mittag-Leffler function which describes physical systems with memory. The order of fractional derivative characterizes the fractality of time and being considered in the interval $0 < \alpha \leq 1$. The traditional conclusions are recovered for $\alpha = 1$, where time becomes homogenous and system has Markovian nature. By using time fractional approach, the discrepancies between the experimentally measured data and the theoretical calculations have been removed.

*Keywords:* RC circuit; fractional calculus; Caputo fractional derivative; Mittag-Leffler function; Planck units.

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## 1. Introduction

A series combination of a resistance and a capacitor is the simplest RC circuit and shown in Fig. 1. The initially uncharged capacitor begins to charge when the switch is closed. $V(t)$ is the time dependent voltage (potential difference) across the capacitor and can be found by using Kirchhoff's current law. These results are in the following differential equation which is linear and first order

$$RC\frac{dV(t)}{dt} + V(t) = \varepsilon. \tag{1}$$

The solution of Eq.(1) yields the following formula

$$V(t) = \varepsilon\left[1 - e^{-\frac{1}{RC}t}\right]. \tag{2}$$

As can be seen in Fig. 2, if the circuit consists of only an initially charged capacitor and a resistance, the capacitor begins to discharge its stored energy through the resistance over time. The differential equation written by using Kirchhoff's current law is as follows,

$$RC\frac{dV(t)}{dt} + V(t) = 0. \tag{3}$$

By solving this equation, it can be seen that the time dependent voltage across the capacitor yields the formula for exponential decay:

$$V(t) = \frac{Q}{C}e^{-\frac{t}{RC}}, \tag{4}$$

where, Q is the capacitor charge at time $t = 0$ [1].

Almost all of the physical processes have a non-conservative feature, since they involve irreversible dissipative effects such as friction. As a result of these dissipative effects, time-reversal symmetry fails for non-conservative systems. Therefore, the equations which determine the non-conservative systems should be non-local in time. But, the well known equations used in standard calculations are local in time, and inadequate to take into account the non-conservative nature of physical processes. Hence, the experimental results do not comply with standard theoretical calculations. As an example, the dissipative effects caused by electrical resistance or Ohmic friction are also found in the electrical circuits [2,3]. In the literature, there exist numerous studies with the aim of placing these dissipation effects on to a relevant theoretical basis. With this purpose, the fractional calculus is applied to a variety of electrical circuit problems

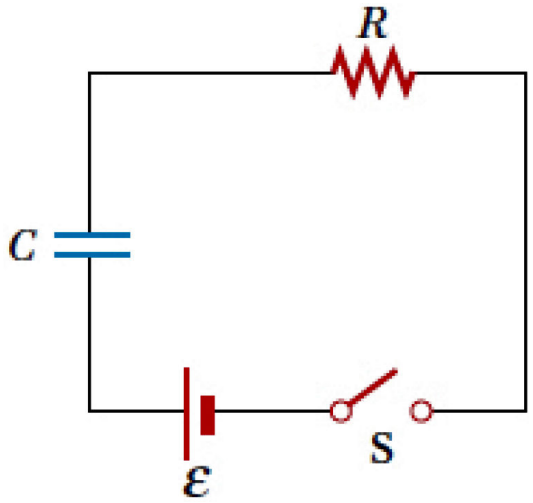


FIGURE 1. Charging a capacitor in RC circuit.

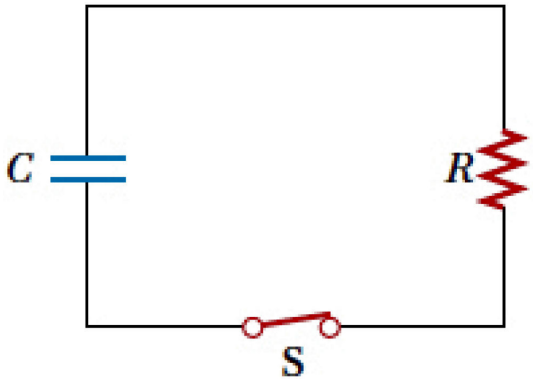


FIGURE 2. Discharging a capacitor in RC circuit.

as a useful mathematical tool. The best known of these studies is Podlubny's applications [4–6]. Another study on this subject was carried out by Rousan *et al*. In that study, the differential equations related to RC and RL circuits were merged in a single equation [7]. In addition to these studies, singular fractional linear systems and positivity, and reachability of fractional electrical circuits were studied by Kaczorek [8, 9]. Applications of integer and fractional models in electrochemical systems, the simple free running multivibrator built around a single fractional capacitor and evolution of a current in a resistor were examined by Jesus and Machado [10], Maundy *et al*. [11] and Obeidat *et al*. [12], respectively. A new and simple algorithm was proposed to obtain analytical solution of the time fractional Fokker–Plank equation which arises in circuit theory by Kumar [13].

The purpose of this study is to provide a mathematical formalism in order to compensate the incompatibility between theoretical and experimental results. It is an interesting contradiction that Eq. (1) and Eq. (3) are linear and local in time, while the realistic behavior of the $RC$ electric circuit is non-linear and non-local in time due to ohmic friction and temperature. It could be said that linear approach is only an idealization of the description of the $RC$electrical circuit. According to us, the non-locality arising from time fractality and the non-linear behavior of $RC$ electrical circuit can be described in a realistic manner with the help of fractional derivatives. In the previous studies for this purpose, linear equation, namely Eq. (1), is considered with fractional derivatives, but these equations are not acceptable physically due to the dimensional incompatibility of the solutions [14–16].

Gomez *et al*. have proposed a fractional differential equation for mechanical oscillations of a simple system [17], transmission line model [18], RLC circuit [19] and RC circuit [20–22]. To keep the dimensionality of the differential equations a new parameter $\sigma$ was introduced. The existence of fractional structures in the system was characterized by the parameter $\sigma$. They defined a relation between the fractional derivative order $(\alpha)$ and this new parameter $(\sigma)$.

Different from the aforementioned studies, in the present study, Planck units are used to preserve the dimensional compatibility in Eq. (1). Thus, both sides of the fractional form of the Eq. (1) have same dimension. The Planck time in our calculations corresponds to the $\sigma$ parameter introduced by Gomez *et al*. [17–20]. The empirical relationship between $\alpha$ and $\sigma$ was defined as $\alpha = \sigma/RC$ in [20]. The $\sigma$ parameter, characterizing the fractional structures in system, is changed with respect to the fractional derivative order $\alpha$. However, Planck time $t_p$ used in this study is constant.

The present paper is organized as follows. In Sec. 2, a brief overview of fractional calculus is presented. In Sec. 3, Eq. (1) is fractionalized by using a Caputo fractional derivative operator, and the solution of fractional equation is obtained in terms of Mitag-Leffler function. In Sec. 4, by making use of different capacitors and resistances, the numerical calculations have been performed for charging and discharging RC circuits, and the non-linear behavior of RC electrical circuit is investigated with the help of graphical representations. Finally, conclusions are summarized in the last section.

## 2. Mathematical Preliminary

Fractional calculus is a field of mathematical analysis where integral and derivative operators of non-integer order take place. More realistic description of the physical systems has been proposed by making use of the derivatives of fractional order. Some fundamental definitions used in the literature are the Riemann-Liouville, the Grünwald-Letnikov, the Weyl and the Caputo fractional derivatives [4, 23–28]. In the previous section, we will prefer to use Caputo fractional derivative to formulate the initial value problems related to RC electrical circuit. The reason of this choice is that the initial conditions of the problems taken into account by using Caputo fractional derivative are physically acceptable. On the other hand, when the Riemann-Liouville fractional derivative is used, one encounters physically unacceptable initial conditions [4].

In the Riemann-Liouville formalism, the fractional integral of order $\alpha$ is given by

$$J^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1}f(\tau)d\tau, \tag{5}$$

where, $\alpha$ is any positive real number and $\Gamma(\alpha)$ is Gamma function. By using the Riemann-Liouville fractional integral, Caputo's fractional derivative is defined as follows:

$$D_{c}^{\alpha}f(t)=J^{m-\alpha}D^{m}f(t), \tag{6}$$

$$D_{c}^{\alpha}f(t):=\begin{cases}\frac{1}{\Gamma(m-\alpha)}\int_{0}^{t}\frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}d\tau, & m-1<\alpha\leq m,\\ \frac{d^{m}f(t)}{dt^{m}} & \alpha=m,\end{cases} \tag{7}$$

where, $\alpha>0$ is the order of fractional derivative and $m$ is the smallest integer greater than $\alpha$, namely $m-1<\alpha\leq m$. The Laplace transform of the Caputo's fractional derivative is formulated as follows:

$$L(D_{c}^{\alpha}f(t))=s^{\alpha}F(s)-\sum_{k=0}^{m-1}s^{\alpha-k-1}(D^{(k)}f(t)\mid_{t=0}), \tag{8}$$

where, $F(s)$ is the Laplace transform of the function $f(t)$, and $s$ is the Laplace transform parameter. Before concluding this section, it is useful to briefly introduce the Mittag-Leffler function which plays an important role in the theory of fractional differential equations. This function is defined by the series expansion

$$E_{\alpha}(x)=\sum_{n=0}^{\infty}\frac{x^{n}}{\Gamma(n\alpha+1)}, \tag{9}$$

and is a generalization of the exponential function, *i.e.*, $E_{1}(x)=\exp(x)$ [23-26].

## 3. Fractional Calculus Analysis of Electrical RC Circuit

For analyzing the physical processes by using the related differential equations, Planck units have a special importance since they describe the mathematical expressions of physical laws in a non-dimensional form. In this context, Eq. (1) which represents the charging RC circuit given in Fig. 1, can be rewritten in a non-dimensional form by using Planck time unit as follows:

$$t_{p}\frac{dV(t)}{dt}+t_{p}\frac{1}{RC}V(t)=t_{p}\frac{\varepsilon}{RC} \tag{10}$$

where, $t_{p}=\sqrt{\hbar G/c^{5}}$ is Planck time, $\hbar$, $G$ and $c$ are reduced Planck constant, gravitational constant and speed of light, respectively. In Eq. (10), the first order time derivative can be changed to a fractional time derivative of order $\alpha$. Thus, the time-fractional differential equation of electrical RC circuit is given by

$$t_{p}^{\alpha}D_{c}^{\alpha}V(t)+t_{p}\frac{1}{RC}V(t)=t_{p}\frac{\varepsilon}{RC} \tag{11}$$

where, $D_{c}^{\alpha}$ denotes the Caputo fractional derivative. The Planck time $t_{p}$ in Eq. (11) varies with power $\alpha$ in order to preserve the units of $V(t)$ [24]. For $\alpha=1$, Eq. (11) reduces to the standard one. This situation is acceptable physically since the both sides of the Eq. (11) have same dimensions, namely volt.

The solution of Eq. (11) can be obtained by performing the Laplace transform to Eq. (11) which leads to

$$\widetilde{V}(s)=\frac{t_{p}^{1-\alpha}\varepsilon}{RC}\frac{1}{s^{\alpha+1}}\left(\frac{s^{\alpha}}{s^{\alpha}+\frac{t_{p}^{1-\alpha}}{RC}}\right) \tag{12}$$

where, $\widetilde{V}(s)$ is the Laplace transform of the $V(t)$. By following the way introduced in [24], Eq. (12) can be expressed as a series expansion

$$\widetilde{V}(s)=-\varepsilon\sum_{n=1}^{\infty}\frac{(-1)^{n}\left(\frac{t_{p}^{1-\alpha}}{RC}\right)^{n}}{s^{n\alpha+1}}. \tag{13}$$

Then performing the inverse Laplace transform to each terms of Eq. (13) leads to the following results;

$$V(t)=\varepsilon\left[1-\sum_{n=0}^{\infty}\frac{\left(-\frac{t_{p}^{1-\alpha}}{RC}t^{\alpha}\right)^{n}}{\Gamma(n\alpha+1)}\right], \tag{14}$$

$$V(t)=\varepsilon\left[1-E_{\alpha}\left(-\frac{t_{p}^{1-\alpha}}{RC}t^{\alpha}\right)\right], \tag{15}$$

where, $E_{\alpha}\left(-\frac{t_{p}^{1-\alpha}}{RC}t^{\alpha}\right)$ represents the Mittag-Leffler function.

## 4. Results and Discussion

During the charging capacitor, experimental values and calculated results obtained from Eq. (2) are not exactly equivalent. To eliminate this inconsistency, Eq. (1) has been redefined by using Planck time unit and Caputo definition of fractional derivative as Eq. (11), and the solution of this new equation has been obtained as Eq. (15).

The laboratory experiments have been performed in Electronic Laboratory of Physics Department in Dumlupınar University. Eq. (2) and Eq. (15) have been used for comparison of the standard and fractional calculations with experimental results respectively.

To obtain the measurement data, the RC circuit in Fig. 1 has been set up by making use of $\varepsilon=1$ V, $R=10$ k$\Omega$ and different capacitors such as $C=0.047$ F, $C=0.0423$ F, $C=0.0376$ F, $C=0.0329$ and $C=0.0282$ F. In calculations, the value of internal resistance of battery is included in the $R$ value of resistance. In Fig. 3, for the aforementioned

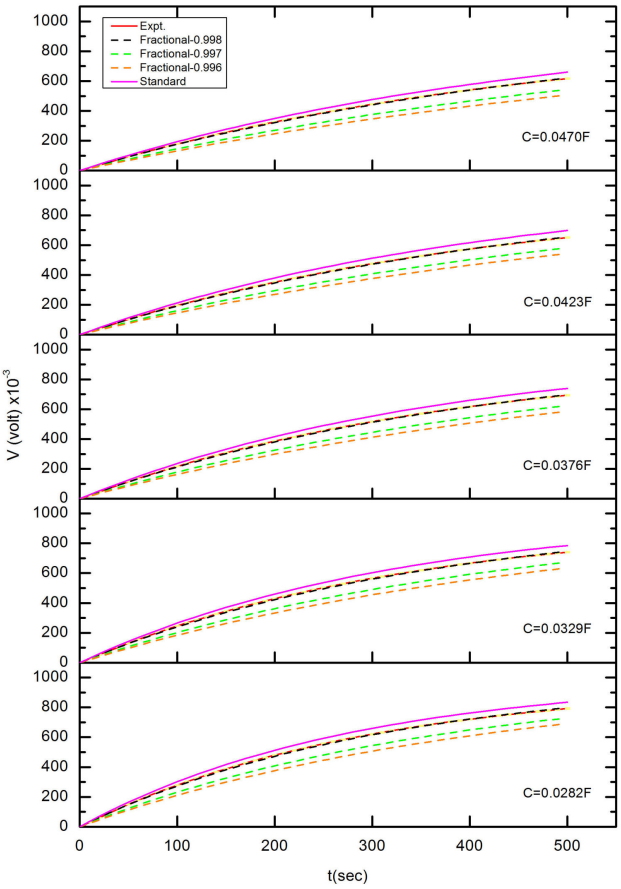


FIGURE 3. During charging a capacitor, the changing of voltage of capacitor in the course of time for $\varepsilon = 1$ V, $R = 10$ kΩ and different capacitors which are $C = 0.047$ F, $C = 0.0423$ F, $C = 0.0376$ F, $C = 0.0329$ and $C = 0.0282$ F. (The error bars have been shown with yellow).

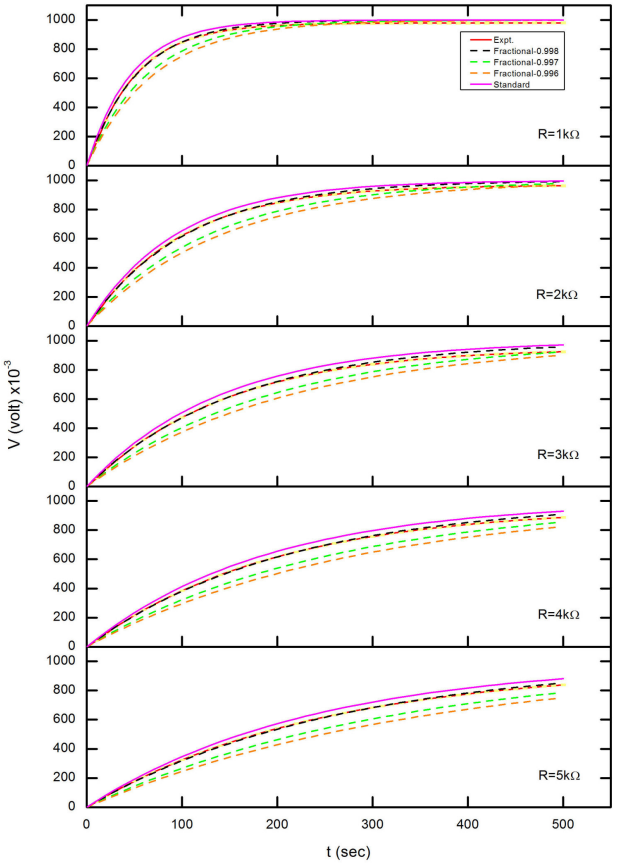


FIGURE 4. During charging a capacitor, the changing of voltage of capacitor in the course of time for $\varepsilon = 1$ V, $C = 0.047$ F and different resistances which are $R = 1$ kΩ, $R = 2$ kΩ, $R = 3$ kΩ, $R = 4$ kΩ and $R = 5$ kΩ. (The error bars have been shown with yellow).

five different capacitors, experimental results have been compared with the standard and fractional calculations respectively. In fractional calculations, the values of fractional derivative order are taken as $\alpha \approx 0.998$. As can be seen in Fig. 3, the experimental values are smaller than the standard ones. However, for the fractional derivative order $\alpha \approx 0.998$, the fractionally calculated results are almost equivalent to the experimental values.

For the values of $\varepsilon = 1$ V, $C = 0.047$ F and five different resistance such as $R = 1$ kΩ, $R = 2$ kΩ, $R = 3$ kΩ, $R = 4$ kΩ and $R = 5$ kΩ the comparisons of standard and fractionally obtained results with the experimental data are shown in Fig. 4. As can be seen in Fig. 4, the voltage values obtained from standard calculations are bigger than the experimental ones, whereas the fractionally obtained results are consistent with the experimental data for the $\alpha \approx 0.998$ value.

This discrepancy between the experimental data and the results obtained from standard calculations means that, some losses are not taken into account in the standard approach. We think that, the resistances caused by circuit components lead to these losses. The standard mathematical approach presumes this event as a linear process, and recommends using a linear differential equation *i.e.*, Eq. (1). Whereas the physical process observed in electrical RC circuit is nonlinear and non-local in time. Hence, using a linear differential equation is not an appropriate mathematical tool for description of this process. The realistic manner of the process is that, the resistance value $(R)$ of circuit elements is not a constant parameter, and typically increases with increasing temperature. The temperature dependence of resistance is usually defined in the following equation

$$R = R_0 \left[1 + a \left(T - T_0\right)\right] \tag{16}$$

where, $a$ is called the temperature coefficient of resistance, $T_0$ is a reference temperature (room temperature), and $R_0$ is the resistance at temperature $T_0$. Here, $a$ is a fitting parameter obtained from measurement data, and varies with the reference temperature. Therefore, Eq. (16) is only a linear approximation, and assuming the temperature coefficient of resistance $a$ as a constant parameter is an idealization for description the physical process [25]. It is concluded from this fact that, the parameter $a$ is not a constant, and the temperature dependence of the resistance has a nonlinear form in a realistic manner. Since the values of resistance increase

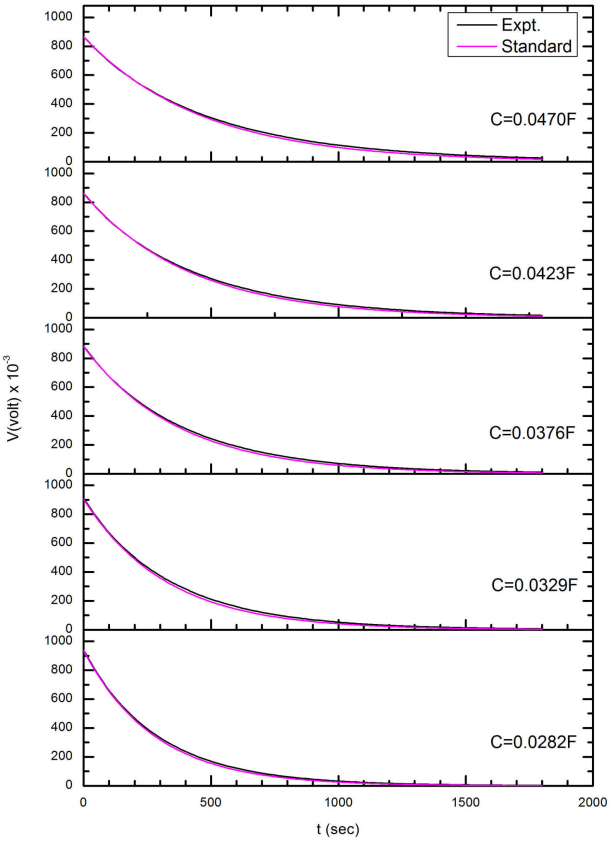


FIGURE 5. During discharging a capacitor, the changing of voltage of capacitor in the course of time for $R = 10$ kΩ and different capacitors which are $C = 0.047$ F, $C = 0.0423$ F, $C = 0.0376$ F, $C = 0.0329$ and $C = 0.0282$ F.

nonlinearly with increasing temperature, the voltage value of capacitor leads to a lower value than the predicted value by Eq. (2). This loss of voltage due to the temperature is taken into account by using fractional calculus approach. As seen in Figs. 3 and 4, fractional calculus approach places the experimental results on a more realistic theoretical basis, and defines the nonlinear behavior of the physical process more close to reality. From these results, it could be said that there is a close relationship between temperature and fractional derivative order alpha. From microscopic point of view, it is well known that temperature could be considered as a measure of energy. According to Heisenberg uncertainty principle, energy and time are closely related to each other. So, it is plausible to think that a similar relation could be found between temperature and time. According to our opinion, increasing temperature of the resistance gives rise to fractionalization of time. In fractional formalism, this fractality of time is measured by fractional derivative order alpha, and the non-locality behavior of the physical process in time can be determined by Eq. (11). Since the temperature coefficient of resistance $a$ depends on the reference temperature, a similar relationship may also be exist between the $a$ coefficient and the fractional derivative order $\alpha$. Consequently, fractional derivative order $\alpha$ can be accepted as a measure of nonlinearity of the physical process and non-locality behavior in time. The case of $\alpha = 1$ corresponds to linear situation (ideal case or local behavior in time) where the resistance values $(R)$ are considered as a constant parameter, and the temperature (energy) and time are not conjugate to each other, whereas the case of $\alpha \neq 1$ corresponds to nonlinear situation of the physical process (realistic behavior that is non-local in time).

For the circuit shown in Fig. 2, it is seen from Fig. 5 that the experimental data are equivalent to the results obtained from the solution of Eq. (3). In this case, since there is no power supply in circuit, the effect of temperature could be omitted. Hence there is no need to make time-fractional approach $(\alpha = 1)$.

## 5. Conclusion

In this study, different from the Refs. 17 to 20, the Planck-time has been used to preserve the dimensional compatibility in equations. Hence, fractional differential equation of the electrical RC circuit is determined as Eq. (11). Dimension of each side of this equation is equal to each other.

As can be seen in Figs. 3 and 4, the traditional (standard) results are bigger than the experimental results for both different capacitors and resistances. This situation shows that there are some dissipative effects which are not considered in the standard theoretical calculations. These dissipative effects, which occur due to the ohmic friction and temperature, make the behavior of the electrical RC circuit be nonlinear and non-local in time. Consequently, instead of the standard calculations, Eq. (11) is more useful to describe the nonlinear behavior of the electrical RC circuit in a more realistic manner. In this context, the measured experimental results could be exactly obtained within the fractional calculus approach for the order $\alpha \approx 0.998$ of the fractional derivative. For the case of $\alpha = 1$, the fractionally obtained solutions which emerge in the scope of the study recover the solutions of the traditional calculations.

In the microscopic level, energy and time cannot be handled independently from each other. From thermostatistics point of view, energy and temperature are also closely related to each other. Hence, it is plausible to say that temperature and time are conjugate to each other, and the change of temperature gives rise to fractionalization of time. Consequently, the energy losses, caused by dissipative effects such as ohmic friction (or temperature dependent electrical resistance), could be taken into account by time fractional approach.

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