The Hagen-Beverloo law for outflow of granular solids from holes on side walls

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In this work we have analyzed experimentally the lateral mass outflow rate $\dot{m}$, for cohesionless granular solids (coarse grains) through circular orifices of diameter D made on side walls of bins. Experiments were performed to determine the influence of the wall thickness of the bin w, and of the grain diameter d on $\dot{m}$. Geometrical and physical arguments are given to get a general correlation for $\dot{m}$ as a function of D, d, and w. This correlation can be termed the Hagen-Beverloo law for orifices on side walls.

Keywords: Mass flow rate; Hagen-Beverloo law; non cohesive granular media.

En este trabajo analizamos experimentalmente el flujo másico de salida lateral $\dot{m}$ para sólidos granulares sin cohesión (granos gruesos) a través de orificos circulares de diámetro D realizados sobre las paredes laterales de los recipientes. Se realizaron experimentos para deterrminar la influencia del grosor de la pared del recipiente, w, y del diámetro de grano, d, sobre $\dot{m}$. Se presentan argumentos geométricos y físicos para obtener una correlación general para $\dot{m}$ como una función de D, d y w. Tal correlación puede ser llamada la ley de Hagen-Beverloo para orificios laterales.

Descriptores: Flujo másico; ley de Hagen-Beverloo; medios granulares no cohesivos.

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1. Introduction

The silo discharge through a bottom circular outlet is one of the oldest and most widely studied problems in granular flow owing to the simple setup and geometry of the system [1–10]. It has been extensively investigated both experimentally [1–7] and computationally [8, 9], and many granular, gravity flow theories use silo discharge as a validation benchmark [10–12]. Conversely, the study of the mass flow rate from orifices on side walls has been largely neglected and, to our knowledge, only few works have been published [13–17], perhaps due to the asymmetric flow profile which occurs close to the vertical wall. Despite of this, the practical use of lateral outlets is very frequent, for instance, in household silos [18].

Recently, in a couple works [19, 20], we derived a formula for the mass flow rate $\dot{m}$ through orifices on side walls, that embraces the dependence on the orifice diameter D and the wall thickness w for granular solids in the limit where $D \gg d$; where d is the mean grain diameter. In such a case we performed experiments with sand and granulated sugar (where $D/d \sim O(10^2)$) and found that the mass flow rate obeys the relation $\dot{m} \sim \rho g^{1/2} D^{5/2}[\arctan(D/w) - \theta_r]$, where ρ is the bulk density of the granular material, g is the acceleration due to gravity and θ_r is the angle of repose. In this later result the wall thickness of the bin, w, has an important influence on occurrence of the gravity flow and it has not information about the mean grain size of the material and, consequently, it is a model of the continuum theory.

The aim of this work is to study experimentally the discharge rate of granular material through vertical walls of open-top bins for different orifice sizes, several wall thicknesses, and coarse grains ($D/d \sim O(10)$). Indeed, it means that we will show how the specific influence of the mean diameter of the grains, d, must be included in the equation for the mass flow rate. Finally, when coarse grains are involved in any process, its mean size must satisfy that $d < (D/6)$, in order to avoid the blockage due to the formation of arches close to the exit holes [11].

The plan of this work is as follows. Firstly, in the next Section we report experiments of discharge rates from bottom and lateral exit holes, where the influence of D, d and w is examined. There we propose, on the basis of our experimental results, a simple correlation that includes a dependence on the three parameters. Finally, in Sec. 3, we give the main conclusions of this study.

2. Mass Flow rate estimations

In their seminal papers on the flow of cohesionless granular solids through circular orifices at the bottom of silos, Hagen [1], and many years later Beverloo, Leniger and van de Velde [3], have reported that the most suitable correlation to predict the mass flow rate from bottom exits in open-top bins, silos and hoppers is the so called Hagen-Beverloo correlation [1, 3], which has the form

$$\dot{m}_0 \sim \rho g^{1/2}(D-kd)^{5/2} \quad (1)$$

where k is a dimensionless fit parameter with typical values $k \sim 1-3$. Equation (1) is valid for materials with big grains and it contains the effect of excluded volume of a grain, *i.e.*, the space that is inaccessible to the other grains in the system as a result of the presence of the first grain and the wall. In this case, very recently, models based on detailed experimental observations [12], have shown that there are important changes in the packing of grains at the middle part of the holes and close to the limit of the orifice.

Conversely, when the grains size is very small respect to the diameter of the outlet ($D \gg d$) the term $(D-kd) \to D$, and consequently Eq. (1) transforms into $\dot{m}_0 \sim \rho g^{1/2} D^{5/2}$. In this Section we will analyze if the grain size affects the flow rate through holes on the side walls of bins with several wall thicknesses.

2.1. Mass flow rate from bottom holes

In our previous works [19, 20] we found that, for granular solids, where $D \gg d$, the mass flow rate through lateral holes obeys the relation $\dot{m} \sim \rho g^{1/2} D^{5/2}[\arctan(D/w)-\theta_r]$. This result was obtained by using square bins with vertical walls where holes of different diameters were drilled. In order to evaluate the influence of the wall thickness each side face has a specific wall thickness, w. See Fig. 1. The term

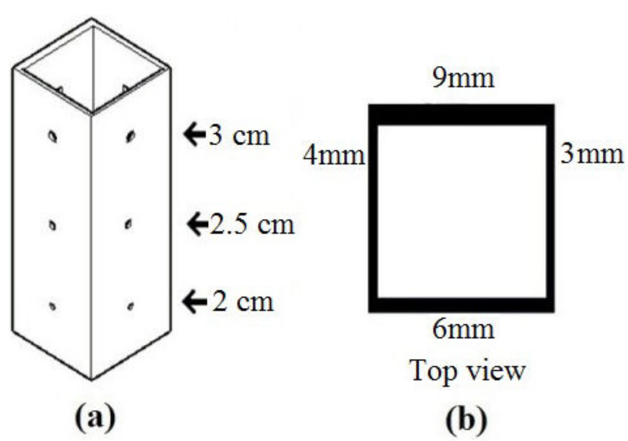

FIGURE 1. (a) Scheme of a bin with staggered holes of different size D; (b) Top view of the bin showing the four different wall thicknesses.

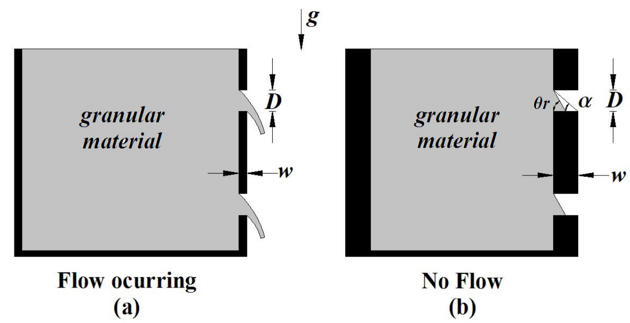

FIGURE 2. Schematic view of the bin and the holes on the side wall when granular material crosses it. In (a) the wall thickness w is small, consequently, $\alpha > \theta_r$ and the granular material will cross the orifice. In (b) the wall thickness produces the condition $\alpha < \theta_r$, hence, there is no flow. The granular flow is arrested in such a form that the slope of the granular material in the exit hole is featured by the repose angle.

$\arctan(D/w)$ can easily identified as the angle of wall, α, *i.e.*, $\alpha = \arctan(D/w)$. In Fig. 2 we illustrate that if $\alpha > \theta_r$ the gravity flow occurs meanwhile when $\theta_r > \alpha$ the flow will be arrested.

Now, we report experiments for non cohesive granular materials made of grains with large diameters. Specifically, we used mustard grains and tapioca pearls. Mustard has the following properties: mean diameter $d = 1.85$ mm, bulk density $\rho = 0.72$ gr/cm^3 and angle of repose $\theta_r = 22$ degs. Tapioca pearls have mean diameter $d = 2.2$ mm, bulk density $\rho = 0.57$ gr/cm^3 and angle of repose $\theta_r = 27$ degs. The section of the laboratory in which the experiments were done was climate controlled ($25° \pm 1°$ C and $45 \pm 10\%$ R.H.). Experiments were made upon a transparent box, 10×10 cm inner cross-section and 50 cm height.

As sketched in Figs. 1 and 2, in the acrylic-made bin were drilled circular orifices of diameters $D = 2$, 2.5, and 3 cm. It yields that typically the ratio $D/d \sim O(10)$. The wall thicknesses were $w = 0.3$, 0.4, 0.6, and 0.9 cm.

In Fig. 3 we show the corresponding particle size distributions which were determined using the microscope software (Steindorff digital microscope), which allowed to find the surface area of each particle and its surface diameter. This method yields the average (median) particle sizes which were $d = 1.85$ mm for mustard, and 2.2 mm for tapioca pearls. Insets in each plot correspond to pictures of the respective round grains.

In agreement with the experimental procedure established elsewhere [19, 20] we need to measure the mass flow rate for exit holes at the bottom, $\dot{m}_0$. This procedure allowed us to found that $\dot{m}_0 \sim \rho g^{1/2}(D-kd)^{5/2}$ for both types of grains. The method used to determine k is simmilar to that used by other authors [21]; moreover, we have found that $k = 1.51$ for mustard (a typical value for smooth near spherical grains) and $k = 3$ for tapioca. High values of k are typical of ellipsoidal particles [21], but additionaly tapioca is very rough.

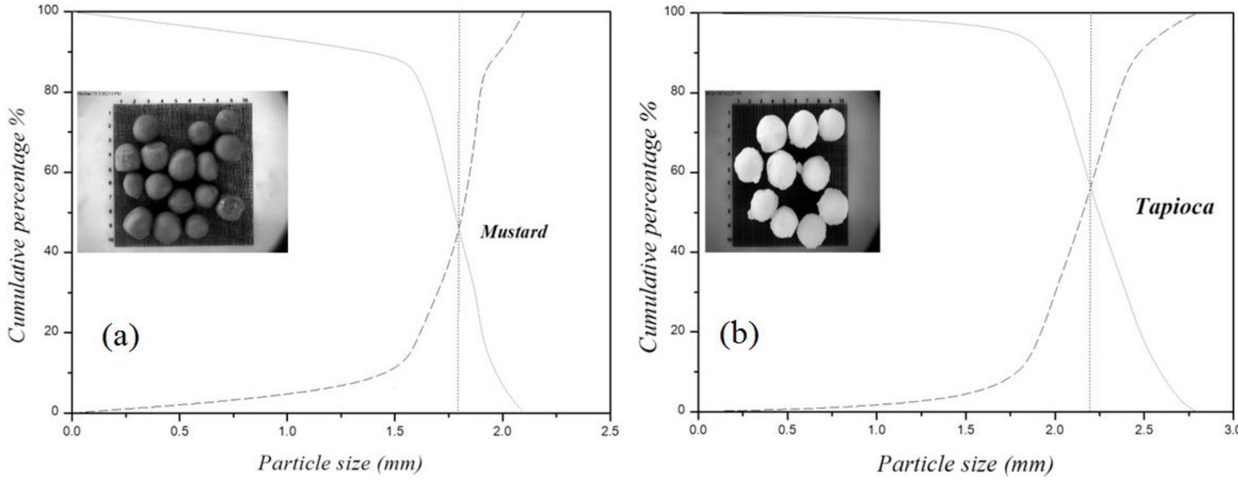

FIGURE 3. Micrographs of the particle shapes of samples of mustard (a) and tapioca (b) used in our experiments.

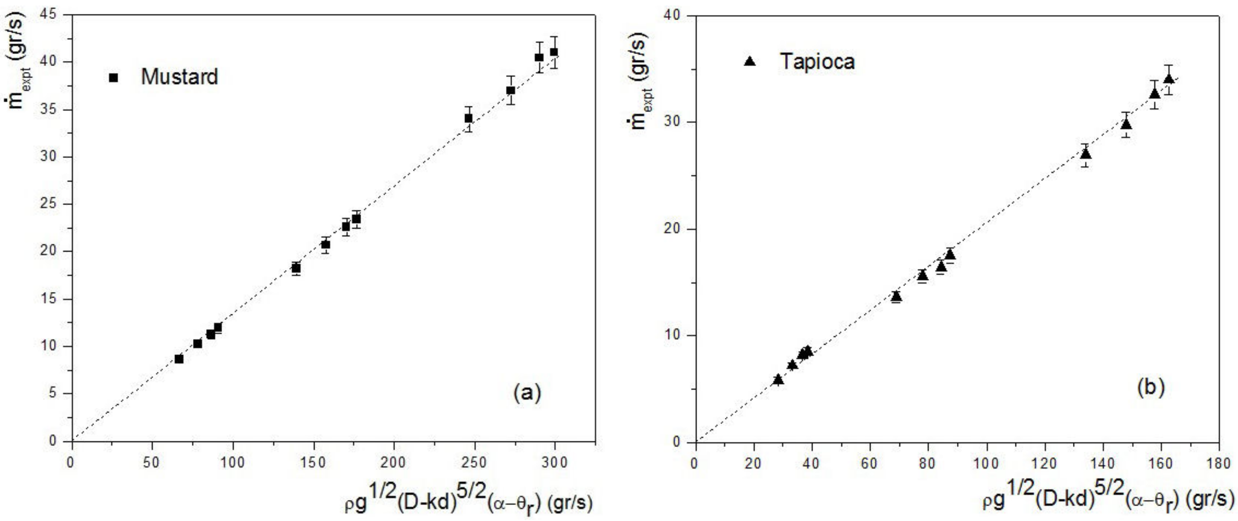

FIGURE 4. Plots of the experimental mass flow rates through holes on side walls, $\dot{m}_{\exp t}$, as a function of $\rho g^{1/2}(D-kd)^{5/2}[\alpha-\theta_r]$. The linear fit given by Eq. (2) yields that $c=0.13$ for mustard (a), and $c=0.20$ for tapioca (b). Erros bars are of around 4%.

2.2. Mass flow rate from lateral holes

In Fig. 4 we plot the mass flow rate measured for holes on the side walls, $\dot{m}_{\exp t}$, as a function of $\rho g^{1/2}(D-kd)^{5/2}[\arctan(D/w)-\theta_r]$. These plots show that experimental data for both granular materials were well fitted by straight lines, the best relation that fits the experimental data has the form

$$\dot{m}_{\exp t}=c\rho g^{1/2}(D-kd)^{5/2}\left[\alpha-\theta_r\right], \tag{2}$$

where the dimensionless discharge coefficient c has the value $c=0.13$ for mustard, and $c=0.20$ for tapioca.

Equation (2) also can be written as

$$\dot{m}_{\exp t}=c\dot{m}_0\left[\alpha-\theta_r\right], \tag{3}$$

where $\dot{m}_0=\rho g^{1/2}(D-kd)^{5/2}$, is the Hagen-Beverloo law. Finally, we can establish that Eq. (2) (or their equivalent form, Eq. (3)) is a generalized Hagen-Beverloo law for the mass flow rate from holes on the side walls.

An important consequence of Eqs. (2) or (3), is if them allows us to determine the critical value, w_c, for which the outflow will be arrested, *i.e.*, there is no flow at all when $\dot{m}_{\exp t}=0$. The unique actual possibility to get a condition for the halt of the flow, in such equations, is that the term $[\alpha-\theta_r]=0$, because the other possible case, $(D-kd)=0$, does not is an universally valid limit in the Hagen-Beverloo approach [11]. Moreover, detailed studies of the arrest condition for the gravity flows from bottom exits of bins refer to the arch formation close to the exit hole. These arches induce jams if the critical size of the hole is below of $D=4.94d$, for smooth and spherical grains [22, 23]. If the grains have friction or non spherical shape the latest condition is invalid [23]. Consequently, the use the condition $(D-kd)=0$ as a halt condition is no recommended.

Despite that it is very recommendable to get a halt condition for side wall holes, we believe that a lot of more experimental studies are required to reach a robust correlation valid for a large enough variety of cases. In this sense we only report that we have observed qualitatively in many cases that the arrest of the flow occurs when $\alpha \approx \theta_r$. This result is similar to the halt condition for the flow rate of granular solids (where the parameter d does not appears in the formula for the mass flow rate) [19, 20].

3. Conclusions

In this work we studied experimentally the problem of the flowability of big grains through the measurements of flow rates of non cohesive granular materials emerging from circular orifices on side walls of bins. Specifically, we have studied, simultaneously, the dependence of $\dot{m}_{\exp t}$ on the diameter of the orifice, the grain's diameter and the wall thickness by using, well characterized, round grains of mustard and tapioca.

Experiments show that the excluded volume effect occurs for big grains through the term $(D - kd)^{5/2}$. Thus, the estimation of the mass flow rate for holes on side walls satisfies, very accurately, the formula $\dot{m}_{\exp t} = c\rho g^{1/2}(D - kd)^{5/2}[\arctan(D/w) - \theta_r]$, where in our experiments the dimensionless discharge coefficient has the value $c \sim 0.1$ for mustard ($c \sim 0.2$ for tapioca), when cgs units are used. Finally, all of our work suggests that Eq. (2) is a general formula valid in a wide range of practical configurations.

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1. G.H.L. Hagen, *Bericht uber die zur Bekanntmachung geeigneten Verhandlungen der Koniglich Preussischen Akademie der Wissenschaften zu Berlin*, **35** (1852) 35-42.
2. F.C. Franklin, L.N. Johanson, *Chem. Eng. Scie.* **4** (1955) 119-129.
3. W.A. Beverloo, H.A. Leniger, J. van de Velde, *Chem. Eng. Sci.* **15** (1961) 260-269.
4. K. Wieghardt, *Ann. Rev. Fluid Mech.* **7** (1975) 89-114.
5. C.R. Woodcock, J.S. Mason, *Bulk solids handling: and introduction to the practice and technology*, Blackie and Son Ltd, (London 1987). Chap. 2 pp. 47-63.
6. D. Schulze, *Powders and Bulk Solids*, Springer-Verlag, (Berlin 2008).
7. S. Dorbolo *et al*., *Granular Matter* **15** (2013) 263–273.
8. D. Hirshfeld, D.C. Rapaport, *Eur. Phys. J. E* **4** (2001) 193–199.
9. L. Staron, P.-Y. Lagree, S. Popinet, *Eur. Phys. J. E* **37** (2014) 1-12.
10. See, for instance: R.M. Nedderman, *Statics and Kinematics of granular materials*, (Cambridge University Press, Cambridge 1992). pp. 292.
11. R.L. Brown, J.C. Richards, *Principles of Powder Mechanics*, (Pergamon Press, Oxford 1970).
12. A. Janda, I. Zurigel, D. Maza, *Phys. Rev. Lett.* **108** (2012) 248001.
13. I.I. Bagrintsev, S.S. Koshkovskii, *Jour. Chem. Petroleum Eng.* **6** (1977) 503-505.
14. C.E. Davies, J. Foye, *Trans. Inst. Chem. Engrs.* **69** (1991) 369-373.
15. C.S. Chang, H.H. Converse, J.L. Steele, *Trans. Am. Soc. Agr. Eng.* **34** (1991) 1789-1796.
16. C.E. Davies, M. Desai, *Powder Tech.* **183** (2008) 436-440.
17. H.G. Sheldon, D.J. Durian, *Granular Matter* **12** (2010) 579-585.
18. Household metal silos, Agricultural and Food Engineering Technologies Service, Food and Agriculture Organization of the United Nations, Viale delle Terme di Caracalla, 00153, Rome, Italy, (2008), download from: http://www.fao.org/fileadmin/user_upload/ags/publications/silos_E _light.pdf
19. A. Medina, D.A. Serrano, G.J. Gutierrez, K. Kesava Rao, C.A. Vargas, *Rev. Mex. Fis.* **59** (2013) 287–291.
20. A. Medina, D. Cabrera, A. López-Villa, M. Pliego, *Powder Tech.* **253** (2014) 270–275.
21. S.D. Liu, Z.Y. Zhou, R.P. Zou, D. Pinson, A.B. Yu, *Powder Tech.* **253** (2014) 70-79.
22. I. Zuriguel, L.A. Pugnaloni, A. Garcimartín, D. Maza, *Phys. Rev. E* **68** (2003) 030301 (R).
23. I. Zuriguel, A. Garcimartín, D. Maza, L.A. Pugnaloni, J.M. Pastor, *Phys. Rev. E* **71** (2005) 051303.