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Mathematical differences and physical similarities between Eliezer-Ford-O'connell equation and Landau-Lifshitz equation

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Recently, for high intensity electromagnetic waves, it has been proven that the solutions for the Eliezer-Ford-O'Connell equation and the Landau-Lifshitz equation coincide within a physically detectable range. For large-scale temporal effects, similar results are obtained for the central force problem. However, in the case of a constant magnetic field, the frequencies which describe the motion in both equations differ. Nonetheless, quantum constraints avoid the measurement of such difference making both equations physically equivalent for all the scale of energies and fields within Classical Mechanics regime.

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1. Introduction

For more than a century, the knowledge of the reaction force for a charged particle has represented an open problem due to unphysical results derived from the different proposals which depict this phenomenon. An interesting article which describes the distinct equations and their own physical problematic was recently done by Hammond [1]. Nowadays, among all the different approaches and despite of the new one proposed by Hammond [2-4], the Landau-Lifshitz equation of motion is considered as the better option to describe the motion of a spinless charged point particle [5-12]. On the other hand, in order to avoid Quantum effects [13], the charge must be constrained to the so-called Shen's zone [14]. This zone represents an area in an energy *versus* field diagram, where Relativistic and Non Relativistic Classical Mechanics drive the motion of a charged particle, without being affected by quantum aspects. Indeed, the energies and fields used in Plasma Physics belong to the Shen's zone. Therefore, not only relativistic situations must be analyzed, but also non relativistic cases must be investigated, since they are of major importance. An example of this is the case of the central force which has been recently studied by Rajeev [15]. Since his objective was to calculate the line-widths of an hydrogen-like atom and also to understand by means of a simple model the capture of a star by a black hole, it was naturally necessary to derive a non relativistic version of the Landau-Lifshitz equation . The method seems to be simple, because it consists in neglecting all the terms depending on $(v/c)^2$ in the Landau-Lifshitz equation. Alternatively, to find the non relativistic version, it is possible to solve the Landau-Lifshitz equation and then make a first order expansion in v/c. Also, another rough method would lead to the Ford Equation [16].

On the other hand, by considering a generalized quantum Langevin equation and by giving a structure to the electron with a factor form and a finite cut-off parameter, Ford and O'Connell derived a non relativistic equation of motion for charged particles. Generalizing this equation to Special Relativity [17] they obtained an equation, that Eliezer derived fifty years before using distinct arguments. Both equations, Landau-Lifshitz and Eliezer-Ford-O'Connell, are second order differential equations which do not present unphysical solutions, as preaccelerations and runaway solutions.

However, even if these equations are considered by many authors as identical, they come from different physical assumptions. Indeed, for Spohn [6] and Rohrlich [7,8], the Landau-Lifshitz equation is obtained if the Lorentz-Dirac equation is restricted to its critical surface. This indicates a fundamentally distinct origin from the one for Eliezer-Ford-O'Connell equation, as we mentioned above. Also, for Parrot [19], the Eliezer equation and other equations, as Mo and Papas equation [20], were derived as variants of Lorentz-Dirac equation by considering the acceleration to be proportional to the applied force. Moreover, Landau and Lifshitz derived their equation by substituting the Lorentz equation in the Lorentz-Dirac reaction term, and consequently it could be also considered as an approximation or a first order expansion in τ_o ($\tau_o = 2q^2/3mc^3$ is called the characteristic time) of the Lorentz-Dirac equation.

As we will see in this paper, although Landau-Lifshitz and Eliezer-Ford-O'Connell equations are mathematically different, they physically coincide. Some authors have made a comparison between the Lorentz-Abraham-Dirac equation and the non relativistic Landau-Lifshitz equation. For example Griffiths [12], and Rajeev [15] considered Ford equation as the non-relativistic case of the Landau-Lifshitz equation, which is true for the examples used by them. Although, comparisons between the Landau-Lifshitz equation and Eliezer-Ford-O'Connell equation have been done for high intense pulses and high energies [21,22]; such a comparison between Ford equation and the non-relativistic Landau-Lifshitz equation has not been done for large-scale temporal effects. In fact, it seems that many authors consider that Ford equation and the non relativistic Landau-Lifshitz equation are identical. Kravets *et al* [21] mentioned that the Eliezer-Ford-O'Connell equation has received less attention than it perhaps deserves, due in part to the confusion arising from its apparent equivalence to the Landau-Lifshitz approximation (for an intense laser pulse, Landau-Lifshitz and Eliezer-Ford-O'Connell solutions are almost equal).

In recent years, due to the appearance of higher laser intensities, the reaction force has been studied from the point of view of the relativistic case [21,13,22,2]. In fact, since for the case of a constant electric field there is an apparent lack of energy balance in the solutions of the Lorentz-Dirac equation, of the Landau-Lifshitz equation and of the Eliezer-Ford-O'Connell equation, an interesting approach has been developed by Hammond [4,3,2,1]. In particular, he found an interesting solution, for low and high velocities, which explains the radiated energy and the motion of the charge for the electric constant field case.

The radiation reaction force effects are very significant in the regime of very high intensities since such fields lead the particle to relativistic motion. Therefore, apparently to focus our interest on non-relativistic equations cannot be justified in first instance. However, non-relativistic effects can be detected for large-scale time as it is proved in many articles [15,23]. Moreover, in order to deal with trajectories, quantum restrictions indicate that for relativistic situations the fields must be less intense and consequently the large-scale temporal effects are more easily detected in the non relativistic cases [14,24].

This paper consists in showing that Ford equation (Eliezer-Ford-O'Connell equation in the relativistic case) and the non relativistic Landau-Lifshitz equation (Landau-Lifshitz equation in the relativistic case) are not mathematically equal but they are physically equivalent. Even if it is considered that both equations are equivalent to first order in τ_o (the characteristic time of the particle) [1], the dependence on the velocity in the applied force is the source of the difference. This will be twice shown by demonstrating the difference between both equations, and also by giving a counterexample (the constant magnetic case). First, we deduce the Landau-Lifshitz solution neglecting $(v/c)^2$ and then we solve the problem by means of Ford equation. Since the results are different, we will be able to understand the difference between both equations. However, the quantum constraints prevent to measure the large-scale temporal effects and thereby both equations can be seen as physically equivalent.

The paper is organized as follows: in Sec. 2, the Landau-Lifshitz equation is deduced and a simple representation is presented. Also, the solutions for the constant electric case and constant magnetic case are found. In Sec. 3, the non-relativistic Landau-Lifshitz equation is deduced. The comparison with Ford equation shows that both equations are mathematically different when the force depends on the speed. In Sec. 4, Ford equation is solved for the constant electric and the constant magnetic cases. An analysis between both results, the non relativistic Landau-Lifshitz solution and the Ford solution, is done. In Sec. 5, by using a special representation of Eliezer equation [19], the equivalence between Eliezer equation and Ford-O'Connell equation is showed and at the same time the mathematical difference with the Landau-Lifshitz equation is exposed. A simplified representation of both equations is deduced. In Sec. 6, the expression for the large distance radiated power is analyzed giving the same result for both equations, even if the force depends on the speed. However, since each equation has different solutions the corresponding radiated powers are distinct. In Sec. 7, in Concluding Remarks, the differences between both equations are summarized. Also, numerical physical situations are studied considering the quantum constraints.

2. Landau-Lifshitz Equation

In 1938, Dirac [25] proposed an equation of motion which pretends to relativistically describe the motion of a spinless point-like charged particle including the reaction force; that is:

$$ma^{\mu} = (q/c)F^{\mu\nu}v_{\nu} + \tau_o m \left[\dot{a}^{\mu} + \frac{a^2}{c^2}v^{\mu}\right], \qquad (1)$$

where q, c and $F^{\mu\nu}$ represent the charge of the particle, the speed of light and the field-strength tensor, respectively. The dot "$\cdot$" means derivative with respect to the proper time of the particle. Since many unphysical results can be derived from this equation, as for example the runaway solutions and the preaccelerations, many others proposals appeared in order to avoid such inconveniences. The Landau-Lifshitz equation [5] represents the most acceptable model since it is a second order differential equation which does not possess any of the problems just cited [6-11].

Let us deduce the equation in such a manner that it will allow to obtain a simple representation. First, consider the Lorentz equation

$$ma_L^{\mu} = (q/c)F^{\mu\nu}v_{\nu}, \qquad (2)$$

where a_L^{μ} represents the particle's acceleration when it is guided just by the Lorentz force and for this reason we use the subscript "L". Second, let us consider the trajectory of a charged particle which satisfies the Landau-Lifshitz equation. In each point of such trajectory, we can define the following $4-$vector fields:

$$v_L^{\mu} = v^{\mu}, \qquad a_L^{\mu} = (q/cm)F^{\mu\nu}v_{\nu}$$

$$\text{and} \qquad b_L^{\mu} = \frac{q}{cm}\left[\frac{\partial F^{\mu\nu}}{\partial x^{\alpha}}v^{\alpha}v_{\nu} + F^{\mu\nu}a_L^{\mu}\right]. \qquad (3)$$

This could be done, since a Lorentz trajectory crosses each point of the Landau-Lifshitz trajectory. Such Lorentz trajectories can be described by a proper Lorentz time, $\tau_L = \tau_L(x^\mu, v^\mu)$. In other words, the particle's proper time when it is restricted to follow the Lorentz force. Therefore, it is clear that the fields v_L^μ and a_L^μ described by Eqs. (3) correspond to the derivative with respect to the proper Lorentz time τ_L,

$$\frac{dx^\mu}{d\tau_L} = v_L^\mu \qquad \text{and} \qquad \frac{dv^\mu}{d\tau_L} = a_L^\mu = \frac{q}{cm} F^{\mu\nu} v_\nu. \tag{4}$$

Also, it is observed that:

$$\begin{aligned}\overset{\circ}{a}{}_L^\mu &= \frac{d\frac{dv^\mu}{d\tau_L}}{d\tau_L} = \frac{da_L^\mu}{d\tau_L} = (q/cm)\frac{d\left[F^{\mu\nu}v_\nu\right]}{d\tau_L} \\ &= (q/cm)\left[\frac{dF^{\mu\nu}}{d\tau_L}v_\nu + F^{\mu\nu}\frac{dv_\nu}{d\tau_L}\right] \\ &= \frac{q}{cm}\left[\frac{\partial F^{\mu\nu}}{\partial x^\alpha}\frac{dx^\alpha}{d\tau_L}v_\nu + F^{\mu\nu}a_L^\mu\right] \\ &= \frac{q}{cm}\left[\frac{\partial F^{\mu\nu}}{\partial x^\alpha}v^\alpha v_\nu + F^{\mu\nu}a_L^\mu\right] = b_L^\mu \end{aligned} \tag{5}$$

Therefore, the proper Lorentz time and the $4-$vector fields in Eqs. (3) and (5) are well-defined along the Landau-Lifshitz trajectory.

If we substitute in the Lorentz-Dirac radiation term, $\tau_o m[\overset{\circ}{a}{}^\mu + (a^2/c^2)v^\mu]$, a^μ and $\dot{a}^\mu$ by the Lorentz acceleration a_L^μ, the result is

$$\begin{aligned} ma^\mu &= \frac{q}{c}F^{\mu\nu}v_\nu + \tau_o\left[\frac{da_L^\mu}{d\tau_L} + \frac{a_L^2}{c^2}v^\mu\right] \\ &= \frac{q}{c}F^{\mu\nu}v_\nu + \tau_o\left[\frac{d}{d\tau_L}\left[\frac{q}{c}F^{\mu\nu}v_\nu\right] + \frac{q^2}{c^4 m}F^2 v^\mu\right] \end{aligned} \tag{6}$$

where $F^\mu = F^{\mu\beta}v_\beta$, $F^2 = F^\mu F_\mu = F^{\alpha\beta}v_\beta F_{\alpha\eta}v^\eta$ and τ_L represents an invariant quantity defined in each point of the real trajectory of the particle which coincides with the proper time of a virtual charged particle whose motion is in accordance with Lorentz equation and with the same $4-$velocity v^μ at the crossing point between the Landau-Lifshitz and Lorentz trajectories. Then, since

$$\tau_o\frac{d}{d\tau_L}\left[\frac{q}{c}F^{\mu\nu}v_\nu\right] = \tau_o\frac{q}{c}\left[\frac{\partial F^{\mu\nu}}{\partial x^\alpha}v^\alpha v_\nu + F^{\mu\nu}a_L^\mu\right]. \tag{7}$$

By using Eq. (4), we obtain

$$\begin{aligned}\tau_o\frac{d}{d\tau_L}\left[\frac{q}{c}F^{\mu\nu}v_\nu\right] \\ = \tau_o\frac{q}{c}\left[\frac{\partial F^{\mu\nu}}{\partial x^\alpha}v^\alpha v_\nu + \frac{q}{cm}F^{\mu\nu}F_{\nu\alpha}v^\alpha\right].\end{aligned} \tag{8}$$

Finally, due to the antisymmetry property of the field-strength tensor $F^{\mu\nu}$, we arrive at

$$\begin{aligned}\tau_o\frac{d}{d\tau_L}\left[\frac{q}{c}F^{\mu\nu}v_\nu\right] \\ = \tau_o\frac{q}{c}\left[\frac{\partial F^{\mu\nu}}{\partial x^\alpha}v^\alpha v_\nu - \frac{q}{cm}F^{\mu\nu}F_{\alpha v}v^\alpha\right].\end{aligned} \tag{9}$$

Introducing this last result in Eq. (6), we obtain the Landau-Lifshitz equation of motion,

$$\begin{aligned} ma^\mu &= (q/c)F^{\mu\nu}v_\nu \\ &\quad + \tau_o\Bigg[\frac{q}{c}\left[\frac{\partial F^{\mu\nu}}{\partial x^\alpha}v^\alpha v_\nu - (q/cm)F^{\mu\nu}F_{\alpha v}v^\alpha\right] \\ &\quad + (q^2/c^4 m)F^2 v^\mu\Bigg], \end{aligned} \tag{10}$$

where $F^2 = F^{\mu\nu}v_\nu F_{\mu\lambda}v^\lambda$. This last equation seems to be very difficult to solve. However, Eq. (6) represents the same equation as the Landau-Lifshitz equation, Eq. (10), but it is presented in such a form that it is easier to solve. Moreover, starting from Eq. (6), another representation of the Landau-Lifshitz equation can be deduced in order to simplify the technique to obtain its solutions.

2.1. A simple representation of the Landau-Lifshitz equation

By defining the constant $k = q/cm$, Eq. (6) can be rewritten as

$$a^\mu = kF^{\mu\nu}v_v + \tau_o\frac{d}{d\tau_L}\left[kF^{\mu\nu}v_\nu\right] + \frac{\tau_o k^2}{c^2}F^2 v^\mu. \tag{11}$$

In order to develop a technique to solve the Landau-Lifshitz equation of motion, the two $4-$vectors a_L^μ and $da_L^\mu/d\tau_L$ can be written as:

$$a_L^\mu = kF^{\mu\nu}v_\nu, \tag{12}$$

and

$$\begin{aligned}\frac{da_L^\mu}{d\tau_L} &= \frac{d}{d\tau_L}\left[kF^{\mu\nu}v_\nu\right] \\ &= k\left[\frac{\partial F^{\mu\nu}}{\partial x^\alpha}v^\alpha v_\nu - kF^{\mu\nu}F_{\alpha v}v^\alpha\right].\end{aligned} \tag{13}$$

These vectors represent the acceleration and the rate of change of the acceleration with respect to the proper time of a charged particle that moves following the trajectory generated by the Lorentz force in each point of the Landau-Lifshitz trajectory of the charge. Substituting these vectors in Eq. (11), we obtain the following equation,

$$a^\mu = a_L^\mu + \tau_o\left[\frac{da_L^\mu}{d\tau_L} + \frac{a_L^2}{c^2}v^\mu\right]. \tag{14}$$

This last equation is also a representation of the Landau-Lifshitz equation and it is the desired equivalent form. It has to be pointed out that Landau-Lifshitz equation has the same form than the Lorentz-Dirac equation but with a^μ substituted by a_L^μ in the right member of the equation. In order to obtain a solution for the Landau-Lifshitz equation, it is necessary to firstly solve the Lorentz equation, which represents a strong simplification. Indeed, it has to be noted that Eq. (14) is a second order equation because the term $da_L^\mu/d\tau_L$ is just a $4-$vector which depends on the $4-$vector position x^μ and the $4-$vector velocity v^μ of the charge (see Eq. (13)). Consequently, since we have noticed its equivalence with the Landau-Lifshitz equation, runaway solutions and preaccelerations are avoided. The interesting property of this representation consists of not having to calculate all the terms that appear in the regular expression of Landau-Lifshitz equation. This point will be shown in the next subsections.

2.2. The constant electric field

As we have mentioned before, the solutions for the constant electric field case have already been found by many authors for the Lorentz-Dirac equation and the Landau-Lifshitz equation. For both equations, the motion of the charged particle is not affected by the reaction force, which vanishes. This has led to a series of discussions of the interpretation of the principle of equivalence [3] and of the role of the attached fields [9]. Due to this reason, a new proposal has been done by Hammond [2,3,4,1]. However, in order to apply our new representation of the Landau-Lifshitz equation, we want to discuss the simple case of one charged particle in a constant electric field whose intensity is given by E. Let us constrain the motion and the field to the x axis. We begin by solving the Lorentz equation for this case. We obtain the two Lorentz coordinate equations,

$$a_L^0 = \frac{d^2 ct}{d\tau_L^2} = wE\frac{d}{d\tau_L}x$$
$$a_L^x = \frac{d^2}{d\tau_L^2}x = wEc\frac{d}{d\tau_L}t, \tag{15}$$

By using the last result, a_L^2 is

$$a_L^2 = \left[c\frac{d^2}{d\tau_L^2}t\right]^2 - \left[\frac{d^2}{d\tau_L^2}x\right]^2 = -w^2E^2c^2. \tag{16}$$

Then,

$$\frac{a_L^2}{c^2} = -w^2E^2. \tag{17}$$

On the other hand, noticing that in order to calculate $d^2x/d\tau_L^2$ and $d^2ct/d\tau_L^2$, Eqs. (15) must be used, we have

$$\frac{d}{d\tau_L}a_L^0 = wE\frac{d^2}{d\tau_L^2}x = w^2E^2c\dot{t}$$
$$\frac{d}{d\tau_L}a_L^x = wE\frac{d^2}{d\tau_L^2}ct = w^2E^2\dot{x}, \tag{18}$$

where it is necessary to remember that the dot "$\cdot$" represents the real proper time derivative ($d\,/d\tau$) and by definition $dx^\mu/d\tau_L$ coincides with the 4-velocity v^μ at each point of the real trajectory. Therefore, we obtain

$$\tau_o\left[\frac{da_L^\mu}{d\tau_L} + \frac{a_L^2}{c^2}v^\mu\right] = 0. \tag{19}$$

This means that the reaction term vanishes in this case. Therefore, when a constant electric field parallel to the motion of a charge is applied, the solution of Landau-Lifshitz equation coincides with the Lorentz one. It has to be noted that for Lorentz-Dirac equation, the Lorentz-Dirac reaction term also vanishes in this case [26,27].

2.3. The constant magnetic field

In order to also show the simplicity of the method, let us consider a constant magnetic field B in the z axis. Without loosing generality, we can constrain the problem to two dimensions, and by writing $w = qB/cm$, the three components of the Lorentz equation of motion are

$$\frac{d^2}{d\tau_L^2}ct = 0 \qquad \frac{d^2}{d\tau_L^2}x = w\dot{y}$$
$$\frac{d^2}{d\tau_L^2}y = -w\dot{x}. \tag{20}$$

Therefore,

$$a_L^2 = -w^2(\dot{x}^2 + \dot{y}^2). \tag{21}$$

So the Landau-Lifshitz equation of motion can be expressed as

$$c\ddot{t} = -\tau_o w^2\frac{1}{c^2}\left(\dot{x}^2 + \dot{y}^2\right)c\dot{t}$$
$$\ddot{x} = w\dot{y} - \tau_o w^2\dot{x}\left(1 + \frac{1}{c^2}\left(\dot{x}^2 + \dot{y}^2\right)\right),$$
$$\ddot{y} = -w\dot{x} - \tau_o w^2\dot{y}\left(1 + \frac{1}{c^2}\left(\dot{x}^2 + \dot{y}^2\right)\right). \tag{22}$$

As we can notice these equations do not depend on the position, x and y. Therefore, we can define the vectors,

$$\overrightarrow{v} = \dot{x}\widehat{i} + \dot{y}\widehat{j} = v(\cos\theta\widehat{i} + \sin\theta\widehat{j}),$$
$$\widehat{e}_v = \left|\frac{\partial\overrightarrow{v}}{\partial v}\right|^{-1}\frac{\partial\overrightarrow{v}}{\partial v} = \cos\theta\widehat{i} + \sin\theta\widehat{j}, \tag{23}$$
$$\widehat{e}_\theta = \left|\frac{\partial\overrightarrow{v}}{\partial\theta}\right|^{-1}\frac{\partial\overrightarrow{v}}{\partial\theta} = -\sin\theta\widehat{i} + \cos\theta\widehat{j}.,$$

where v represents the magnitude of the relativistic velocity, that is: $v = (\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}}$, and θ is the angle of the vector $\overrightarrow{v}$ with the x axis. Therefore, Eq. (22) can be written as

$$\left[\dot{v}\right]\widehat{e}_v + \left[v\dot{\theta}\right]\widehat{e}_\theta = -\tau_o w^2 v\left(1 + \frac{v^2}{c^2}\right)\widehat{e}_v - wv\widehat{e}_\theta. \tag{24}$$

Consequently, we can assure that

$$\dot{\theta} = -w = \text{constant}$$
$$\frac{1}{v\left(1 + \frac{v^2}{c^2}\right)}dv = -\tau_o w^2 d\tau. \tag{25}$$

By integrating the last equation, we arrive at

$$v = \frac{v_o c \exp -\tau_o w^2 \tau}{\left[c^2 + v_o^2 \left[1 - \exp -2\tau_o w^2 \tau\right]\right]^{\frac{1}{2}}}. \tag{26}$$

It is observed that similar results for the Lorentz-Dirac equation have been found [24,28].

Finally, the speed of the charge is

$$\overrightarrow{v} = \frac{v_o c \exp -\tau_o w^2 \tau}{\left[c^2 + v_o^2 \left[1 - \exp -2\tau_o w^2 \tau\right]\right]^{\frac{1}{2}}} \times \left[\cos(w\tau + \delta)\widehat{i} + \sin(w\tau + \delta)\widehat{j}\right], \tag{27}$$

where v_o and δ represent the initial speed of the particle (with respect to the proper time) and the phase of the trigonometric functions at $\tau = 0$, respectively. It has to be noted that this last solution implies that a drift of the center of motion of the charged particle appears [24]. Moreover, in a typical TOKAMAK environment, the magnetic field is around $10^5 G$ ($10T$) [29], the decay time is around 10^{-1} sec for the electrons ($t_{ed} \simeq 1/\tau_{eo} w^2$, with $\tau_{eo} = 2e^2/3m_e c^3$ and m_e the electron mass) and the electrons will lose all the energy. Consequently, the confinement of the plasma could be affected due to the fact that the electron energy will decay faster than the energy of a proton (or a similar ion) around 10^9 sec ($t_{pd} \simeq 1/\tau_{po} w^2$ with $\tau_{po} = 2e^2/3m_p c^3$ and m_p the proton mass).

3. Deduction of the Non Relativistic Landau-Lifshitz Equation

If we consider the reaction term in the new form of the Landau-Lifshitz equation,

$$\tau_o \left[\frac{da_L^\mu}{d\tau_L} + \frac{a_L^2}{c^2} v^\mu\right]. \tag{28}$$

First, it has to be noted that the Lorentz-Dirac reaction term transforms as $\tau_o d\mathbf{a}/dt$, when it is considered the non relativistic case [30]. Then, it is clear that the Landau-Lifshitz reaction term in the non-relativistic case can be expressed as

$$\tau_o \frac{d\mathbf{a}_L}{dt_L} = \frac{\tau_o}{m} \frac{d\mathbf{F}}{dt_L}, \tag{29}$$

where t_L represents the Lorentz time, and therefore,

$$\frac{dv^i}{dt_L} = \frac{F^i}{m}. \tag{30}$$

Here the Lorentz time has a similar role to the Lorentz proper time, but in the non relativistic case. This gives the following non relativistic equation for a charged particle,

$$m\mathbf{a} = \mathbf{F} + \tau_o \frac{d\mathbf{F}}{dt_L}. \tag{31}$$

This equation can be called the non-relativistic Landau-Lifshitz equation. Apparently, this is the Ford equation. However, this is not true due the presence of the Lorentz-time t_L. Indeed, when the force only depends on the time and the coordinates, or in other words $\mathbf{F} \neq \mathbf{F}(\mathbf{v})$, and considering that $dx^i/dt_L = dx^i/dt$, we have

$$\frac{d\mathbf{F}}{dt_L} = \frac{\partial \mathbf{F}}{\partial x^i}\frac{dx^i}{dt_L} + \frac{\partial \mathbf{F}}{\partial t} = \frac{\partial \mathbf{F}}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial \mathbf{F}}{\partial t} = \frac{d\mathbf{F}}{dt}. \tag{32}$$

In this case Eq. (31) coincides with Ford equation. It has to be noted that in their first articles, Ford and O'Connell just considered forces depending on the time [16], but later they generalized their proposal to the relativistic case obtaining the Eliezer equation of motion [18], where no constraint about the force dependence appears [17]. Therefore, the generalization of Ford equation including forces depending on the coordinates is natural. However, when the force depends on the velocity, things are different. For such a dependence in the velocity, the non relativistic Landau-Lifshitz equation, Eq. (31), must be expressed as

$$m\mathbf{a} = \mathbf{F} + \tau_o \frac{d\mathbf{F}}{dt_L} = \mathbf{F} + \tau_o \left[\frac{\partial \mathbf{F}}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial \mathbf{F}}{\partial t} + \frac{\partial \mathbf{F}}{\partial v^i}\frac{dv^i}{dt_L}\right]. \tag{33}$$

Using Eq. (30), it is obtained:

$$m\mathbf{a} = \mathbf{F} + \tau_o \left[\frac{\partial \mathbf{F}}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial \mathbf{F}}{\partial t} + \frac{\partial \mathbf{F}}{\partial v^i}\frac{F^i}{m}\right]. \tag{34}$$

On the other side, Ford equation will be represented by:

$$m\mathbf{a} = \mathbf{F} + \frac{\tau_o}{m} \left[\frac{\partial \mathbf{F}}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial \mathbf{F}}{\partial t} + \frac{\partial \mathbf{F}}{\partial v^i}\frac{dv^i}{dt}\right]. \tag{35}$$

Therefore, the difference between both equations consists in noticing that

$$\frac{\partial \mathbf{F}}{\partial v^i}\frac{dv^i}{dt_L} = \frac{\partial \mathbf{F}}{\partial v^i}\frac{F^i}{m} \neq \frac{\partial \mathbf{F}}{\partial v^i}\frac{dv^i}{dt}, \tag{36}$$

because

$$\frac{dv^i}{dt_L} = \frac{F^i}{m} \neq \frac{dv^i}{dt} = a^i. \tag{37}$$

The reason of the difference is that the acceleration of the trajectories are driven by the Landau-Lifshitz equation or by Ford equation, depending on which equation has been chosen, and not by the Lorentz equation. In the following section, some examples of this nature will be exposed.

4. Ford Equation vs Non Relativistic Landau-Lifshitz Equation in Typical Cases

In this section the difference between the non-relativistic Landau-Lifshitz equation and Ford equation is going to be analyzed through a pair of typical physical situations. Specifically, the cases of a particle under a constant electric field or a constant magnetic field are studied.

4.1. Ford equation with a constant electric field

Let us consider a non relativistic charged particle submitted to a constant electric field in the $x-$axis, which motion is described by Ford equation.

$$m\mathbf{a} = qE\widehat{i} + \tau_o \frac{d\,(qE)}{dt}\widehat{i}. \tag{38}$$

Since E is constant, the solution is the regular constant acceleration that comes from the corresponding Lorentz equation,

$$m\mathbf{a} = qE\widehat{i}. \tag{39}$$

Even if the reaction force vanishes there is some radiated power, as we will see in the next section.

On the other hand, if it is considered that the equation of motion is the non relativistic Landau-Lifshitz equation, then:

$$m\mathbf{a} = qE\widehat{i} + \tau_o \frac{d\,(qE)}{dt_L}\widehat{i}. \tag{40}$$

Since the Lorentz force is constant and consequently it does not depend on the velocity, Eq. (40) also coincides in this case with the Lorentz equation, as it happens when Ford equation is used. Therefore, as we have predicted, in this case since the force does not depend on the velocity, then the solution coincides for both equations, Ford equation and the non relativistic Landau-Lifshitz equation. In fact, the solution coincides with four equations,because it is also solution for the Lorentz and the Lorentz-Abraham equations.

4.2. Ford equation with a constant magnetic field

Consider now a non relativistic charged particle submitted to a constant magnetic field, $\mathbf{B} = B\widehat{k}$. By using Ford equation, we obtain:

$$m\ddot{x} = w\dot{y} + \tau_o w \frac{d\dot{y}}{dt} \qquad \text{and} \qquad \ddot{y} = -w\dot{x} - \tau_o w \frac{d\dot{x}}{dt}. \tag{41}$$

It has to be observed that $\ddot{y} = d\dot{y}/dt$ cannot be identified with $-w\dot{x}$, since this would imply that the particle's motion will be Lorentz-like. Then, Eq. (41) can be solved by putting

$$\dot{x} = a \exp -iw_m t \qquad \text{and} \qquad \dot{y} = b \exp -iw_m t. \tag{42}$$

Therefore, a secular equation is obtained, from where it is obtained:

$$w_m^2 - w^2(1 - i\tau_o w_m)^2 = 0. \tag{43}$$

Finally, the motion can be described by

$$\begin{aligned}\dot{x} &= A \exp\left[-\tau_o \frac{w^2}{1+\tau_o^2 w^2} t\right] \sin\left[\frac{w}{1+\tau_o^2 w^2} t + \delta\right]\\ \dot{y} &= A \exp\left[-\tau_o \frac{w^2}{1+\tau_o^2 w^2} t\right] \cos\left[\frac{w}{1+\tau_o^2 w^2} t - \delta\right],\end{aligned} \tag{44}$$

where A and δ are real constants determined by the initial conditions. The interesting fact is that the oscillating term has a frequency given by

$$\Omega_F = \frac{w}{1+\tau_o^2 w^2} \tag{45}$$

and a decay time of the form:

$$t_{Fd} = \frac{1+\tau_o^2 w^2}{\tau_o w^2} \tag{46}$$

If we analyze the expression of the relativistic case using Landau-Lifshitz equation, Eq. (27), and we take the non relativistic limit, the frequency and the decay time are written as:

$$\Omega_{LL} = w \tag{47}$$

and

$$t_{LLd} = \frac{1}{\tau_o w^2} \tag{48}$$

which differ by the factor $\tau_o^2 w^2$.

Our principal purpose is to compare Ford equation with the non relativistic Landau-Lifshitz equation. In order to confirm our solution, let us solve Eq. (34) and check that it coincides with the result of Eqs. (47) and (48). The non relativistic Landau-Lifshitz equation in this case is

$$\begin{aligned}m\ddot{x} &= w\dot{y} + \tau_o w \frac{d\dot{y}}{dt_L} \qquad \text{and}\\ \ddot{y} &= -w\dot{x} - \tau_o w \frac{d\dot{x}}{dt_L}.\end{aligned} \tag{49}$$

It has to be noted that in this case $d\dot{x}/dt_L$ and $d\dot{y}/dt_L$ must be replaced by the Lorentz force, that is:

$$\frac{d\dot{x}}{dt_L} = w\dot{y} \qquad \text{and} \qquad \frac{d\dot{y}}{dt_L} = -w\dot{x} \tag{50}$$

Therefore,

$$m\ddot{x} = w\dot{y} - \tau_o w^2 \dot{x} \quad \text{and} \quad m\ddot{y} = w\dot{x} - \tau_o w^2 \dot{y} \tag{51}$$

From where a secular equation is obtained:

$$w_m^2 + 2iw_m \tau_o w^2 + w^2(1 - \tau_o^2 w^2) = 0, \tag{52}$$

which implies

$$w_m = w + i\tau_o w^2 \tag{53}$$

Thus, the solution is of the form

$$\begin{aligned}\dot{x} &= A \exp\left[-\tau_o w^2 t\right] \cos\left(wt + \delta\right)\\ \dot{y} &= A \exp\left[-\tau_o w^2 t\right] \sin\left(wt - \delta\right),\end{aligned} \tag{54}$$

where A and δ are also constants given by the initial conditions. The frequency and the decay time coincide, as we expected, with Eqs. (47) and (48).

Finally, it has been highlighted that the Ford solution and the non relativistic Landau-Lifshitz solution differ as it is proved in Eq. (36).

5. Equivalence between Eliezer and Ford-O'Connell Equation

In 1948, by making an approximation of the Lorentz-Dirac equation, Eliezer [18] deduced a second order equation which does not present unphysical solutions. On the other hand, by using a generalized quantum Langevin equation, and giving a structure to the electron with a factor form and a finite cut-off parameter, Ford and O'Connell obtained a non relativistic equation, the so-called Ford equation. Also, by generalizing Ford equation to Special Relativity a relativistic equation was deduced, the Ford-O'Connell equation [17,16]. Albeit Ford and O'Connell obtained the same equation as Eliezer, they claimed that their deduction was much physical because it is based on quantum principles and not in an approximation. The reason why Eliezer equation and Ford-O'Oconnell equation are frequently not recognized as equivalent is because they are sometimes differently expressed. Let us proof that both equation are the same starting by using Parrot [19] expression of the Eliezer equation,

$$ma^\mu = \frac{q}{c}F^{\mu\nu}v_\nu + \tau_o\left[\frac{q}{c}\frac{d\left(F^{\mu\nu}v_\nu\right)}{d\tau} + \frac{q}{c}\left(F^{\nu\lambda}v_\lambda a_\nu\right)\frac{v^\mu}{c^2}\right]. \tag{55}$$

If we develop the second part of the radiation term, $\left(q/c^3\right)\left(F^{\nu\lambda}v_\lambda a_\nu\right)v^\mu$, and noticing that $v_\nu v_\lambda dF^{\nu\lambda}/d\tau$ vanishes due to the antisymmetry of the Strength Tensor, $F^{\nu\lambda}$, we obtain:

$$\begin{aligned}\frac{q}{c}\left(F^{\nu\lambda}v_\lambda a_\nu\right)\frac{v^\mu}{c^2} &= \frac{q}{c}\left(F^{\nu\lambda}v_\lambda a_\nu\right)\frac{v^\mu}{c^2} - \frac{v^\mu}{c^2}\left[v_\nu\frac{dF^{\nu\lambda}}{d\tau}v_\lambda\right]\\ &= \frac{q}{c}\left(F_{\nu\lambda}v^\lambda a^\nu\right)\frac{v^\mu}{c^2} - \frac{q}{c}\frac{v^\mu}{c^2}\left[v^\nu\frac{dF_{\nu\lambda}}{d\tau}v^\lambda\right]\\ &= -\frac{q}{c^3}v^\mu v^\nu\frac{d\left(F_{\nu\lambda}v^\lambda\right)}{d\tau}.\end{aligned} \tag{56}$$

Therefore, we arrive at

$$ma^\mu = \frac{q}{c}F^{\mu\nu}v_\nu + \tau_o\left[\frac{q}{c}\frac{d\left(F^{\mu\nu}v_\nu\right)}{d\tau} - \frac{q}{c}\frac{v^\mu v^\nu}{c^2}\frac{d\left(F_{v\lambda}v^\lambda\right)}{d\tau}\right], \tag{57}$$

which represents the Ford-O'Connell equation. Therefore, since both representations of the same equation were deduced by using different physical arguments, the equation may be called the Eliezer-Ford-O'Connell equation.

By using the same method of Subsec. 2.1, we can express the Eliezer-Ford-O'Connell equation as

$$a^\mu = a_L^\mu + \tau_o\left[\frac{da_L^\mu}{d\tau} - \frac{v^\mu v^\nu}{c^2}\frac{da_\nu^L}{d\tau}\right]. \tag{58}$$

Analyzing the second part of the radiation term, we have

$$\begin{aligned}v^\nu\frac{da_\nu^L}{d\tau} &= \frac{dv^\nu a_\nu^L}{d\tau} - a^\nu a_v^L\\ &= \frac{q}{cm}\frac{d\left(v^\nu F_{\nu\lambda}v^\lambda\right)}{d\tau} - a^\nu a_v^L = -a^\nu a_v^L,\end{aligned} \tag{59}$$

where the antisymmetry of $F_{\nu\lambda}$ has been used. Therefore, a better representation of the Eliezer-Ford-O'Connell equation is:

$$a^\mu = a_L^\mu + \tau_o\left[\frac{da_L^\mu}{d\tau} + \frac{v^\mu}{c^2}a^\nu a_\nu^L\right]. \tag{60}$$

This representation of the Eliezer-Ford-O'Connell equation is very similar to the expression of the Landau-Lifshitz equation in Eq. (14), but it differs in the following points: first, in Eq. (60) the derivative with respect to the Lorentz-proper time does not appear; secondly, a_L^2 in Eq. (14) is substituted by $a^\nu a_\nu^L$ in Eq. (60). These differences show that there is no equivalence between the Landau-Lifshitz and the Eliezer-Ford-O'Connell equations.

6. Expressions for the Radiated Power

Let us analyze the physical consequences of the reaction term in each equation. Some authors have proposed a new radiation rate of energy (or a radiated power expression) for the Landau-Lifshitz equation, [9,31,10,11] which differs from the classical relativistic Larmor formula. Indeed, when the Lorentz-Dirac equation is considered, the time coordinate of the reaction term corresponds to the radiated power. However, it is composed of two terms, as Rohrlich has interpreted [30], one corresponds to the attached fields, $\tau_o m\dot{a}^0$, which follows the charge, and other one that can be measured experimentally, the large distance radiated power $\tau_o m(a^2/c^2)v^o$. Following Rohrlich ideas and taking into account the $0-$component of the Landau-Lifshitz reaction term,

$$\begin{aligned}G^0 &= \tau_o\left[\frac{d}{d\tau_L}\frac{q}{c}F^{0\nu}v_\nu + \frac{q^2}{c^4 m}F^2 v^0\right]\\ &= \tau_o m\left[\frac{da_L^0}{d\tau_L} + \frac{a_L^2}{c^2}v^0\right],\end{aligned} \tag{61}$$

we can propose the large distance radiated power as:

$$P_{lar} = -\tau_o\frac{q^2}{c^3 m}F^2 v^0 = -\tau_o m\frac{a_L^2}{c}v^0, \tag{62}$$

and the attached radiated power like:

$$P_{att} = -\tau_o c\frac{d}{d\tau_L}\left[\frac{q}{c}F^{0\nu}v_\nu\right] = -\tau_o mc\frac{da_L^o}{d\tau_L} \tag{63}$$

The new expressions for the large distance and the attached radiated powers were introduced to show that there is a consistence between the radiated power and the Landau-Lifshitz equation [9,10]. When the constant electric field case is analyzed, since the reaction term vanishes, it can be thought that there is no radiation. Nevertheless, the large distance radiated power does not vanishes. In fact, the attached energy provides the energy to the large distance radiated power. This means that there is an arrangement of the energy [9]. The essential idea consists of proposing that the radiation emitted by a point charge is due exclusively to the external exerted electromagnetic forces on the charge.

On the other hand, based on energy conservation, Ford and O'Connell have proposed a radiated power for their equation [32], and it is expressed as

$$P = \tau_o \frac{F^2}{m}. \tag{64}$$

If we take the limit of Eq. (62) when $(v/c)^2$ tends to 0, the corresponding radiated power for the non relativistic Landau-Lifshitz case coincides with Ford proposal, Eq. (64). Therefore, even if Ford equation and the non relativistic Landau-Lifshitz equation are different, they have in common the radiated power. For example, in the case of the constant magnetic case, in the Ford case, the force must be considered as

$$F^2 = w^2 v^2. \tag{65}$$

For Landau-Lifshitz case,

$$F^2 = w^2 v^2. \tag{66}$$

In both cases, the expression is the same as it has been expected, but since the trajectories differ the power will be different. Indeed, the Ford power will be

$$P_F = \tau_o \frac{F^2}{m} = \tau_o \frac{A \exp\left[-2\tau_o \frac{w^2}{1+\tau_o^2 w^2} t\right]}{m}, \tag{67}$$

while the Landau-Lifshitz power will be

$$P_{LLF} = \tau_o \frac{F^2}{m} = \tau_o \frac{A \exp\left[-2\tau_o w^2 t\right]}{m}. \tag{68}$$

Even if both expression are not equal, they must be evaluated in physical situations. This will be done, among others remarks, in the following section. However, for the $0-$component of the Eliezer-Ford-O'Connell radiation term,

$$G^0_{EFO} = \tau_o m \left[\frac{da^0_L}{d\tau} + \frac{v^0}{c^2} a^\nu a^L_\nu\right], \tag{69}$$

we can propose the large distance radiated power as:

$$P^{EFO}_{lar} = -\tau_o m \frac{v^0}{c^2} a^\nu a^L_\nu, \tag{70}$$

and the attached radiated power like:

$$P^{EFO}_{att} = -\tau_o m \frac{da^0_L}{d\tau}. \tag{71}$$

It is obvious that for the relativistic case the difference between both proposals are bigger than in the non relativistic case. A fine experimental measurement of the large distance radiated power will select which one is the better equation. However, as we will see in Sec. 7, Concluding Remarks, quantum constraints prevent any comparison.

By reviewing Eq. (62), it is easy to deduce that the non relativistic case of the Landau-Lifshitz equation will lead us to a non relativistic radiated power equal to the expression described in Eq. (64), but looking at Eq. (70) the corresponding radiated power for Ford equation must be expressed as

$$P^F = \tau_o \mathbf{a} \cdot \mathbf{F}, \tag{72}$$

and not by using Eq. (26) as Ford claimed [32].

7. Concluding Remarks

It has been demonstrated that Ford equation coincides with the non-relativistic Landau-Lifshitz equation just in the case when the force acting on the particle does not depend on its the velocity. In the case of velocity dependent forces, the trajectories predicted by Ford equation or by the non relativistic Landau-Lifshitz equation are distinct. However, at the moment of measuring such differences, physical constraints prevent to detect them. Indeed, trying to measure the decay time, the frequency, the radiated power or the critical frequency of each solution, in the proportional differences $\Delta t_{\text{dec}}/t_{\text{dec}}$, $\Delta w/w_{LL}$, $\Delta P/P_{LL}$, $\Delta w_c/w_{cLL}$, the term $\tau_o^2 w^2$ makes the difference; that is:

$$\frac{\Delta w}{w_{LL}} = \frac{w_{LL} - w_F}{w_{LL}} = \frac{w - \dfrac{w}{1+\tau_o^2 w^2}}{w} = \frac{1}{1+\tau_o^2 w^2}.$$

In order to detect the difference, $\tau_o^2 w^2$ must be at least 10^{-3}. This identity also holds for relativistic motion. This implies that

$$\tau_o^2 w^2 \simeq 10^{-4}. \tag{73}$$

That is,

$$\tau_o w = \frac{2q^3 B}{3m^2 c^4} \simeq 10^{-2}. \tag{74}$$

Since for electrons $\tau_{eo} \simeq 6.26 \times 10^{-24}\,\text{sec}$, we have to deal with magnetic fields of the order of

$$B \simeq 10^{14} G \tag{75}$$

and for protons,

$$B \simeq 10^{20} G. \tag{76}$$

Nevertheless, in order to be able of dealing with a trajectory, quantum effects must be negligible. This is accomplished if the magnetic field satisfies the following requirements [24]:

1- De Broglie wavelength $\ll$ Characteristic length

$$\frac{B}{B_q} \ll \gamma^2 \tag{77}$$

with $B_q \simeq 4.4 \times 10^{13} G$

2- Radiation effects

$$\frac{B}{B_q} \ll \frac{1}{\gamma} \tag{78}$$

For the relativistic case the second identity outweighs, Eq. (78). This means that more energy requires less field. This is the reason why it is better to analyze the non relativistic case. Therefore, it is necessary to deal with magnetic fields smaller than B_q; that is:

$$B \ll 10^{13} G.$$

For protons the situation is worst and it will not be possible to experimentally differentiate both solutions. Even if in some astrophysical situations some magnetic fields have been found to be of the order of $10^{13} G$ (see for example, Camilo *et al* [33]), quantum effects will dominate the behavior of the charge and it will not make sense to analyze the trajectories.

Finally, the purpose of the article was to check that, even if the two equations were different to first order in τ_o, the large-scale temporal behavior of the solutions were able to differentiate between both of them. For high relativistic motion, the electric constant case, high intense planes waves, and crossed electric fields do not present such differences [21,22]. For central forces, the equations coincide even if the large-scale temporal effects are studied [15]. For low energy motion in electric or magnetic fields, quantum restrictions prohibit to find experimentally a difference. We can conclude Eliezer-Ford-O'Connell and Landau-Lifshitz equations are physically equivalent within the Shen's Zone [14,24].

Finally, after 100 hundred years of discussion about the correct equation of motion for a charged particle, the Landau-Lifshitz equation (or the equivalent Eliezer-Ford-O'Connell) represents the most acceptable approach. Even more, Quinn and Wald [34] proposed a general relativistic equation including the reaction force with the tail term which coincides with Landau-Lifshitz equation for Minkowski space. However, Hammond's method [1-4] which appeared in order to discuss the balance of energy of the electric constant case, represents an interesting alternative to be discussed. For the followers of Landau-Lifshitz equation, there exists a rearrangement of the radiated energy between the large distance radiation and the radiation term due to the attached fields. While for Hammond, the energy loss comes from Larmor formula and this leads to another representation of the equation. However, an equivalent method for Hammond proposal for the general relativistic case has to be improved and compared with Hobbs [35] and Quinn and Wald approaches [34,36].

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