Fractional viscoelastic models applied to biomechanical constitutive equations

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The aim of this work consist to compare the traditional viscoelastic material models vs the fractional ones, determinate the fractional order of the differential operator that characterize the mechanical stress-strain relation, the stress relaxation and the creep compliance of this models for a biological soft tissue, in particular a femoral artery. Apply the Laplace transform for Mittag-Leffler function type and the convolution on fractional standard lineal solid differential equation, known as Zener model, to obtain analytical solution. Simulated the force-pressure related by singular blood flow pulse and the displacement response.

Keywords: Fractional; viscoelastic; biomechanics; soft tissues.

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1. Introduction

In the last years the fractional calculus has demonstrated a huge range of applicability, for example on the electronic field, the theory of control [1] and the circuit's theory [2-4], in mechanics the principal developments are in mechanical systems [5-8]. On the Biomechanics field the things are not so much different [9], the fractional differential and integral operators have a great development specially in the task of characterize the mechanical behavior of soft tissues [10] like the brain [11], liver [12,13], arteries [14-17] and the human calcaneal fat pad [18]. Biological soft tissues are mainly made of collagen, elastin and muscular fiber [19] which bring special mechanical properties. This kind of material behavior is known as viscoelasticity [20,21]. In general, viscoelastic behavior may be imagined as a spectrum with elastic deformation as one limiting case and viscous flow on the other extreme case, with varying combinations of the two spread over the range between. Thus, valid constitutive equations for viscoelastic behavior embody elastic deformation and viscous flow as special cases and at the same time provide for response patterns that characterize behavior blends of the two. Intrinsically, such equations will involve not only stress and strain, but time-rates of both stress and strain as well [22].

Many of the basic ideas of viscoelasticity can be introduce within the context of a one-dimensional state of stress, and once we obtain the relaxation modulus, the creep compliance and the complex modulus, this functions can be include by a subroutine on a FEM software that includes the geometry restrictions [23] and the viscoelastic relaxation modifications, or by an finite element model specially develop for fractional differential and integral operators [24].

2. Fractional Calculus

Let $f(t) \in \mathcal{C}^2$ where $f(t) : \mathbb{R}^+ \to \mathbb{R}$, according to the Riemann-Liouville approach to fractional calculus the notion of fractional integral of order $\nu > 0$, is a natural analogue of the Cauchy formula, Eq. (1).

$$ {}_0\mathrm{I}_t^n = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)\mathrm{d}\tau, \quad n \in \mathbb{Z}^+ \tag{1}$$

In a natural way, one is lead to extend the above formula (1) from positive integers values of the index to any positive real values by using the Gamma function and ν an arbitrary positive real number, one defines the Riemann-Liouville fractional integer of order $\nu > 0$,

$$ {}_0\mathrm{I}_t^\nu = \frac{1}{\Gamma(\nu)}\int_0^t (t-\tau)^{\nu-1} f(\tau)\mathrm{d}\tau, \quad t>0 \quad \nu \in \mathbb{R}^+ \tag{2}$$

For complementation of the Eq. (2), we need to define ${}_0\mathrm{I}_t^0 = \mathbf{I}$ such that ${}_0\mathrm{I}_t^0 f(t) = f(t)$.

The local operator of the standard derivative of order $n \in \mathbb{Z}^+$ for a given t is just the left inverse of the non-local operator of the n-fold integral ${}_a\mathrm{I}_t^n$, having as a starting point any finite $a < t$.

$$\mathrm{D}_t^n \circ {}_a\mathrm{I}_t^n, \quad t > a$$

and

$$ {}_a\mathrm{I}_t^n \circ \mathrm{D}_t^n = f(t) - \sum_{k=0}^{n-1} f^k(a^+)\frac{(t-a)^k}{k!}, \quad t>a \tag{3}$$

As a consequence, taking $a = 0$, we required that ${}_0\mathrm{D}_t^\nu$ be defined as left inverse to ${}_0\mathrm{I}_t^\nu$. For this purpose we first introduce the positive integer $m \in \mathbb{Z}^+$ such that $m-1 < \nu < m$, and then we define the Riemann-Liouville fractional derivative of order $\nu > 0$,

$$
\begin{aligned}
{}_0\mathrm{D}_t^\nu f(t) &= \mathrm{D}_t^m \circ {}_0\mathrm{I}_t^{m-\nu} f(t), \\
\text{with,} \qquad & m-1 < \nu < m, \qquad (4)
\end{aligned}
$$

namely:

$$
{}_0\mathrm{D}_t^\nu f(t) = \frac{1}{\Gamma(m-\nu)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_0^t \frac{f(\tau)}{(t-\tau)^{\nu+1-m}} \mathrm{d}\tau,
$$

if $m-1 < \nu < m$ and $\mathrm{d}^m/\mathrm{d}t^m f(t)$, if $\nu = m$

For complementation, we define ${}_0\mathrm{D}_t^0 = \mathrm{I}$.

By interchanging in the Eq. (4) the process of differentiation and integration we are lead to the so called *Caputo fractional derivative*, defined properly and in extensive way in [25,26], of order $\nu > 0$ defined as:

$$
\begin{aligned}
{}_0^*\mathrm{D}_t^\nu f(t) &= {}_0\mathrm{I}_t^{m-\nu} \circ \mathrm{D}_t^m f(t), \\
\text{with,} \quad & m-1 < \nu < m, \qquad (5)
\end{aligned}
$$

namely:

$$
{}_0^*\mathrm{D}_t^\nu f(t) = \frac{1}{\Gamma(m-\nu)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\nu+1-m}} \mathrm{d}\tau,
$$

$$
\text{if} \quad m-1 < \nu < m \quad \text{and} \quad \frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t), \quad \text{if} \quad \nu = m
$$

For $\nu \notin \mathbb{Z}^+$ the *Caputo* definition (5), requires the absolute integrability of the derivative of order m.

The Caputo fractional derivative represent a sort of regularization in the time origin for the Riemann-Liouville fractional derivative. Note that for its existence all the limiting values

$$
f^{(k)}(0^+) = \lim_{t \to 0^+} \mathrm{D}_t^k f(t), \ k = 0, 1, 2, ..., m-1
$$

are required to be finite.

2.1. Laplace transform for fractional derivatives

We point out the major usefulness of the Caputo fractional derivative in treating initial value problems for physical applications where the initial conditions are usually expressed in terms of integer order derivatives. For the Caputo derivative of order ν with $m-1 < \nu < m$, we have

$$
\mathcal{L}\{{}_0^*\mathrm{D}_t^\nu f(t); s\} = s^\nu \bar{f}(s) - \sum_{k=0}^{m-1} s^{\nu-1-k} f^{(k)}(0^+), \qquad (6)
$$

$$
f^{(k)}(0^+) = \lim_{t \to 0^+} \mathrm{D}_t^k f(t)
$$

When all the limiting values $f^{(k)}(0^+) = 0$ for $k = 1, 2, ...$ are zero the Eq. (6), simplifies into:

$$
\mathcal{L}\{{}_0^*\mathrm{D}_t^\nu f(t); s\} = s^\nu \bar{f}(s), \qquad m-1 < \nu < m \qquad (7)
$$

The Laplace transform can be expressed in terms of functions of Mittag-Leffler type,

$$
\mathcal{L}\left\{t^{\varphi-1} E_{\nu,\varphi}(-\omega t^\nu); s\right\} = \frac{s^{\nu-\varphi}}{s^\nu + \omega} \qquad (8)
$$

where

$$
E_{\nu,\varphi}(-\omega t^\nu) = \sum_{n=0}^{\infty} \frac{(-\omega t^\nu)^n}{\Gamma(\nu n + \varphi)} \qquad (9)
$$

with $\nu, \varphi \in \mathbb{R}^+$ and $\omega \in \mathbb{R}$.

Another Laplace transform that we need to remember, is the notion of Laplace convolution

$$
\begin{aligned}
\mathcal{L}\{f(t) * g(t)\} &= \mathcal{L}\left\{ \int_0^t f(t-\xi) g(\xi) d\xi; \ s \right\} \\
&= \bar{f}(s) \cdot \bar{g}(s) \qquad (10)
\end{aligned}
$$

3. Fractional viscoelastic material models

3.1. Integer order viscoelastic models

Viscoelasticity is one of the major fields in the application of the fractional differential and integral operators [27-29]. A material that exhibit elements of both the elastic and viscous behaviors is known as a viscoelastic material. The stress - strain behavior of such materials can be model by combining the relationships between the solids Hooke's law $\sigma(t) = e\epsilon(t)$, represented by springs and for Newtonian fluids $\sigma(t) = \eta\dot{\epsilon}(t)$, represented by dash pots, where e, η are constants in several ways as shown in Fig. 1.

The viscoelastic models may be use to represent the rate-dependent behavior [30] where the stress-strain relation is a function of the rate of strain, the creep compliance $\mathrm{J}(t)$ behavior where the strain increases under a constant applied load and the stress relaxation $\mathrm{G}(t)$ behavior where the stress decreases under a constant applied strain. When the material its subjected to harmonic stress or strain another important part of the theory of viscoelasticity are the useful concept [23] of the complex modulus $\mathrm{G}^*(i\omega)$ where the real part, $\mathrm{G}'(\omega)$, of this modulus is associated with the amount of energy stored in the element during a complete loading cycle and is called the storage modulus. The imaginary part, $\mathrm{G}''(\omega)$, relates to the energy dissipated per cycle and is called the loss modulus.

We mentioned before that viscoelastic model are arrays of springs and dash pots like the shown in Fig. 1, this arrays can be express mathematically by the Eq. (11) for the Kevin model (KM) [31] and the equation (12) for the standard lineal solid, also know as Zener model (ZM) [32].

$$
\dot{\sigma}(t) + \frac{e_1 + e_2}{\eta} \sigma(t) = e_1 \dot{\epsilon}(t) + \frac{e_1 e_2}{\eta} \epsilon(t) \qquad (11)
$$

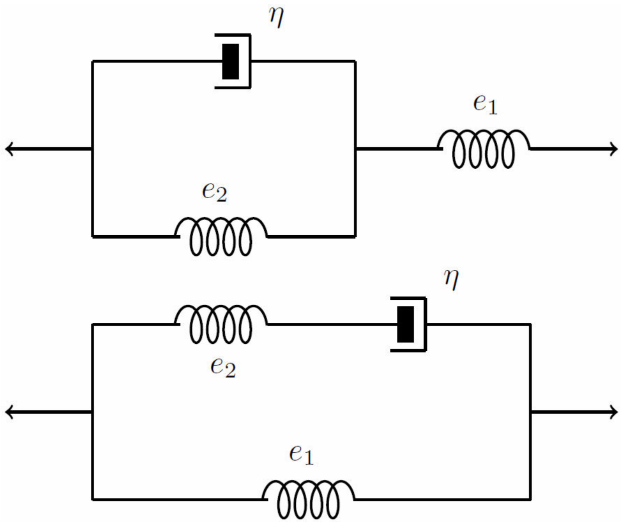

FIGURE 1. Kelvin model (KM) on top, Standard lineal solid model other known as Zener model (ZM).

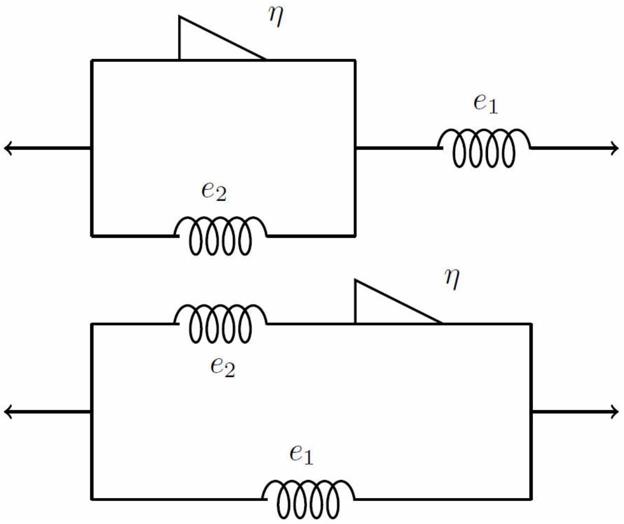

FIGURE 2. Fractional Kelvin model (FKM) on top and at the bottom Zener fractional model (FZM), can be observe that the basic idea consist in to replace the dash pot for a new element know as spring-pot.

$$\dot{\sigma}(t) + \frac{e_2}{\eta}\sigma(t) = (e_1 + e_2)\,\dot{\epsilon}(t) + \frac{e_1 e_2}{\eta}\epsilon(t) \qquad (12)$$

3.2. Fractional viscoelastic models

We now consider the fractional generalization of the KM and ZM, shown in Fig. 2. For this purpose it's sufficient to replace the first order derivative with the fractional derivative of order $\nu \in (0,1)$ in their constitutive equations. We obtain the following stress-strain relationship and corresponding material functions.

The Eqs. (13) and (14) are basically the same shown before, but here we replace the first derivative with the Caputo fractional differential operator

$${}_0^*\mathrm{D}_t^\nu \sigma(t) + \frac{e_1 + e_2}{\eta}\sigma(t) = e_1\, {}_0^*\mathrm{D}_t^\nu \epsilon(t) + \frac{e_1 e_2}{\eta}\epsilon(t) \qquad (13)$$

$${}_0^*\mathrm{D}_t^\nu \sigma(t) + \frac{e_2}{\eta}\sigma(t) = (e_1 + e_2)\, {}_0^*\mathrm{D}_t^\nu \epsilon(t) + \frac{e_1 e_2}{\eta}\epsilon(t) \qquad (14)$$

3.3. Analytical solution

Applying the Laplace transform (7) to booth sides of the Eq. (13) we obtain,

$$\left[s^\nu + \left(\frac{e_1 + e_2}{\eta}\right)\right]\bar{\sigma}(s) = \left[e_1 s^\nu + \left(\frac{e_1 e_2}{\eta}\right)\right]\bar{\epsilon}(s)$$

solving for $\bar{\epsilon}(s)$

$$\bar{\epsilon}(s) = \frac{s^\nu + \alpha}{e_1 s^\nu + \beta}\bar{\sigma}(s)$$

where $\alpha = (e_1 + e_2/\eta)$ and $\beta = e_1 e_2/\eta$, applying the Laplace inverse transform and the convolution law (10)

$$\epsilon(t) = \left[\frac{1}{e_1}\mathcal{L}^{-1}\left\{\frac{s^\nu}{s^\nu + \frac{\beta}{e_1}}\right\} + \frac{\alpha}{e_1}\mathcal{L}^{-1}\left\{\frac{1}{s^\nu + \frac{\beta}{e_1}}\right\}\right] * \sigma(t)$$

this functions don't have the require form to apply the laplace transform of the Mittag-Leffler function (8), then we rescribe on the form

$$\epsilon(t) = \left[\frac{1}{e_1}\mathcal{L}^{-1}\left\{\frac{s \cdot s^{\nu-1}}{s^\nu + \zeta}\right\}\right.$$
$$\left. + \frac{\alpha}{e_1}\mathcal{L}^{-1}\left\{\frac{s^{-\nu+1} \cdot s^{\nu-1}}{s^\nu + \zeta}\right\}\right] * \sigma(t)$$

now we can separate

$$\epsilon(t) = \left[\frac{1}{e_1}\left(\mathcal{L}^{-1}\{s\} \cdot \mathcal{L}^{-1}\left\{\frac{s^{\nu-1}}{s^\nu + \zeta}\right\}\right) + \frac{\alpha}{e_1 \Gamma(n-1)}\right.$$
$$\left. \times \left(\mathcal{L}^{-1}\left\{\frac{\Gamma(\nu-1)}{s^{\nu-1}}\right\} \cdot \mathcal{L}^{-1}\left\{\frac{s^{\nu-1}}{s^\nu + \zeta}\right\}\right)\right] * \sigma(t)$$

By the Mittag-Leffler Laplace transform (8) and the convolution (10) we have:

$$\epsilon(t) = \left[\frac{1}{e_1}\int_0^t \delta'(t-\xi) \cdot E_\nu\left(-\zeta\xi^\nu\right) d\xi + \frac{\alpha}{e_1 \Gamma(\nu-1)}\right.$$
$$\left. \times \int_0^t (t-\xi)^{\nu-2} E_\nu(-\zeta\xi^\nu) d\xi\right] * \sigma(t)$$

in terms of Eq. (9) we obtain the analytical solution for Kelvin fractional model, in similarly way can be obtained the solution for the ZFM,

$$\epsilon(t) = \left[\frac{\delta(t)}{e_1} + \frac{1}{e_1 t}\sum_{n=1}^{\infty}\frac{(-\zeta t^\nu)^n}{\Gamma(n\nu)}\right.$$
$$\left. + \frac{\alpha t^{\nu-1}}{e_1}\sum_{n=0}^{\infty}\frac{(-\zeta t^\nu)^n}{\Gamma(\nu(n+1))}\right] * \sigma(t)$$

were $\delta(t)$ it's the traditional Dirac's delta.

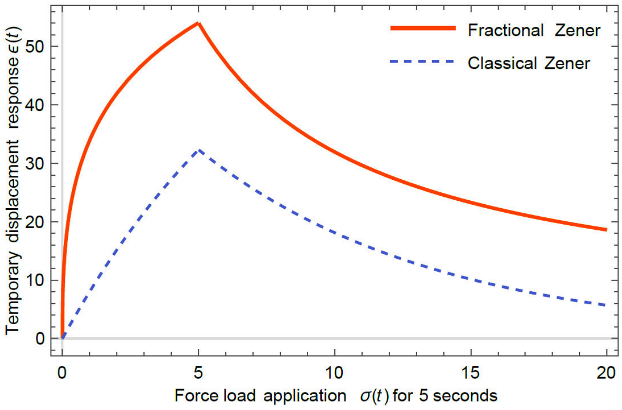

FIGURE 3. Displacement response for the Zener model (fractional and integer order), for arbitrary constant values and fractional order $\nu = 0.4$. The election of this value will be justify latter.

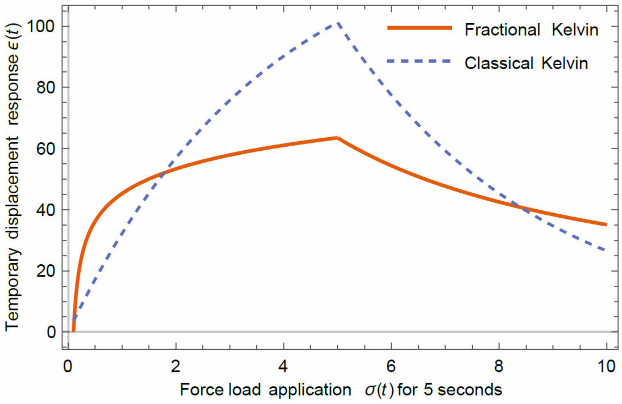

FIGURE 4. Displacement response for the Kelvin model. Fractional and integer.

4. Results

4.1. Model analysis

Now we can obtain the temporary deformation response for the integer and fractional viscoelastic models, first we apply an 50 MPa step function in a time lapse of 5 seconds. The displacement response for the ZFM an ZM integer order viscoelastic constitutive equation is shown on Fig. 3.

On Fig. 3, we can appreciate for the ZFM, the classic viscoelastic ideal behavior, the almost one on one force-displacement relationship and the relaxation phenomena once the load its retire, other hand ZM integer order have an almost linear response in the beginning, not a one on one relationship and a minor relaxation time. From here we can deduce that ZFM presents a more accurate description of the biological soft tissue that ZM, wich posses a viscoelastic material behavior.

On the Kelvin model, Fig. 4, we observe a more accurate behavior for the integer model with respect to the fractional one, but the response almost duplicate the force function. For that reason we choose the Zener fractional model for the task

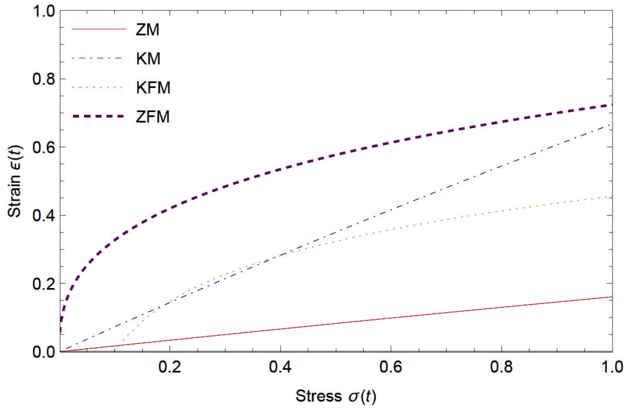

FIGURE 5. Stress-strain curve for the ZM, KM, KFM and ZFM.

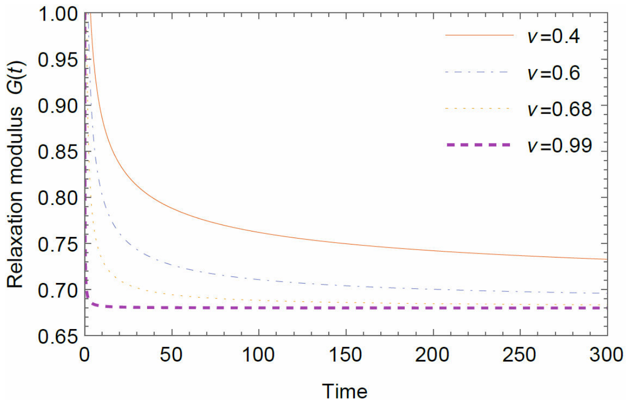

FIGURE 6. Fractional relaxation modulus, can be observed that for a near integer order, $\nu = 0.99$ the relaxation function have an exponential behavior.

of characterize the viscoelastic mechanical description of the artery segment. For last we plot the stress-strain relationship for the four material models on Fig. 5, where we can observe that only the fractional models corresponds with the classical curve that expected, but the KFM have an initial translation that it not desire.

4.2. Artery characterization

Now we have described the material model functions like the relaxation modulus, creep compliance and complex modulus. The relaxation modulus for the ZFM have the form,

$$\mathrm{G}(t) = e_1 + e_2 \cdot E_\nu \left[- \left(\frac{e_2}{\eta} t^\nu \right) \right]$$

In Fig. 6 is plot the relaxation modulus function for four fractional order values.

Using experimental data [33,34] and the application of the *Levenberg-Marquardt* algorithm based on a Gauss Newton method for least-squares problems, we can determinate the fractional order $\nu = 0.4$ that better fit the relaxation modulus from the experimental obtained data and the value its in the range that finding different researches for all kind of biological tissue [15,32].

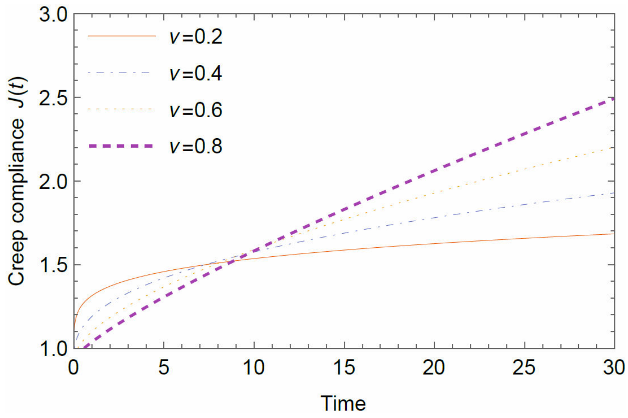

FIGURE 7. Creep compliance function, here we observe that for values close to real order $\nu = 1$ the behavior its almost lineal.

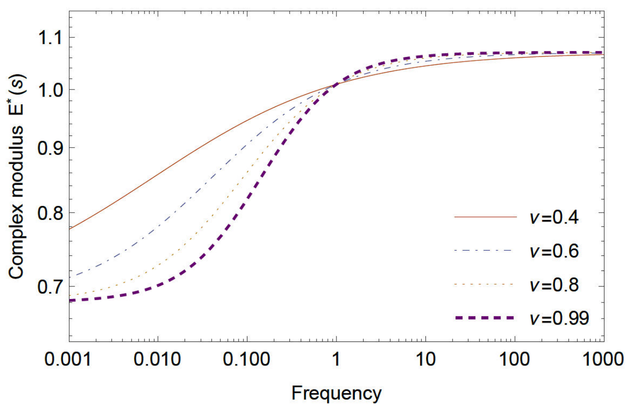

FIGURE 8. Complex modulus, observe that for values nearest to integer order one, the graphic present the traditional form.

In the same way, we obtain and plot the creep compliance function, that it's illustrate on Fig. 7 for different fractional values ν, the creep compliance function $\mathrm{J}(t)$ have the form:

$$\mathrm{J}(t) = \mu + \left(\frac{1}{e_1} - \mu\right)\left[1 - E_\nu\left[-\left(\frac{e_1 e_2 \mu}{\eta} t^\nu\right)\right]\right]$$

were $\mu = 1/e_1 + e_2$.

The complex modulus, present on Fig. 8, complete the set of basic functions require for biomechanical characterization.

4.3. Blood flow response

Describe the mechanical stress-strain relationship for the blood pulse, its one of the principal biomechanical objectives [35,36], in that sense we model a blood flow pulse (BFP) of 125/80 diastolic sistolic, that its the regular pulse stage, Fig. 9.

The BFP its parametrize by the function,

$$B(t) = \begin{cases} \frac{11}{10}\sin\left(\frac{\pi t}{0.36}\right) + \frac{9}{10} & 0 < t < 0.18 \\ \frac{11}{10}\cos\left(\pi\left(\frac{100t}{64} - \frac{28}{100}\right)\right) + \frac{9}{10} & 0.18 < t < 0.5 \\ \frac{9}{10} - \frac{1}{20}\sin(\pi(5t - \frac{5}{2})) & 0.5 < t < 0.9 \\ \frac{9}{10} & 0.9 < t < 1 \end{cases}$$

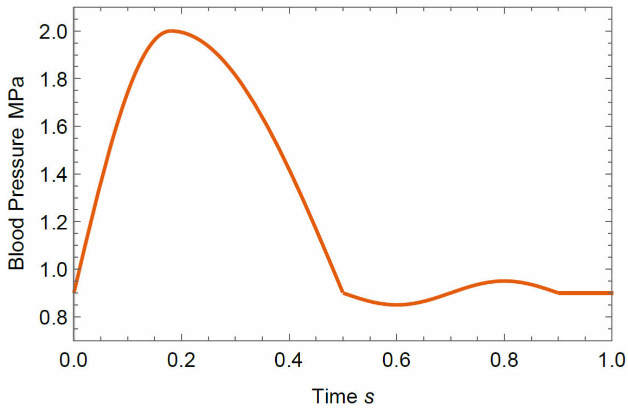

FIGURE 9. 125/80 mmHg blood pulse in Mega Pascals.

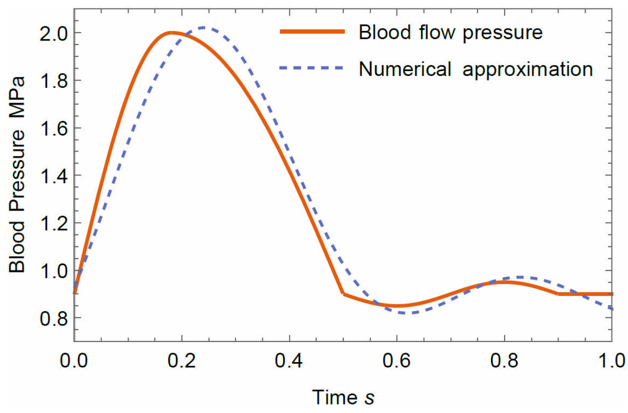

FIGURE 10. Numerical approximation of regular blood flow, this new function have continuity derivatives.

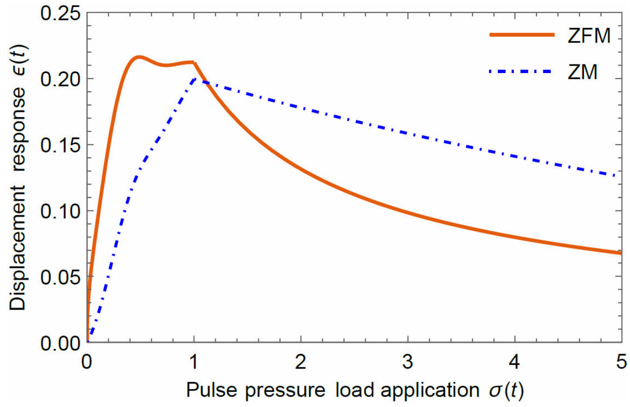

FIGURE 11. ZFM and ZM response for a single blood pulse.

but $B(t) \notin \mathcal{C}^2$, to avoid that problem we use the *Levenberg-Marquardt* algorithm to approximate the function to a new one $\breve{B}(t) \in \mathcal{C}^2$, the numerical approximation is represent on Fig. 10.

The function that represents the blood flow and that be use like force load on ZFM, has the form

$$\breve{B}(t) = 1.26\cos(3.79t) - 0.34\cos(10.82t) + 2.01\sin(1.97t) + 0.13\sin(12.11t)$$

The displacement response shown in Fig. 11 its only the ideal response for a singular blood flow pulse, can we see the pulse response on top during the second and a expected relaxation before that, compare with the ZM that only have a curve displacement and a relaxation linear stage.

5. Conclusions

We demonstrate that the fractional model preserves the relaxation phenomena that characterize the viscoelastic materials and special the biological ones, like the artery that is the research object, due to the relaxation soft decay and the logarithmic increment on the initial case of load. An a certain way, with the fractional differential and integral operators, is easily to obtain the principal functions relatives to the material behavior, like the relaxation modulus, creep compliance and the complex modulus, that are very useful in the case that we need realize adjustment to data obtained from experimental creep or relaxation test, to in vivo material, or in the case of the complex modulus that is in terms of the frequency, and are very helpful when the soft tissue are stimulated by an harmonic function, like the pulse or when try to obtain mechanical constants via ultrasound stimulation.

Using fractional differential and integer operators, and all the background related, we get a reliable model for the mechanical properties characterization that in the future research permits the integration on a finite element method software, this aloud to include the geometry of the artery and all the geometrical restrictions, in that sense we can obtain the dynamical response of the artery under different load events, or the inclusion of different kind and prolonged load blood flow rates by new numerical methods for fractional differential equations [37,38] or finite elements paradigmas [23,24].

For last we parametrize a blood flow pulse obtained from real medical researches [35,36], and development a numerical adjustment with continuum derivatives and using the analytical solution obtained in the present research we can able to determinate the displacement response for the pulse load application.

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