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Equilibrium profiles of liquids in tilted Taylor-Hauksbee cells

A. Jara, A Medina
*Escuela Superior de Ingeniería Mecánica y Eléctrica Azcapotzalco, Instituto Politécnico Nacional
Av. de las Granjas 682, Col. Sta. Catarina Azcapotzalco, México D.F. 02250, México*

C.A. Vargas
*Laboratorio de Sistemas Complejos, Departamento de Ciencias Básicas,
Universidad Autónoma Metropolitana-Azcapotzalco,
Av. San Pablo 180, Col. Reynosa, 02200 México, D.F., México.*

F.J. Higuera
*Escuela Técnica Superior de Ingenieros Aeronáuticos, Universidad Politécnica de Madrid
Plaza Cardenal Cisneros 3, 28040 Madrid, Spain.*

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In this work we study theoretically and experimentally the equilibrium profiles attained under spontaneous capillary rise of viscous liquids in the wedge-shaped narrow gap between two vertical plates intersecting at a tight angle $\alpha \ll 1$. We contrast the differences among the case with vertical edge and those in which the arista is tilted to the vertical. Our theoretical description agrees well with experimental data for several analyzed inclinations.

Keywords: Capillary flows; hydrostatics; flow visualization and imaging.

En este trabajo estudiamos teórica y experimentalmente los perfiles de equilibrio alcanzados en virtud de ascenso capilar espontáneo de líquidos viscosos en el pequeño espacio en forma de cuña creado entre dos placas verticales que se unen en un ángulo estrecho $\alpha \ll 1$. Contrastamos las diferencias entre el caso con la arista vertical y aquellos en los que la arista está inclinada respecto a la vertical. Nuestra descripción teórica concuerda bien con los datos experimentales para varias inclinaciones analizadas.

Descriptores: Flujos capilares; hidrostática; imágenes de flujo.

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1. Introduction

The problem of the spontaneous capillary rise of a liquid within a vertical cell made by two plates touching on an edge and having a very tight angle among them (wedge) was initially studied experimentally by Taylor [1] and Hauksbee [2] three centuries ago to show the existence of the capillary driven force; as well as the formation of equilibrium profiles. In both works, the reported equilibrium profiles were rectangular hyperbolas with a very important peculiarity; near the edge (arista), liquid tends to reach an infinite height, which in nature is very suitable to supply water and sap (which contains the nutrients) to the upper leaves of tall trees. In practice, the formation of the equilibrium capillary profiles has also been used, for instance, as a method to measure the surface tension accurately [3] and in miniature grooved heat pipes and heat spreaders [4]. Additionally, the study of the equilibrium profiles from the point of view of the applied mathematicians, has allowed them to find interesting theoretical criteria for the occurrence and stability of the profiles in systems with tight or wide angles of aperture [5–7].

Despite the interest and utility of the Taylor-Hauksbee configuration, the studies of other variants of this problem are scarce [8, 9]. In this work, we report a series of experiments where the wedge's arista is tilted with respect to the vertical. This change originates new shapes of the equilibrium profiles with respect to the vertical ones. Here, we are interested in finding the equilibrium shapes of the free surfaces, theoretically and experimentally. To reach such a goal, this work is divided as follows: Firstly we revisit the problem of the equilibrium profiles in the Taylor-Hauksbee cell. Later on, we propose a theoretical description of the equilibrium profiles in cells with a tilted edge. In Sec. 3, experiments yield that our description is a suitable way to trace such profiles. Finally, in Sec. 4, we present the main conclusions for this work.

2. Equilibrium profiles in the Taylor-Hauksbee cell

2.1. Cell with a vertical edge

The wedge-shaped gap between two vertical plates intersecting at an angle $\alpha \ll 1$ is initially empty. At a certain instant, the lower edges of the plates get in touch with a liquid of density ρ, dynamic viscosity μ, and surface tension σ. The liquid wets the plates with a contact angle $\theta < \pi/2$ and therefore rises between the plates by capillary action up to reach the equilibrium, as shown in Fig. 1(a). The ratio of the two principal curvatures of the free surface of the liquid between the plates is small, of the order of α. The normal section of maximum curvature, by a plane nearly normal to the plates, is approximately an arc of circle of radius $\alpha x/2\cos\theta$, where x is the distance to the line of intersection of the plates. By

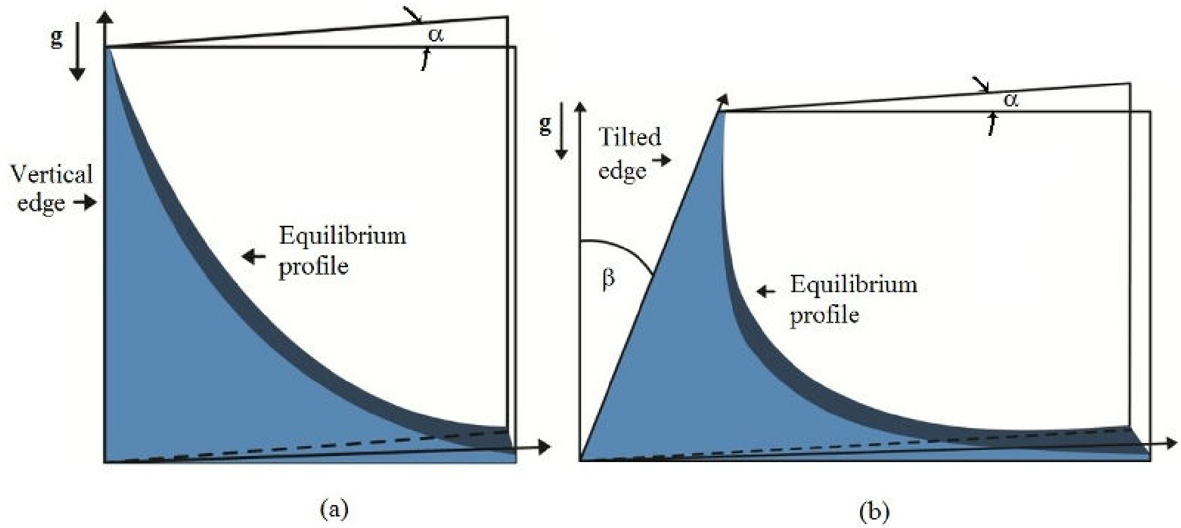

FIGURE 1. Scheme of the equilibrium profiles in the Taylor-Hauksbee cell: (a) cell with the vertical arista and (b) cell with a tilted arista.

using the Laplace equation [10], the pressure jump across the surface is approximately

$$\Delta p = \frac{2\sigma\cos\theta}{\alpha x}. \tag{1}$$

At equilibrium, the height $H_e(x)$ of the meniscus above the level of the outer liquid is determined by the balance

$$\Delta p = \rho g H_e, \tag{2}$$

where g is the acceleration due to gravity. This balance gives the rectangular hyperbola

$$H_e = \frac{2\sigma\cos\theta}{\rho g \alpha x}. \tag{3}$$

which is a well known result in literature [3].

2.2. Cell with a tilted edge

Now, we consider the cases of cells with tilted edges with respect to the vertical, see Fig. 1(b). The usual Cartesian coordinates (x, y) appears no suitable to depict the equilibrium profiles because there are two possible values of $H(x)$ for some values of x. Thus, the concept of injective function can be lost.

Instead, we choose the rotated coordinate system (X, Y) to describe the equilibrium profile. We analyze the problem of the equilibrium profile for the point (s, Y_s) shown in Fig. 2. s is the distance from the axis y to any point $x = s$, whereas Y_s is the distance to the equilibrium profile along the line parallel to the tilted edge (Y) and that starts at the point $X = s\cos\beta$ and $Y = s\sin\beta$. Thus the pressure jump across the surface at Y_s is approximately

$$\Delta p = \frac{2\sigma\cos\theta}{\alpha s\cos\beta}, \tag{4}$$

meanwhile the equilibrium profile is determined by the balance

$$\Delta p = \rho g\cos\beta Y_s. \tag{5}$$

This balance gives the hyperbola

$$Y_s = \frac{2\sigma\cos\theta}{\rho g\cos^2\beta\alpha s}, \tag{6}$$

Another way to get the profile is by using the vertical height $H_s(s)$ in the (x, y) system. Therefore, the balance equation yields that

$$\rho g H_s = \frac{2\sigma\cos\theta}{\alpha(s - Y_s\sin\beta)}. \tag{7}$$

here we shall be cautious because for some values of s there could be two values of H_s while for others none. Consequently, it is better to compute the function $s(H_s)$ which can

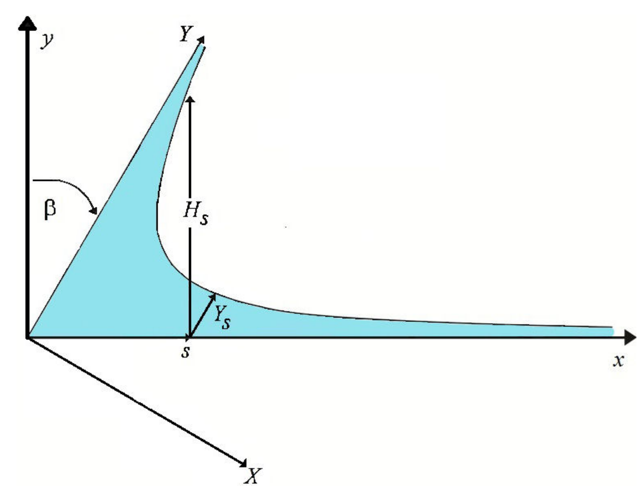

FIGURE 2. Schematic of the equilibrium profile described from the coordinate system (x, y) or from the rotated (X, Y) system. The rotation angle is β. In this case Y_s is the distance along the line parallel to the tilted edge and that starts at $x = s$ and $y = 0$.

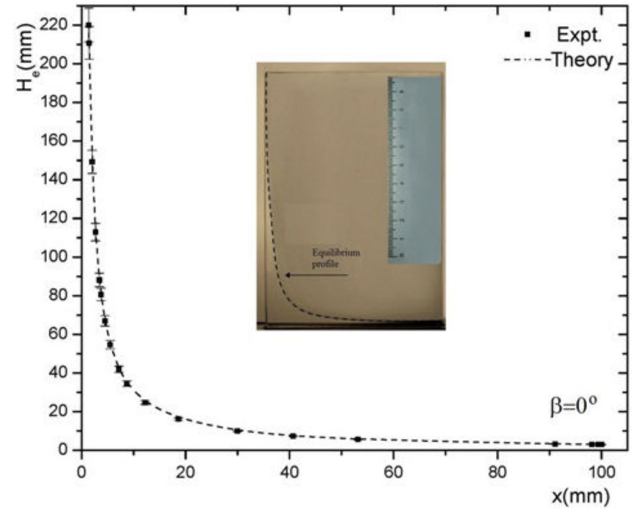

FIGURE 3. Plot of the theoretical hyperbola (dashed lines) built with Eq. (3) and the experimental data (symbols). The inset depicts a picture of a typical equilibrium profile (visible numbers are in cm scale). The error bars show a 5% error margin.

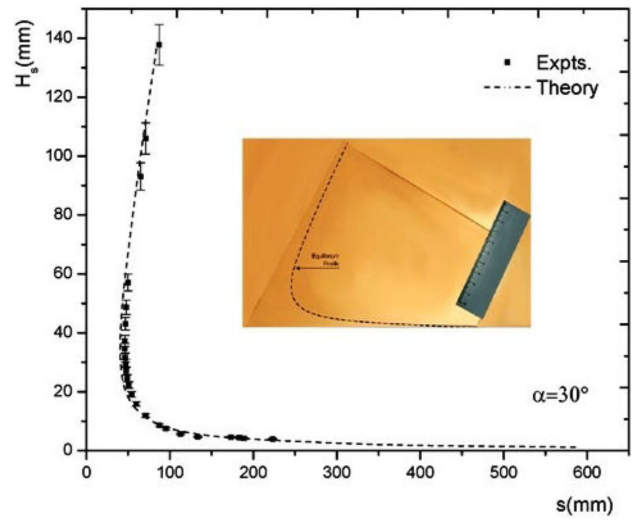

FIGURE 4. Plot of the theoretical profile (dashed lines) build with Eq. (8) and the experimental data (symbols). The picture on the inset is typical of the experimental equilibrium profiles for $\beta = 0.523$ rad (30°) and $\alpha = 0.0092$ rad (visible numbers are in cm scale). The error bars show a 5% error margin.

be found by using Eq. (7) and the relation $H_s = Y_s \cos\beta$, it results

$$s = \frac{2\sigma\cos\theta}{\rho g \alpha H_s \cos\beta} + H_s \tan\beta. \tag{8}$$

In summary, the equilibrium profile can be plotted by using Eq. (6) or Eq. (8).

3. Experiments

In order to compare the theoretical profiles with those obtained experimentally, we performed a series of experiments by using silicone oil, as working fluid, of viscosity

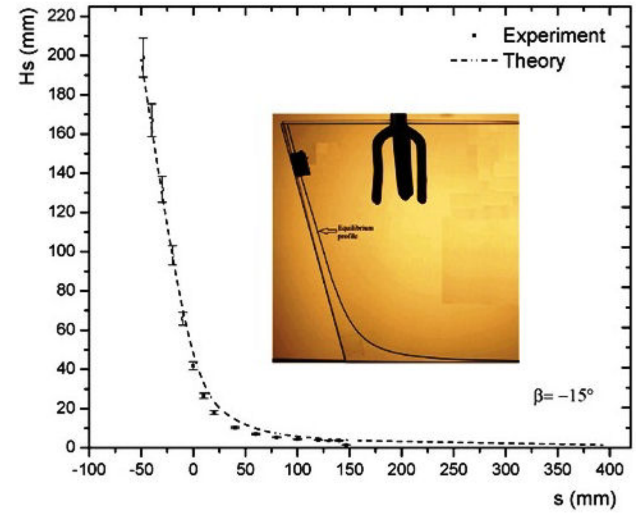

FIGURE 5. Plot of the theoretical profile (dashed lines) built with Eq. (8) and the experimental data (symbols). In the inset the picture shown is typical of the experimental equilibrium profile for $\beta = -0.261$ rad (-15°) and $\alpha = 0.0083$ rad (black objects are fasteners). The error bars show a 5% error margin.

$\mu = 100$ cP, surface tension $\sigma = 0.0215$ N/m and density $\rho = 971$ Kg/m^3. *Ad hoc* cells having different inclinations of the arista were made with flat glass plates (3 mm thickness) of different sizes. In our experiments we took measurements of the contact angle of the silicone oil over the flat glass by using the static sessile drop method. The angle average that we found was $\theta = 0.122 \pm 0.006$ rad (7°) at a room temperature $T_{\text{room}} = 296.15$ K. Wedges with tight angles were made by keeping the plates in contact along the arista and by introducing, among the glass plates, a metallic slim sheet, parallel to the arista and at a short distance from it. Special care was taken to avoid any contact of the sheet with the silicone oil during the profile formation. Each angle α, of a given cell, was measured by using a vernier to measure of the narrow space among the plates. On the contrary, the tilted angle β it was measured by using a inclinometer

In our experiments, we used a digital Cannon Réflex T3i camera to take pictures of the capillary profiles. The equilibrium profiles were obtained, in all cases, three hours after the lower edges of the plates were gently introduced in the silicone oil and later there wasn't any change observed in the profiles. The experimental data given in each plot were obtained through the average of four independent experiments. Thus, Fig. 3 shows the theoretical profile plot of liquid in a vertical cell (dashed lines) given by Eq. (3) and the experimental data (symbols); the picture on the inset is representative of a given experimental profile. In this case the fit is very good, here the aperture angle was of $\alpha = 0.0166 \pm 0.0009$ rad (0.95°) .

In order to show the accurateness of the model made to predict the equilibrium profiles in tilted cells, in Fig. 4 we show the plot of the profile in a cell having $\beta = 0.523 \pm 0.003$ rad (30°) and $\alpha = 0.0092 \pm 0.0005$ rad (0.52°). This profile (dashed lines) was obtained by using Eq. (8) and symbols correspond to experimental data. In Fig. 5 it is given the plot of the theoretical profile (dashed lines) for a negative tilted angle $\beta = -0.261 \pm 0.003$ rad $(-15°)$ and $\alpha = 0.0083 \pm 0.0004$ rad (0.47°), symbols correspond to data and a typical experimental profile is given in the inset.

After these comparisons, we can establish that due to the good agreement among the shown theoretical profiles and the experimental data, it is correct to affirm that Eq. (8) describes the hyperbolas in tilted Taylor-Haksbee cells.

4. Conclusions

In this work we have analyzed the problem of the algebraic shapes of the equilibrium profiles of viscous liquids in vertical and tilted Taylor-Hauskbee cells. We have proposed a simple model to describe the actual equilibrium profiles. The experiments allowed us to show that the model predicts pretty well the profiles in such cells. Moreover, the profiles of the current work resemble those occurring locally in periodically corrugated parallel plates [11]. By the way, our results can be useful to extend the computation of the effective contact angle in vertical plates with roughness because vertical structures, like micro-v grooves, may simulate this roughness. For vertical plates such a roughness increases the effective contact angle [9]. Now, following such a model, roughness on tilted plates can produce larger or smaller effective contact angles depending on the tilted angle being positive or negative. Experimental work along this line is now in progress.

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1. B. Taylor, *Philos. Trans. R. Soc. London* **27** (1712) 538.
2. F. Hauksbee, *Philos. Trans. R. Soc. London* **27** (1712) 539.
3. A. Ferguson, I. Voguel, *Proc. Phys. Soc. London* **38** (1925) 193.
4. D. Deng, Y. Tang, J Zeng, S Yang, H Shao, *Int. J. Heat Mass Transfer* **77** (2014) 311.
5. P. Concus, R. Finn, *Proc. Natl. Acad. Sci. U.S.A.* **63** (1969) 292.
6. R. Finn, *Equilibrium Capillary Surfaces*, Springer-Verlag, (Berlin, 1986).
7. C. F. Scott, G. C. Sander, and J. Norbuty, *Q.J. Mech. Appl. Math.* **60** (2004) 1.
8. W. J. O'Brien, R. G. Craig and A. Peyton, *J. Colloid Interface Sci.* **26** (1968) 500.
9. M. L. Anderson, A. P. Bassom, N. Fowkes, *Proc. R. Soc. A* **462** (2006) 3645.
10. P.-G. de Gennes, F. Brochard-Wyart, and D. Quéré, *Capillarity and Wetting Phenomena: Drops, Bubbles, Pearls, Waves*, Springer, (Berlin, 2004).
11. G. J. Gutierrez, A. Medina, F. J. Higuera, *J. Colloid Interface Sci.* **338** (2009) 519.