Dimensional regularization with non Beta-functions

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The most general method to regularize Feynman's integrals in quantum field theory is Dimensional Regularization, in which the most common way to evaluate the associated integral involves Beta functions. We present a new method to evaluate the integral through the residue theorem. We apply our method to a toy model on universal extra dimensions and show that radiative corrections changes the shift-mass between zero and Kaluza-Klein excited modes.

Keywords: Quantum field theory; renormalization; radiative corrections.

Regularización Dimensional suele ser el método mas utilizado para regularizar integrales de Feynman, comúnmente en dicho método, las integrales asociadas implican funciones Beta. Presentamos un método para evaluar tales integrales mediante el Teorema del Residuo. El método se aplica sobre un modelo de juguete con dimensiones extras universales, observando como las correcciones radiativas envuelven un corrimiento de masa entre el modo cero y los modos de Kaluza-Klein.

Descriptores: Teoría cuantica de campos; renormalización; correcciones radiativas.

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1. Introduction

It is well known that Quantum Field Theory (QFT) is plagued with divergences, that appear through the integrals associated to the Feynman diagrams of beyond tree level corrections. Dealing with those divergences, in order to obtain physical (meaningful) results out of perturbation theory is the main goal of renormalization theory [1,2], in which, regularizing the divergent integrals becomes the first crucial step.

In four dimensions, for instance, it is common that the integrals associated to two and four-point correlations functions diverge. However, through *regularization procedures* they can be made finite. To achieve this a regularization parameter is introduced, in away that the divergences of the integrals would now appear as singularities that depend on this parameter.

For example, in Dimensional regularization (DR), which was introduced by 't Hooft and Veltman [3-5], the measure of momentum integration is changed by allowing the dimension D in the integrals to be an arbitrary complex number. There are several attractive features on DR. First, it preserves all symmetries for a non-supersymmetric theory (under this scheme the gauge fields and the momentum integrals are promoted to D dimensions, and the mismatch between the number of degrees of freedom of gauge fields(D) and gauginos(4) breaks supersymmetry [6]). Second, it allows an easy identification of the divergences. Third, it suggests in a natural way a *minimal subtraction scheme* (MS scheme) [7], that greatly simplifies renormalization calculation and it is capable of regularizing IR-divergences in massless theories.

In order to see how DR operates, consider for example the simplest Feynman integral

$$I\left(D\right)=\int\frac{d^{D}p}{\left(2\pi\right)^{D}}\frac{1}{\mathbf{p}^{2}+m^{2}},\tag{1}$$

which typically arises at on loop order corrections in scalar field theories. There are many ways to evaluate such an integral. One of them is through the use of *proper time representation* of Feynman integrals. Other option is the introduction of polar coordinates where one rewrites the integral as

$$I\left(D\right)=\frac{S_{D}}{\left(2\pi\right)^{D}}\int\limits_{0}^{\infty}dp\frac{p^{D-1}}{p^{2}+m^{2}},\tag{2}$$

where the overall coefficient

$$S_{D}=\frac{2\pi^{D/2}}{\Gamma\left(D/2\right)},\tag{3}$$

is the surface of a unit sphere in D dimensions. Then, the resulting one-dimensional integral can be casted into the form of an integral for the Beta function which can be expressed in terms of the Gamma functions, and thus, the value for (1) is given by

$$I\left(D\right)=\frac{\left(m^{2}\right)^{D/2-1}}{\left(4\pi\right)^{D/2}}\Gamma\left(1-D/2\right).\tag{4}$$

One of the disadvantages of these methods arises when two or more propagators are considered, because in the case of the proper time representation it is necessary to consider two or more integrations, whereas, in the case of Beta function it is required the use of Feynman's parametric integral formulas.

In this work we present a new method that make use of Residue theorem which allows to evaluate the one-dimensional integral without considering Beta functions. Instead, it is only necessary the use of one specific contour for the integration. As we will discuss, this provide an easier way for the evaluation of the integrals with two or more propagators (similar techniques are shown in Ref. 8).

The outline of our paper is as follows: Sec. 2 presents the contour and their use to evaluate the one-dimensional integral. Sec. 3 shows a detailed example where the method is used to evaluate the radiative corrections for the scalar field mass due to a Yukawa interaction. Here as we will show explicitly, the use of extra integrations becomes unnecessary. Next, in Sec. 4 we apply the method to another example of physical interest. There, we consider a toy model on an extra-dimensional space-time [9] to compute the radiative corrections for the zero mode scalar mass reduced by their interactions with the excited modes. Sec. 5 lists the conclusions.

2. Evaluation of Integral

Definite integrals appear repeatedly in problems of mathematical physics. Three moderately techniques are used in evaluating definite integrals: (1) Numerical quadrature which is not commonly used on QFT; (2) conversion to Gamma or Beta functions which, as we mentioned in the introduction, is rather very used on the literature, and (c) contour integration which is rarely used in the problem at hand. Demonstrating the advantages of the use of the last technique to evaluate the usual integral

$$A = \int_0^\infty \frac{t^{D-1}}{t^2+m^2}dt, \tag{5}$$

is the purpose of this section.

To proceed, we choose the contour shown in Fig. 1 to avoid the pole in the origin (because in it arises a branch) and the possible poles over the real positive axis. Then, a given integral on such a contour is generally expressed as

$$\begin{aligned}\oint f(z)\,dz &= e^{i\alpha}\int_\epsilon^R f\left(e^{i\alpha}t\right)dt \\ &+ iR\int_0^{2\pi-\alpha} f\left(Re^{i\theta}\right)e^{i\theta}d\theta \\ &+ e^{i(2\pi-\alpha)}\int_R^\epsilon f\left(e^{i(2\pi-\alpha)}t\right)dt \\ &+ i\epsilon\int_{2\pi-\alpha}^0 f\left(\epsilon e^{i\theta}\right)e^{i\theta}d\theta \\ &= 2\pi i\sum \text{residues (within the contour)},\end{aligned} \tag{6}$$

where we will, of course, take $f(z) = z^{D-1}/(z^2+m^2)$.

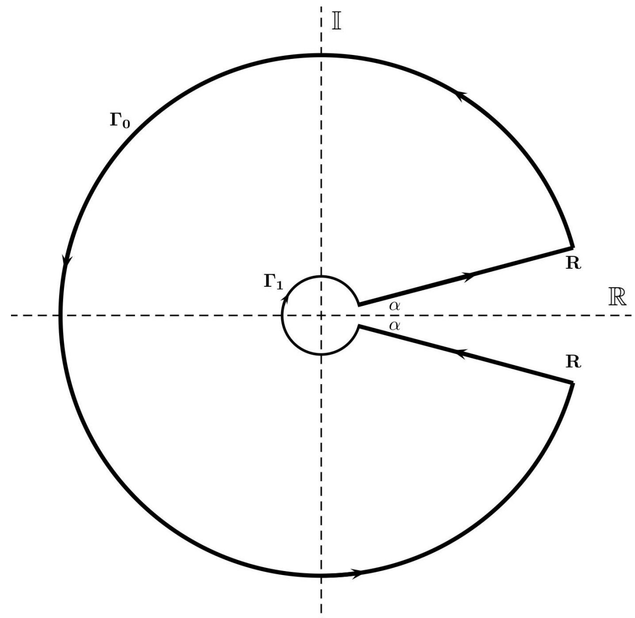

FIGURE 1. Contour used to evaluate the integral (5).

Next, by Jordan's lemma when $R \to \infty$, we have

$$\int_{\Gamma_0} \frac{z^{D-1}}{z^2+m^2}dz = 0; \qquad z = Re^{i\theta}, \tag{7}$$

whereas, when $\epsilon \to 0$

$$\int_{\Gamma_1} \frac{z^{D-1}}{z^2+m^2}dz = 0; \qquad z = \epsilon e^{i\theta}. \tag{8}$$

It is important to remark the necessity to use analytic continuation to solve the integral (5), because the Eqs. (7) and (8) are valid only for values $D < 2$.

From this the integral is reduced to

$$\begin{aligned}\oint f(z)\,dz = e^{iD\alpha}\int_0^\infty \frac{t^{D-1}}{t^2e^{i2\alpha}+m^2} \\ - e^{iD(2\pi-\alpha)}\int_0^\infty \frac{t^{D-1}}{t^2e^{i2(2\pi-\alpha)}+m^2},\end{aligned}$$

and thus, in the limit $\alpha \to 0$, we get

$$\int_0^\infty \frac{t^{D-1}}{t^2+m^2}dt = -\frac{\pi i m^{D-2}}{(1-e^{i2\pi D})}\left(i^D + (-i)^D\right). \tag{9}$$

Therefore,

$$I(D) = -\left(\frac{2\pi i m^{D-2}}{1-e^{i2\pi D}}\right)\frac{i^D+(-i)^D}{2^D\pi^{\frac{D}{2}}\Gamma\left(\frac{D}{2}\right)}. \tag{10}$$

This Feynman integral is UV-divergent in even dimensions, which is reflected by the poles in $\left(1-e^{i2\pi D}\right)^{-1}$. The poles can be subtracted in various ways, parametrized by an arbitrary mass parameter μ. Let us take the limit $D = 4 - \epsilon$ to write

$$\mu^{\epsilon} I\left(D\right) = -\frac{2\pi i m^2}{1 - e^{-i2\pi\epsilon}} \times \frac{\left[\Gamma\left(2 - \frac{\epsilon}{2}\right)\right]^{-1}}{8\pi^2}\left(-\frac{4\pi\mu^2}{m^2}\right)^{\frac{\epsilon}{2}}. \tag{11}$$

The arbitrary mass parameter μ appears in a dimensionless ratio with the mass. It is this kind of terms which contains IR-divergences in the limit $m^2 \to 0$. They are expanded in powers of ϵ:

$$\left(-\frac{4\pi\mu^2}{m^2}\right)^{\frac{\epsilon}{2}} = 1 + \frac{\epsilon}{2}\ln\frac{4\pi\mu^2}{m^2} + O\left(\epsilon^2\right). \tag{12}$$

The ϵ-expansion over the Gamma function in (11) reads

$$\frac{1}{\Gamma\left(2 - \frac{\epsilon}{2}\right)} = 1 + \left(1 - \gamma_E\right)\frac{\epsilon}{2} + O\left(\epsilon^2\right) \tag{13}$$

where γ_E is the Euler-Mascheroni constant.

Next step is to consider the ϵ-series expansion of the function associated with the exponential, for which we take its Laurent expansion (see Appendix A)

$$\frac{1}{1 - e^{-i2\pi\epsilon}} = -\frac{i}{2\pi\epsilon} + \frac{1}{2} + \frac{i\pi\epsilon}{6} + O\left(\epsilon^3\right). \tag{14}$$

Inserting Eqs. (12-14) into (11) we find the Laurent expansion in ϵ:

$$\mu^{\epsilon} I\left(D\right) = -\frac{m^2}{16\pi^2} \times \left(\frac{2}{\epsilon} + 1 - \gamma_E + \ln\frac{4\pi\mu^2}{m^2} + O\left(\epsilon\right)\right). \tag{15}$$

Notice that the residue of the pole is proportional to m^2 and independent of μ.

We should underline that the result of Eq. (15) is the same value obtained through the use of Beta functions, as it could be expected.

3. Two propagators

Next, we demonstrate the method for the integration over two propagators, which is known to be convergent for $D < 4$. So, we consider the integral

$$I'\left(D\right) = \int \frac{d^D p}{\left(2\pi\right)^D}\frac{1}{\mathbf{p}^2 + m^2}\frac{1}{\left(\mathbf{p} - \mathbf{k}\right)^2 + m^2}, \tag{16}$$

where the external momentum k is the incoming momentum.

It is easy to see that one can rewrite it as

$$I'\left(D\right) = \frac{S_D}{\left(2\pi\right)^D}\int\limits_0^{\infty}\frac{p^{D-1}}{\left(p - k\right)^2 + m^2}\frac{dp}{p^2 + m^2}. \tag{17}$$

The poles of the integral on the Eq. (17) are located at

$$\pm im \qquad \text{and} \qquad k \pm im. \tag{18}$$

Thus, using the residue theorem, we straightforwardly find that

$$I'\left(D\right) = \frac{2\pi i}{1 - e^{i2\pi D}}\frac{1}{2^{D-1}\pi^{\frac{D}{2}}\Gamma\left(\frac{D}{2}\right)} \times \left(i\frac{A\left(B^*\right)^{D-1} - A^* B^{D-1}}{2km\left|A\right|^2} - m^{D-2}\frac{Ai^D + A^*\left(-i\right)^D}{2k\left|A\right|^2}\right), \tag{19}$$

where $A = k + 2im$ and $B = k + im$.

Therefore, the divergence for $D = 4$ is contained in the denominator, since the overall coefficient possesses poles at $D = 4, 6, \ldots$. The remaining parameter integral is finite for any D as long as $m^2 \neq 0$. In terms of ϵ and μ parameters introduced above, the expresion for the integral (16) reads (see Appendix B)

$$\begin{aligned}\mu^{\epsilon} I'\left(D\right) = &-\frac{m^2}{16\pi^2\left(k^2 + 4m^2\right)} \times \left(\frac{2}{\epsilon} + \ln\frac{4\pi\mu^2}{m^2} + 1 - \gamma_E + O\left(\epsilon\right)\right) \\ &- \frac{k^4 + 3k^2m^2 - 2m^4}{16\pi^2 km\left(k^2 + 4m^2\right)}\arctan\frac{2km}{k^2 - m^2} \\ &+ \frac{k^2 + 5m^2}{16\pi^2\left(k^2 + 4m^2\right)} \times \left(\frac{2}{\epsilon} + \ln\frac{4\pi\mu^2}{k^2 + m^2} + 1 - \gamma_E + O\left(\epsilon\right)\right).\end{aligned} \tag{20}$$

This analysis exemplifies very well the advantage of the method, because we were not compelled to use the Feynman's parametric integral formula to evaluate the integral (16), but rather proceed through a direct evaluation.

It is straightforward to extend the method to compute integrals which involve more than two propagators, in that case it is necessary to consider an integral with the form

$$J\left(D\right) = \frac{S_D}{\left(2\pi\right)^D}\int\limits_0^{\infty} p^{D-1}dp\prod_{l=1}^{n}\frac{1}{\left(p - k_l\right)^2 + m^2} \tag{21}$$

where n is the number of propagators and $k_l \pm im$ are the poles. As it is showed above, with the new method, the Feynman parametrizations are changed by algebra with complex numbers.

4. $\lambda\phi^4$ on Extra Dimension

An interesting application of the proposed method, is found when dealing with multiple field theories, where such multiplicity amounts to more complicated integrals. Such is the case of models with extra dimensions, whose effective $4D$ theory contains infinite towers of excited states, numbered by a Kaluza-Klein (KK) index [10].

To see how the method works on such cases, we next consider a scalar field toy model on an extra space-time dimensions, whose Lagrangian is given by

$$\mathcal{L} = \partial_M \phi^\dagger \partial^M \phi - \frac{\lambda}{4} \left|\phi\right|^4 . \tag{22}$$

To be specific we assume the compactification is realized over a circle of radius R, and thus, the KK wave functions go as $\phi_n \propto e^{iny/R}$. We are interested in computing the radiative corrections for the mass of the zero mode, which essentially are given by the effective interactions

$$\mathcal{L}_{int} = -\frac{\lambda'}{4} \left|\phi_0\right|^2 \sum_{n=0} \left|\phi_n\right|^2 , \tag{23}$$

where $\lambda' = \lambda/2\pi R$, is the effective $4D$ coupling constant.

The corrections we shall consider are given by two integrals

$$\delta m_0^2 \propto 4 \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2} + \sum_{n=1} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 + \left(\frac{n}{R}\right)^2}. \tag{24}$$

First integral does not contribute, because its pole is not present within the contour. This is actually expected as a result of *Veltman's theorem* [11].

Next we make the use of the new variable $x = pR$ and the sum

$$\sum_{n=1} \frac{1}{x^2+n^2} = \sum_{n=0}^{\infty} \frac{1}{x^2+n^2} - \frac{1}{x^2} = \frac{\pi}{2x} \coth\left(\pi x\right) - \frac{1}{2x^2}, \tag{25}$$

which allows to write the second integral on Eq. (24) as

$$\frac{1}{2R^{D-2}} \int \frac{d^D x}{(2\pi)^D} \left[\frac{\pi}{x} \coth\left(\pi x\right) - \frac{3}{x^2}\right]. \tag{26}$$

After performing the integration with the use of the method, we get

$$\delta m_0^2 \propto -\frac{2\pi i}{1-e^{2\pi D}} \frac{\pi^{-\frac{D}{2}} R^2}{(2R)^D} \sum_{n=1}^{\infty} \frac{(in)^D + (-in)^D}{\Gamma\left(\frac{D}{2}\right) n^2}, \tag{27}$$

and in order to regularize it we take the limit $D \to 4-\epsilon$, such that, above expression becomes

$$\delta m_0^2 \propto -\frac{1}{(4\pi R)^2} \sum_{n=1}^{\infty} n^2 \times \left(\frac{2}{\epsilon} + 1 - \gamma_E + \ln \frac{4\pi\mu^2 R^2}{n^2} + O\left(\epsilon\right)\right). \tag{28}$$

Now we make use of the definition of Riemann's zeta function [12]

$$\sum_{n=1}^{\infty} n^2 = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{-2} = \zeta\left(-2\right) = 0, \tag{29}$$

which implies that the zero mode mass correction is given by

$$\delta m_0^2 = \frac{\lambda'}{8\left(\pi R\right)^2} \sum_{n=1}^{\infty} n^2 \ln n. \tag{30}$$

We should notice that this result is still divergent, due to infinite number of KK modes from the tower. For each level, however, poles have been removed (by renormalization procedures). To attain a finite value we need to explicitly introduce a cut-off on the KK levels. An example that introduce the cut-off is given in Ref. 13 in which the authors present other mechanism to regularize extra-dimensional models.

4.1. Splitting mass between zero mode and the KK modes

As we saw in the last section, there is a change for the zero mode mass due to radiative corrections. That would be also the case for the excited KK modes. However for them, the first integral on the right hand of Eq. (24) is changed into

$$4 \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 + \left(\frac{n}{R}\right)^2}. \tag{31}$$

Therefore after regularizing, the change on the KK excited masses is given by

$$\delta m_n^2 = -\frac{\lambda' n^2}{(4\pi R)^2} \left(\frac{2}{\epsilon} + 1 - \gamma_E + \ln \frac{4\pi\mu^2 R^2}{n^2} + O\left(\epsilon\right)\right) + \frac{\lambda'}{8\left(\pi R\right)^2} \sum_{m=1}^{\infty} m^2 \ln m. \tag{32}$$

That allows to compute the total shift mass between zero and excited KK modes taking into account the radiative corrections under MS scheme

$$m_n^2 - m_0^2 = \frac{n^2}{R^2} \left(1 - \frac{\lambda'}{16\pi^2} \ln \frac{4\pi\mu^2 R^2}{n^2}\right). \tag{33}$$

5. Conclusions

Loop corrections on Quantum Field theories can play an important role in the phenomenology, often these corrections lead to non physical divergences over parameters of the theories, one way to remove these divergences is through renormalization. Any procedure to achieving this requires the use of a regularization method that isolates the divergences from finite physical contributions.

In this paper we present a method to perform Dimensional Regularization without the use of Beta functions nor Feynman's Integral parametrization. We believe this method has the advantage of a direct evaluation of the loop integrals, by using an appropriate integration contour in complex space. To exemplify this we have applied our method to usual loop integrals with one and two propagators. We have also presented the use of this method to regularize models on extra space-time dimensions, as an illustrative example.

Appendix

A. Laurent Series

When it is not possible to express a function in a Taylor serie, we can expand that function by its complex extension and express it by its Laurent serie.

When x is zero, the function $1/(1-e^x)$ is not defined, however we can express it by a Laurent serie, such that

$$\frac{1}{1-e^x} = \left(-\sum_{n=1} \frac{x^n}{n!}\right)^{-1} = -\frac{1}{x\,(1+u)}$$

$$u = \sum_{n=1} \frac{x^n}{(n+1)!} = -\frac{1}{x}\left(1-u+u^2-\ldots\right)$$

$$= -\frac{1}{x} + \frac{1}{2} - \frac{x}{12} + O(x^2). \tag{A.1}$$

Note the existence of the term which includes a negative degree.

B. Complex Logarithm

For the propose of present discussion it is important to remind the reader the properties of the Logarithm over complex numbers, for which

$$\ln(x+iy) = \ln(r) + i\theta, \tag{B.1}$$

where

$$r = \sqrt{x^2+y^2}; \qquad \theta = \arctan\frac{y}{x}, \tag{B.2}$$

and we need to properly choice the branch not to overvalue the angle. This has been used on Sec. 3, where have make use of the algebra

$$\ln\frac{\mu}{p-im} = \ln\frac{\mu\,(p+im)}{p^2+m^2}$$

$$= \ln\frac{\mu\sqrt{p^2+m^2}}{p^2+m^2} + i\arctan\frac{m}{p}.$$

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1. For a review, see for instance L.-F. Li, arXiv:1208.4700 [hep-ph].
2. See also D.I. Kazakov, arXiv:0901.2208 [hep-ph].
3. G. 't Hooft and M. J.G. Veltman, *Nucl. Phys. B* **44** (1972) 189.
4. C.G. Bollini and J.J. Giambiagi, *Nuovo Cim. B* **12** (1972) 20.
5. G. 't Hooft, *Nucl. Phys. B* **61** (1973) 455.
6. W. Siegel, *Phys. Lett. B* **94** (1980) 37.
7. S. Weinberg, *Phys. Rev. D* **8** (1973) 3497.
8. V.A. Smirnov, *Springer Tracts Mod. Phys.* **211** (2004) 1.
9. For a review, see A. Perez-Lorenzana, *J. Phys. Conf. Ser.* **18** (2005) 224.[hep-ph/0503177].
10. H.-C. Cheng, K.T. Matchev and M. Schmaltz, *Phys. Rev. D* **66** (2002) 036005. [hep-ph/0204342].
11. G.'t Hooft and M.J.G. Veltman, *Nucl. Phys. B* **153** (1979) 365.
12. J. Sondow, *Proc. Amer. Math. Soc.* **120** (1994) 421-424
13. T. Varin, J. Welzel, A. Deandrea and D. Davesne, *Phys. Rev. D* **74** (2006) 121702. [hep-ph/0610130].