Hamiltonian dynamics for Proca's theories in five dimensions with a compact dimension

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The canonical analysis of Proca's theory in five dimensions with a compact dimension is performed. From the Proca five dimensional action, we perform the compactification process on a $S^1/\mathbf{Z_2}$ orbifold, then, we analyze the four dimensional effective action that emerges from the compactification process. We report the extended action, the extended Hamiltonian and we carry out the counting of physical degrees of freedom of the theory. We show that the theory with the compact dimension continues laking of first class constraints. In fact, the final theory is not a gauge theory and describes the propagation of a massive vector field plus a tower of massive KK-excitations and one massive scalar field. Finally, we develop the analysis of a 5D BF-like theory with a Proca mass term, we perform the compactification process on a $S^1/\mathbf{Z_2}$ orbifold and we find all the constraints of the effective theory, we also carry out the counting of physical degrees of freedom; with these results, we show that the theory is not topological but reducible in the first class constraints.

Keywords: Proca theory; extra dimensions; Hamiltonian dynamics.

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1. Introduction

Nowadays, the introduction of extra dimensions in field theories have allowed a new way of looking at several problems in theoretical physics. It is well-know that the first proposal introducing extra dimensions beyond the fourth dimension was considered around 1920's, when Kaluza and Klein (KK) tried to unify electromagnetism with Einstein's gravity by proposing a theory in 5D where the fifth dimension is compactified on a circle S^1 of radius R, and the electromagnetic field is contained as a component of the metric tensor [1]. The study of models involving extra dimensions has an important activity in order to explain and solve some fundamental issues found in theoretical physics, such as, the problem of mass hierarchy, the explanation of dark energy, dark matter and inflation etc., [2]. Moreover, extra dimensions become also important in theories of grand unification trying of incorporating gravity and gauge interactions in a theory of everyting. In this respect, it is well known that extra dimensions have a fundamental role in the developing of string theory, since all versions of the theory are formulated in a spacetime of more than four dimensions [3, 4]. For some time, however, it was conventional to assume that in string theory such extra dimensions were compactified to complex manifolds of small sizes about the order of the Planck length, $\ell_P \sim 10^{-33}$ cm [4, 5], or they could be even of lower size independently of the Plank Length [6–8]; in this respect, the compactification process is a crucial step in the construction of models with extra dimensions [9, 10].

By taking into account the ideas explained above, in this paper we perform the Hamiltonian analysis of Proca's theory in 5D with a compact dimension. It is well-known that four dimensional Proca's theory is not a gauge theory, the theory describes a massive vector field and the physical degrees of freedom are three, this is, the addition of a mass term to Maxwell theory breaks the gauge invariance of the theory and adds one physical degree of freedom to electromagnetic degrees of freedom [11, 12]. Hence, in the present work, we study the effects of the compact extra dimension on a 5D Proca's theory. Our study is based on a pure Dirac's analysis, this means that we will develop all Dirac's steps in order to obtain a complete canonical analysis of the theory [13–16]. We shall find the full constraints of the theory; we need to remember that the correct identification of the constraints will play a key role to make progress in the study of the quantization aspects. We also report the extended Hamiltonian and we will determine the full Lagrange multipliers in order to construct the extended action. It is important to comment, that usually from consistency of the constraints it is not possible to determine the complete set of Lagrange multipliers, so a pure Dirac's analysis becomes useful for determining all them. Finally, we study a 5D BF-like theory with a massive term. We perform the compactification process on a $S^1/\mathbf{Z_2}$ orbifold and we obtain a 4D effective action, then we study the action developing the Hamiltonian analysis, we report the full constraints program and we show that the 4D effective theory is a reducible system in the first class constraints. All these ideas will be clarified along the paper.

The paper is organized as follows: In Sec. 1, we analyze a Proca's theory in 5D, after performing the compactification process on a $S^1/\mathbf{Z_2}$ orbifold we obtain a 4D effective Lagrangian. We perform the Hamiltonian analysis and we obtain the complete constraints of the theory, the full Lagrange multipliers associated to the second class constraints and we construct the extended action. In addition, we carry-out the counting of physical degrees of freedom. Additionally, in Sec. 2, we perform the Hamiltonian analysis for a 5D BF-like theory with a Proca mass term; we also perform the compactification process on a $S^1/\mathbf{Z_2}$ orbifold, we find the complete set of constraints and then the effective action is obtained. We show that for this theory there exist reducibility conditions among the first class constraints associated with the zero mode and the excited modes. Finally we carry out the counting of physical degrees of freedom. In Sec. 3, we present some remarks and prospects.

2. Hamiltonian Dynamics for Proca theory in five dimensions with a compact dimension

In this section, we shall perform the canonical analysis for Proca's theory in five dimensions, then we will perform the compactfication process on a $S^1/\mathbf{Z_2}$ orbifold. For this aim, the notation that we will use along the paper is the following: the capital latin indices M, N run over $0, 1, 2, 3, 5$, here as usual, 5 label the compact dimension. The M, N indices can be raised and lowered by the five-dimensional Minkowski metric $\eta_{MN} = (-1, 1, 1, 1, 1)$; y will represent the coordinate in the compact dimension, x^μ the coordinates that label the points of the four-dimensional manifold M_4 and $\mu, \nu = 0, 1, 2, 3$ are spacetime indices; furthermore we will suppose that the compact dimension is a $S^1/\mathbf{Z_2}$ orbifold whose radius is R.

The Proca Lagrangian in five dimensions without sources is given by

$$L_{5p} = -\frac{1}{4} F_{MN}(x,y) F^{MN}(x,y) + \frac{m^2}{2} A_M(x,y) A^M(x,y), \tag{1}$$

where $F_{MN}(x,y) = \partial_M A_N(x,y) - \partial_N A_M(x,y)$.

Because of the compactification of the fifth dimension will be carry out on a $S^1/\mathbf{Z_2}$ orbifold of radius R, such a choose imposes parity and periodic conditions on the gauge fields given by

$$\begin{aligned} A_M(x,y) &= A_M(x, y + 2\pi R), \\ A_\mu(x,y) &= A_\mu(x, -y), \\ A_5(x,y) &= -A_5(x, -y), \end{aligned} \tag{2}$$

thus, the fields can be expanded in terms of Fourier series as follows

$$\begin{aligned} A_\mu(x,y) &= \frac{1}{\sqrt{2\pi R}} A_\mu^{(0)}(x) + \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi R}} A_\mu^{(n)}(x) \cos\left(\frac{ny}{R}\right), \\ A_5(x,y) &= \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi R}} A_5^{(n)}(x) \sin\left(\frac{ny}{R}\right). \end{aligned} \tag{3}$$

We shall suppose that the number of KK-modes is k, and we will take the limit $k \to \infty$ at the end of the calculations, thus, $n = 1, 2, 3 \ldots k-1$. Moreover, by expanding the five dimensional Lagrangian L_{5p}, takes the following form

$$\begin{aligned} L_{5p}(x,y) =& -\frac{1}{4} F_{\mu\nu}(x,y) F^{\mu\nu}(x,y) + \frac{m^2}{2} A_\mu(x,y) A^\mu(x,y) \\ & - \frac{1}{2} F_{\mu 5}(x,y) F^{\mu 5}(x,y) + \frac{m^2}{2} A_5(x,y) A^5(x,y). \end{aligned} \tag{4}$$

Now, by inserting (3) into (4), and after performing the integration on the y coordinate, we obtain the following 4D effective Lagrangian

$$\begin{aligned} L_p(x) = \int \Bigg\{ & -\frac{1}{4} F_{\mu\nu}^{(0)}(x) F^{\mu\nu}_{(0)}(x) + \frac{m^2}{2} A_\mu^{(0)}(x) A^\mu_{(0)}(x) \\ & + \sum_{n=1}^{\infty} \Bigg[-\frac{1}{4} F_{\mu\nu}^{(n)}(x) F^{\mu\nu}_{(n)}(x) + \frac{m^2}{2} A_\mu^{(n)}(x) A^\mu_{(n)}(x) \\ & + \frac{m^2}{2} A_5^{(n)}(x) A^5_{(n)}(x) - \frac{1}{2} \left(\partial_\mu A_5^{(n)}(x) + \frac{m}{R} A_\mu^{(n)}(x) \right) \\ & \times \left(\partial^\mu A_5^{(n)}(x) + \frac{m}{R} A^{\mu(n)}(x) \right) \Bigg] \Bigg\} dx^4. \end{aligned} \tag{5}$$

The terms given by

$$-\left(\frac{1}{4}\right) F_{\mu\nu}^{(0)}(x) F^{\mu\nu}_{(0)}(x) + \left(\frac{m^2}{2}\right) A_\mu^{(0)}(x) A^\mu_{(0)}(x)$$

are called the zero mode of the Proca theory [11,12], and the following terms are identified as a tower of KK-modes [17].

In order to perform the Hamiltonian analysis, we observe that the theory is singular. In fact, it is straightforward to observe that the Hessian for the zero mode given by

$$\begin{aligned} W^{\rho\lambda(0)} &= \frac{\partial^2 L_p}{\partial(\partial_0 A_\rho^{(0)}) \partial(\partial_0 A_\lambda^{(0)})} \\ &= -\frac{g^{\mu\alpha} g^{\nu\beta}}{4} \Bigg[(\delta_\mu^0 \delta_\nu^\lambda - \delta_\nu^0 \delta_\mu^\lambda) \frac{\partial F_{\alpha\beta}^{(0)}}{\partial(\partial_0 A_\rho^{(0)})} \\ & + \left(\delta_\alpha^0 \delta_\beta^\lambda - \delta_\beta^0 \delta_\alpha^\lambda \right) \frac{\partial F_{\mu\nu}^{(0)}}{\partial(\partial_0 A_\rho^{(0)})} \Bigg] \\ &= g^{\rho 0} g^{\lambda 0} - g^{\rho\lambda} g^{00} = g^{\rho 0} g^{\lambda 0} + g^{\rho\lambda}, \end{aligned}$$

has $detW^{(0)}=0$, rank= 3 and one null vector. Furthermore, the Hessian of the KK-modes has the following form

$$W^{HL(l)}=\frac{\partial^2 L_p}{\partial(\partial_0 A_H^{(l)})\partial(\partial_0 A_L^{(l)})}$$
$$=g^{H0}g^{L0}-g^{HL}g^{00}-g^{00}\delta_5^L\delta_5^H$$
$$=g^{H0}g^{L0}+g^{HL}+\delta_5^L\delta_5^H,$$

and has $detW^{(l)}=0$, rank= $4k-4$ and $k-1$ null vectors. Thus, a pure Dirac's method calls the definition of the canonical momenta $(\pi_{(0)},\pi^i_{(n)},\pi^5_{(n)})$ to the dynamical variables $(A^{(0)}_\mu,A^{(n)}_\mu,A^{(n)}_5)$ given by

$$\pi^i_{(0)}=-\partial^i A_0^{(0)}+\partial_0 A^{i(0)}, \tag{6}$$

$$\pi^i_{(n)}=-\partial^i A_0^{(n)}+\partial_0 A^{i(n)}, \tag{7}$$

$$\pi^5_{(n)}=\partial_0 A_5^{(n)}+\frac{n}{R}A_0^{(n)}. \tag{8}$$

From the null vectors we identify the following k primary constraints

$$\phi^1_{(0)}=\pi^0_{(0)}\approx 0, \tag{9}$$

$$\phi^1_{(n)}=\pi^0_{(n)}\approx 0. \tag{10}$$

Hence, the canonical Hamiltonian is given by

$$\begin{aligned}H_c=\int\Bigg[&-A_0^{(0)}(x)\partial_i\pi^i_{(0)}(x)\\
&+\frac{1}{2}\pi^i_{(0)}(x)\pi_i^{(0)}(x)+\frac{1}{4}F^{(0)}_{ij}(x)F^{ij}_{(0)}(x)\\
&-\frac{m^2}{2}A^{(0)}_\mu(x)A^\mu_{(0)}(x)+\sum_{n=1}^{\infty}\Bigg(-A_0^{(n)}(x)\partial_i\pi^i_{(n)}(x)\\
&+\frac{1}{2}\pi^i_{(n)}(x)\pi_i^{(n)}(x)+\frac{1}{2}\pi^5_{(n)}(x)\pi^5_{(n)}(x)\\
&-\frac{n}{R}\pi^5_{(n)}(x)A_0^{(n)}(x)+\frac{1}{4}F^{(n)}_{ij}(x)F^{ij}_{(n)}(x)\\
&-\frac{m^2}{2}A^{(n)}_\mu(x)A^\mu_{(n)}(x)-\frac{m^2}{2}A^{(n)}_5(x)A^5_{(n)}(x)\\
&+\frac{1}{2}\left(\partial_i A_5^{(n)}(x)+\frac{n}{R}A_i^{(n)}(x)\right)\\
&\times\left(\partial^i A_5^{(n)}(x)+\frac{n}{R}A^{i(n)}(x)\right)\Bigg)\Bigg]d^3x,\end{aligned} \tag{11}$$

by using the primary constraints, we identify the primary Hamiltonian

$$\begin{aligned}H_p\equiv H_c&+\int\lambda_1^{(0)}(x)\phi^1_{(0)}(x)d^3x\\
&+\int\sum_{n=1}^{\infty}\lambda_1^{(n)}(x)\phi^1_{(n)}(x)d^3x,\end{aligned} \tag{12}$$

where $\lambda_1^{(0)}$ y $\lambda_1^{(n)}$ are Lagrange multipliers enforcing the constraints.

Therefore, from the consistency of the constraints we find that

$$\begin{aligned}\dot\phi^1_{(0)}(x)&=\{\phi^1_{(0)},H_1\}\\
&=\partial_i\pi^i_{(0)}(x)+m^2A^0_{(0)}(x)\approx 0,\end{aligned}$$

and

$$\begin{aligned}\dot\phi^1_{(n)}(x)&=\left\{\phi^1_{(n)},H_1\right\}\\
&=\partial_i\pi^i_{(n)}(x)+m^2A^0_{(n)}(x)+\frac{n}{R}\pi^5_{(n)}(x)\approx 0.\end{aligned}$$

In this manner, there are the following secondary constraints

$$\phi^2_{(0)}(x)=\partial_i\pi^i_{(0)}(x)+m^2A^0_{(0)}(x)\approx 0, \tag{13}$$

$$\phi^2_{(n)}(x)=\partial_i\pi^i_{(n)}(x)+\frac{n}{R}\pi^5_{(n)}(x)+m^2A^0_{(n)}(x)\approx 0. \tag{14}$$

On the other hand, concistency of secondary constraints implies that

$$\dot\phi^2_{(0)}(x)=m^2\partial_iA^i_{(0)}(x)-m^2\lambda_1^{(0)}(x)\approx 0,$$

hence,

$$\lambda_1^{(0)}(x)\approx\partial_iA^i_{(0)}(x), \tag{15}$$

and

$$\begin{aligned}\dot\phi^2_{(n)}(x)=&m^2\partial_iA^i_{(n)}(x)-\frac{2n}{R}\partial_i\left(\partial^iA_5^{(n)}(x)+\frac{n}{R}A^{i(n)}(x)\right)\\
&-m^2\lambda_1^{(n)}(x)+\frac{m^2n}{R}A^5_{(n)}(x)\approx 0,\end{aligned}$$

thus

$$\begin{aligned}\lambda_1^{(n)}(x)\approx\partial_iA^{i(n)}(x)&-\frac{2n}{m^2R}\partial_i\big(\partial^iA_5^{(n)}(x)\\
&+\frac{n}{R}A^{i(n)}(x)\big)+\frac{n}{R}A^{5(n)}(x).\end{aligned} \tag{16}$$

For this theory there are not third contraints.

By following with the method, we need to identify the first class and second class constraints. For this step we calculate the Poisson brackets among the primary and secondary constraints for the zero mode, obtaining

$$\left(W'^{\alpha\beta(0)}\right) = \begin{pmatrix} \{\phi^1_{(0)}(x), \phi^1_{(0)}(z)\} & \{\phi^1_{(0)}(x), \phi^2_{(0)}(z)\} \\ \{\phi^2_{(0)}(x), \phi^1_{(0)}(z)\} & \{\phi^2_{(0)}(x), \phi^2_{(0)}(z)\} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \{\pi^0_{(0)}(x), \partial_i\pi^i_{(0)}(z) + m^2A^0_{(0)}(z)\} \\ \{\partial_i\pi^i_{(0)}(x) + m^2A^0_{(0)}(x), \pi^0_{(0)}(z)\} & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & m^2\delta^3(x-z) \\ -m^2\delta^3(x-z) & 0 \end{pmatrix} = m^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \delta^3(x-z).$$

The matrix $W^{\alpha\beta'(0)}$ has a rank= 2, therefore the constraints found for the zero mode are of second class. In the same way, the Poisson brackets among the constraints related for the KK-modes, we find

$$\left(W'^{\alpha\beta(n)}\right) = \begin{pmatrix} \{\phi^1_{(n)}(x), \phi^1_{(n)}(z)\} & \{\phi^1_{(n)}(x), \phi^2_{(n)}(z)\} \\ \{\phi^2_{(n)}(x), \phi^1_{(n)}(z)\} & \{\phi^2_{(n)}(x), \phi^2_{(n)}(z)\} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \{\pi^0_{(n)}(x), m^2A^0_{(n)}(z)\} \\ \{m^2A^0_{(n)}(x), \pi^0_{(n)}(z)\} & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & m^2\delta^3(x-z) \\ -m^2\delta^3(x-z) & 0 \end{pmatrix}$$
$$= m^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \delta^3(x-z),$$

the matrix $W'^{\alpha\beta(n)}$ has a rank= $2(k-1)$, thus, the constraints associated to the KK-modes are of second class as well. In this manner, the counting of physical degrees of freedom is given in the following way: there are $10k-2$ dynamical variables, and $2k-2+2=2k$ second class constraints, there are not first class constraints. Therefore, the number of physical degrees of freedom is $4k-1$. It is important to note that for $k=1$ we obtain the three degrees of freedom of a four dimensional Proca's theory identified with the zero mode [11, 12].

Furthermore, we found $2k$ second class constraints, which implies that $2k$ Lagrange multipliers must be fixed; however, we have found only k given in the expressions (15) and (16). Hence, let us to find the full Lagrange multipliers; it is important to comment that usually the Lagrange multipliers can be determined by means consistency conditions, however, for the theory under study this is not possible because some of them did not emerge from the consistency of the constraints. In order to construct the extended action and the extended Hamiltonian, we need identify all the Lagrange multipliers, hence, for this important step, we can find the Lagrange multipliers by means of

$$\dot{\phi}^\alpha(x) = \{\phi^\alpha, H_c\} + \lambda_\beta \left\{\phi^\alpha, \phi^\beta\right\} \approx 0, \tag{17}$$

where ϕ^β are all the constrains found. In fact, by calling $C^{\alpha\beta} = \left\{\phi^\alpha, \phi^\beta\right\}$ and $h^\alpha = \{\phi^\alpha, H_c\}$, we rewrite (17) as

$$h^\alpha + C^{\alpha\beta}\lambda_\beta \approx 0. \tag{18}$$

Therefore the Lagrange multipliers are given by [16]

$$\lambda_\beta = -C^{-1}_{\beta\rho}h^\rho. \tag{19}$$

In this manner, for the zero mode we obtain

$$\left(C^{\alpha\beta(0)}\right) = \begin{pmatrix} \left\{\phi^1_{(0)}(x), \phi^1_{(0)}(z)\right\} & \left\{\phi^1_{(0)}(x), \phi^2_{(0)}(z)\right\} \\ \left\{\phi^2_{(0)}(x), \phi^1_{(0)}(z)\right\} & \left\{\phi^2_{(0)}(x), \phi^2_{(0)}(z)\right\} \end{pmatrix}$$
$$= m^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \delta^3(x-z),$$

and its inverse is given by

$$C^{(0)-1}_{\alpha\beta} = \frac{1}{m^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \delta^3(x-z),$$

so, for the constraints associated with the zero modes, $h^{(0)}$ is obtained from

$$h^{(0)} = \begin{pmatrix} \left\{\phi^1_{(0)}(x), H_c\right\} \\ \left\{\phi^2_{(0)}(x), H_c\right\} \end{pmatrix}$$
$$= \begin{pmatrix} \partial_i\pi^i_{(0)}(x) + m^2A^0_{(0)}(x) \\ m^2\partial_iA^i_{(0)}(x) \end{pmatrix},$$

therefore, by using (19) we can determine the Lagrange multipliers associated with the zero modes,

$$\begin{pmatrix} \lambda^{(0)}_1(x) \\ \lambda^{(0)}_2(x) \end{pmatrix} \approx -\frac{1}{m^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$\times \begin{pmatrix} \partial_i\pi^i_{(0)}(x) + m^2A^0_{(0)}(x) \\ m^2\partial_iA^i_{(0)}(x) \end{pmatrix} \delta^3(x-z),$$

thus

$$\lambda^{(0)}_1(x) \approx \partial_iA^{i(0)}(x), \tag{20}$$

$$\lambda^{(0)}_2(x) \approx -\frac{1}{m^2}\partial_i\pi^{i(0)}(x) - A^{0(0)}(x). \tag{21}$$

In the same way, for the KK-modes we observe that

$$\left(C^{\alpha\beta(n)}\right) = \begin{pmatrix} \left\{\phi^1_{(n)}(x), \phi^1_{(n)}(z)\right\} & \left\{\phi^1_{(n)}(x), \phi^2_{(n)}(z)\right\} \\ \left\{\phi^2_{(n)}(x), \phi^1_{(n)}(z)\right\} & \left\{\phi^2_{(n)}(x), \phi^2_{(n)}(z)\right\} \end{pmatrix} = m^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \delta^3(x-z),$$

where the inverse is

$$C^{(n)-1}_{\alpha\beta} = \frac{1}{m^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \delta^3(x-z),$$

so, for the constraints associated with the excited modes, $h^{(n)}$ are given by

$$h^{(n)} = \begin{pmatrix} \left\{\phi^1_{(n)}(x), H_c\right\} \\ \left\{\phi^2_{(n)}(x), H_c\right\} \end{pmatrix} = \begin{pmatrix} \partial_i \pi^i_{(n)}(x) + m^2 A^0_{(0)}(x) + \frac{n}{R}\pi^5_{(n)}(x) \\ m^2 \partial_i A^{i(n)}(x) - \frac{2n}{R}\partial_i \left(\partial^i A^{(n)}_5(x) + \frac{n}{R} A^{i(n)}(x)\right) + \frac{nm^2}{R} A^{5(n)}(x) \end{pmatrix},$$

additionally by using (19) we obtain

$$\begin{pmatrix} \lambda^{(n)}_1(x) \\ \lambda^{(n)}_2(x) \end{pmatrix} = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_i \pi^i_{(n)}(x) + m^2 A^{0(0)}(x) + \frac{n}{R}\pi^5_{(n)}(x) \\ m^2 \partial_i A^{i(n)}(x) - \frac{2n}{R}\partial_i \left(\partial^i A^{(n)}_5(x) + \frac{n}{R} A^{i(n)}(x)\right) + \frac{nm^2}{R} A^{5(n)}(x) \end{pmatrix} \frac{\delta^3(x-z)}{m^2},$$

thus, the $2k$ Lagrange multipliers associated for the second class constraints of the KK-modes read

$$\lambda^{(n)}_1(x) = \partial_i A^{i(n)}(x) - \frac{2n}{m^2 R}\partial_i \times \left(\partial^i A^{5(n)}(x) + \frac{n}{R} A^{i(n)}(x)\right) + \frac{n}{R} A^{5(n)}(x), \quad (22)$$

$$\lambda^{(n)}_2(x) = -\frac{1}{m^2}\partial_i \pi^i_{(n)}(x) + A^{(n)}_0(x) - \frac{n}{m^2 R}\pi^5_{(n)}(x). \quad (23)$$

Therefore, we have seen that by using a pure Dirac's method we were able to identify all Lagrange multipliers of the theory. We have commented above that Lagrange multipliers are essential in order to construct the extended action and then identify from it the extended Hamiltonian. In fact, the importance for finding the extended action is that it is defined on the full phase space, therefore, if there are first class constraints, then the extended action is full gauge invariant. On the other hand, from the extended Hamiltonian, we are able to obtain the equations of motion which are mathematical different from Euler-Lagrange equations, but the difference is unphysical. It is interesting to point out that in [11, 12] the complete Lagrange multipliers were not reported, thus, our approach extend the results reported in those works.

Furtheremore, by using the matrix $C^{(0)-1}_{\alpha\beta}$ and $C^{(n)-1}_{\alpha\beta}$ it is straightforward to calculate the Dirac brackets of the theory, thus, with the results of this paper we have a complete hamiltonian description of the system. By using all our results, we will identify the extended action, for this aim, we write the second class constraints as

$$\chi^1_{(0)}(x) \equiv \pi^0_{(0)}(x) \approx 0,$$
$$\chi^2_{(0)}(x) \equiv \partial_i \pi^i_{(0)}(x) + m^2 A^0_{(0)}(x) \approx 0,$$

$$\chi^1_{(n)}(x) \equiv \pi^0_{(n)}(x) \approx 0,$$
$$\chi^2_{(n)}(x) \equiv \partial_i \pi^i_{(n)}(x) + m^2 A^0_{(0)}(x) + \frac{n}{R}\pi^5_{(n)}(x) \approx 0,$$

now, by using the second class constraints and the Lagrange multipliers found for the zero mode and the excited modes, the extended action has the following expression

$$\begin{aligned} S_E[A,\pi,\bar{v}] = \int \Big\{ & \dot{A}^{(0)}_\mu \pi^\mu_{(0)} + A^{(0)}_0(x)\partial_i \pi^i_{(0)}(x) \\ & - \frac{1}{2}\pi^i_{(0)}(x)\pi^{(0)}_i(x) - \frac{1}{4}F^{(0)}_{ij}(x)F^{ij}_{(0)}(x) \\ & + \frac{m^2}{2}A^{(0)}_\mu(x)A^\mu_{(0)}(x) - \bar{v}^{(0)}_j \chi^j_{(0)} \\ & + \sum_{n=1}^{\infty}\Big[\dot{A}^{(n)}_\mu \pi^\mu_{(n)} + \dot{A}^{(n)}_5 \pi^5_n + A^{(n)}_0(x)\partial_i \pi^i_{(n)}(x) \\ & - \frac{1}{2}\pi^i_{(n)}(x)\pi^{(n)}_i(x) - \frac{1}{2}\pi^5_{(n)}(x)\pi^5_{(n)}(x) \\ & + \frac{n}{R}\pi^5_{(n)}(x)A^{(n)}_0(x) - \frac{1}{4}F^{(n)}_{ij}(x)F^{ij}_{(n)}(x) \\ & + \frac{m^2}{2}A^{(n)}_\mu(x)A^\mu_{(n)}(x) + \frac{m^2}{2}A^{(n)}_5(x)A^5_{(n)}(x) \\ & - \frac{1}{2}\left(\partial_i A^{(n)}_5(x) + \frac{n}{R}A^{(n)}_i(x)\right) \\ & \left(\partial^i A^{(n)}_5(x) + \frac{n}{R}A^{i(n)}(x)\right) - \bar{v}^{(n)}_j \chi^j_{(n)}\Big]\Big\} d^4x, \end{aligned} \quad (24)$$

where $\bar{v}^{(0)}_j$ and $\bar{v}^{(n)}_j$ are Lagrange multipliers enforcing the second class constraints. From the extended action, we are able to identify the extended Hamiltonian given by

$$
\begin{aligned}
H_E = \int \Bigg\{ & A_0^{(0)}(x)\partial_i \pi^i_{(0)}(x) - \frac{1}{2}\pi^i_{(0)}(x)\pi_i^{(0)}(x) \\
& - \frac{1}{4}F_{ij}^{(0)}(x)F^{ij}_{(0)}(x) + \frac{m^2}{2}A_\mu^{(0)}(x)A^\mu_{(0)}(x) \\
& + \sum_{n=1}^{\infty}\Big[A_0^{(n)}(x)\partial_i\pi^i_{(n)}(x) - \frac{1}{2}\pi^i_{(n)}(x)\pi_i^{(n)}(x) \\
& - \frac{1}{2}\pi^5_{(n)}(x)\pi^5_{(n)}(x) + \frac{n}{R}\pi^5_{(n)}(x)A_0^{(n)}(x) \\
& - \frac{1}{4}F_{ij}^{(n)}(x)F^{ij}_{(n)}(x) + \frac{m^2}{2}A_\mu^{(n)}(x)A^\mu_{(n)}(x) \\
& + \frac{m^2}{2}A_5^{(n)}(x)A^5_{(n)}(x) \\
& - \frac{1}{2}\left(\partial_i A_5^{(n)}(x) + \frac{n}{R}A_i^{(n)}(x)\right) \\
& \times \left(\partial^i A_5^{(n)}(x) + \frac{n}{R}A^{i(n)}(x)\right)\Big]\Bigg\} d^3x. \qquad (25)
\end{aligned}
$$

It is worth to comment that there are not first class constraints, therefore there is not gauge symmetry; the system under study is not a gauge theory and we can observe from (5) that the field $A_\mu^{(n)}$ is a massive vector field with a mass term given by $(m^2 + (n^2/R^2))$ and $A^5_{(n)}$ is a massive scalar field with a mass term given by m^2.

3. Hamiltonian Dynamics for a Bf-like theory plus a Proca term in five dimensions with a compact dimension

The study of topological field theories is a topic of great interest in physics. The importance to study these theories arises because they have a close relation- ship with field theories as Yang-Mills and General Relativity [13–15, 18]. In fact, for the former we can cite to Martellini's model; this model consists in expressing Yang Mills theory as a BF-like theory, and the BF first-order formulation is equivalent (on shell) to the usual (second-order) formulation. In fact, both formulations of the theory possess the same perturbative quantum properties [22]. On the other hand, with respect General Relativity we can cite the Mcdowell-Mansouri formulation; this formulation consist in breaking down the group symmetry of a BF theory from $SO(5)$ to $SO(4)$, hence, it is obtained the Palatini action plus the sum of the second Chern and Euler topological invariants [23], and since these topological classes have trivial local variations that do not contribute classically to the dynamics, one obtains essentially General Relativity. In this manner, the study of BF formulations becomes relevant, and we will study in the context of extra dimensions to an abelian BF-like theory with a Proca mass term. In this manner, we shall analyze the following action

$$
S[A,B] = \int_M \left(B^{MN}F_{MN} - \frac{m^2}{4}A_M A^M\right) dx^5, \qquad (26)
$$

here $B^{MN} = -B^{NM}$ is an antisymmetric field, and A_M is the connexion. The Hamiltonian analysis of the BF-like term without a compact dimension has been developed in [18], the theory is devoid of physical degrees of freedom, the first class constraints present reducibility conditions and the extended Hamiltonian is a linear combination of first class constrains. Hence, it is an interesting exercise to perform the analysis of the action (26) in the context of extra dimensions. We expect that the massive term gives physical degrees of freedom to the full action.

We shall resume the complete Hamiltonian analysis of (26); for this aim, we perform the $4+1$ decomposition, and then we will carry out the compactification process on a $S^1/\mathbf{Z_2}$ orbifold in order to obtain the following effective Lagrangian,

$$
\begin{aligned}
L = B^{\mu\nu}_{(0)}F^{(0)}_{\mu\nu} - \frac{m^2}{4}A_\mu^{(0)}A^\mu_{(0)} + \sum_{n=1}^{\infty}\Bigg[& B^{\mu\nu}_{(n)}F^{(n)}_{\mu\nu} \\
- \frac{m^2}{4}A_\mu^{(n)}A^\mu_{(n)} + 2B^{\mu 5}\left(\partial_\mu A_5^{(n)} + \frac{n}{R}A^n_\mu\right)\Bigg]. & \qquad (27)
\end{aligned}
$$

By performing the Hamiltonian analysis of the action (27) we obtain the following results: there are 6 first class constraints for the zero mode

$$
\gamma^{(0)}_{ij} = F^{(0)}_{ij} - \frac{1}{2}\left[\partial_i \Pi^{(0)}_{0j} - \partial_j \Pi^{(0)}_{0i}\right] \approx 0, \qquad (28)
$$

$$
{\gamma'}_{ij}^{(0)} = \Pi^{(0)}_{ij} \approx 0, \qquad (29)
$$

here, $\left(\Pi^{(n)}_{MN}, \Pi^M_{(n)}\right)$ are canonically conjugate to $\left(B^{MN}_{(n)}, A^{(n)}_M\right)$ respectively. Furthermore, these constraints are not independent because there exist the reducibility conditions given by $\partial_i \epsilon^{ijk}\gamma^{(0)}_{jk} = 0$; thus, there are $[6-1] = 5$ independent first class constraints for the zero mode. Moreover, there are 8 second class constraints

$$
\chi_{(0)} = \partial_i \Pi^i_{(0)} - \frac{m^2}{2}A_{0(0)} \approx 0, \qquad (30)
$$

$$
\chi^i_{(0)} = \Pi^i_{(0)} - 2B^{0i}_{(0)} \approx 0, \qquad (31)
$$

$$
\chi^0_{(0)} = \Pi^0_{(0)} \approx 0, \qquad (32)
$$

$$
\chi^{0i}_{(0)} = \Pi^{0i}_{(0)} \approx 0, \qquad (33)
$$

thus, with that information we carry out the counting the physical degrees of freedom for the zero mode, we find that there is one physical degree of freedom. In fact, the massive term adds that degree of freedom to the theory, just like Proca's term to Maxwell theory.

On the other hand, for the exited modes there are $12k-12$ first class constraints given by

$$\gamma_i^{(n)}=\partial_i A_5^{(n)}+\frac{n}{R}A_i^{(n)}-\left[\partial_i\Pi_{05}^{(n)}+\frac{n}{2R}\Pi_{0i}^{(n)}\right]\approx 0, \quad (34)$$

$$\gamma_{ij}^{(n)}=F_{ij}^{(n)}-\frac{1}{2}\left[\partial_i\Pi_{0j}^{(n)}-\partial_j\Pi_{0i}^{(n)}\right]\approx 0, \quad (35)$$

$$\gamma'{}_{ij}{}^{(n)}=\Pi_{ij}^{(n)}\approx 0, \quad (36)$$

$$\gamma_{i5}{}^{(n)}=\Pi_{i5}^{(n)}\approx 0, \quad (37)$$

however, also these constraints are not independent because there exist the following reducibility conditions; there are $k-1$ conditions given by $\epsilon^{ijk}\partial_i\gamma_{jk}^{(n)}=0$, and $3(k-1)$ conditions given by $\partial_i\gamma_j^{(n)}-\partial_j\gamma_i^{(n)}-\frac{n}{R}\gamma_{ij}^{(n)}=0$. Hence, there are $[(12k-12)-(4k-4)]=8k-8$ independent first class constraints. Furthermore, there are $10k-10$ second class constraints

$$\chi^i_{(n)}=\pi^i_{(n)}-2B^{0i}_{(n)}\approx 0, \quad (38)$$

$$\chi^0_{(n)}=\pi^0_{(n)}\approx 0 \quad (39)$$

$$\chi_{0i}^{(n)}=\Pi_{0i}^{(n)}\approx 0, \quad (40)$$

$$\chi_{05}^{(n)}=\Pi_{05}^{(n)}\approx 0, \quad (41)$$

$$\chi^5_{(n)}=\Pi^5_{(n)}-B^{05}_{(0)}\approx 0, \quad (42)$$

$$\chi_{(n)}=\partial_i\pi^i_{(n)}+\frac{n}{R}\pi^5_{(n)}+\frac{m^2}{2}A^0_{(n)}\approx 0. \quad (43)$$

In this manner, by performing the counting of physical degrees of freedom we find that there are $2k-2$ physical degrees of freedom for the excited modes. So, for the full theory, zero modes plus KK-modes, there are $2k-1$ physical degrees of freedom. Therefore, the theory present reducibility conditions among the first class constraints of the zero mode and there are reducibility in the first class constraints of the KK-modes. We observe from the first class constraints that the field $A_\mu^{(n)}$ has a mass term given by m^2 and is not a gauge field. On the other hand, $B_{ij}^{(n)}$ is a massless gauge field.

4. Conclusions and Prospects

In this paper, the Hamiltonian analysis for a 5D Proca's theory in the context of extra dimensions has been developed. In oder to obtain a 4D effective theory, we performed the compactification process on a $S^1/\mathbf{Z_2}$ orbifold. From the analysis of the effective action, we obtained the complete set of constraints, the full Lagrange multipliers and the extended action was found. From our results we conclude that the theory is not a gauge theory, namely, there are only second class constraints. Thus, 5D Proca's theory with a compact dimension, describes the propagation of a massive vector field associated with the zero mode plus a tower of excited massive vector fields and a massive scalar field. Furthermore, we carry out the counting of physical degrees of freedom, in particular, our results reproduce those ones known for Proca's theory without a compact dimension. Finally, in order to construct the extended action, we have identified the complete set of Lagrange multipliers; we observed that usually Lagrange multipliers emerge from consistency conditions of the constraints. However, if the Lagrange multipliers are mixed, then it is difficult identify them. In those cases, it is necessary to perform a pure Dirac's analysis as was developed in this paper, thus, all Lagrange multipliers can be determined.

On the other hand, we developed the Hamiltonian analysis of a 5D BF-like theory with a massive Proca term. From our analysis, we conclude that the theory is not topological anymore. In fact, the massive term breakdown the topological structure of the BF-like term. The theory is reducible, it present first and second class constraints and we used this fact in order to carry out the counting of physical degrees of freedom. The physical effect of the massive term, is that it adds degrees of freedom to the topological BF-like term just like Proca's term adds degrees of freedom to Maxwell theory, in addition, the massive term did not allowed the presence of pseudo-Goldstone bosons just like is present in Maxwell or Stueckelberg theory [24]. Hence, we have in this work all the necessary tools for studying the quantization aspects of the theories annalized along this paper. It is worth to comment that our results can be extended to models that generalize the dynamics of Yang-Mills theory, as for instance the following Lagrangian [19]

$$L=\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}B_{a\mu\nu}F^a_{\rho\sigma} -\frac{e^2}{4}B_{a\mu\nu}B^{a\mu\nu}+\frac{N}{4}\epsilon^{\mu\nu\rho\sigma}F_{a\mu\nu}F^a_{\rho\sigma}. \quad (44)$$

here, a are indices of $SU(N)$ group. Such generalization provide a generalized QCD theory. In fact, it is claimed in [19] that the analysis of this kind of Lagrangians is mandatory for studying the dynamics of the gluons with further interactions, in addition, that QCD generalization could be amenable to experimental test. In this respect, our work can be useful for studying that model within the extra dimensions context.

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