

Solution of the Schrödinger equation making use of time-dependent constants of motion

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It is shown that if a complete set of mutually commuting operators is formed by constants of motion, then, up to a factor that only depends on the time, each common eigenfunction of such operators is a solution of the Schrödinger equation. In particular, the operators representing the initial values of the Cartesian coordinates of a particle are constants of motion that commute with each other and from their common eigenfunction one readily obtains the Green function.

Keywords: Constants of motion; Schrödinger equation; Green's functions.

Se muestra que si un conjunto completo de operadores que conmutan entre sí está formado por constantes de movimiento, entonces, salvo un factor que solo depende del tiempo, cada eigenfunción común de tales operadores es una solución de la ecuación de Schrödinger. En particular, los operadores que representan los valores iniciales de las coordenadas cartesianas de una partícula son constantes de movimiento que conmutan entre sí y de sus eigenfunciones comunes uno obtiene fácilmente la función de Green.

Descriptores: Constantes de movimiento; ecuación de Schrödinger; funciones de Green.

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1. Introduction

In the standard procedure to solve the Schrödinger equation for a time-independent Hamiltonian one makes use of the method of separation of variables, or one starts looking for a set of mutually commuting operators that also commute with the Hamiltonian. In fact, the separable solutions of the Schrödinger equation are common eigenfunctions of a set of mutually commuting operators, with the separation constants being the eigenvalues of such operators. Usually, these operators do not depend explicitly on the time and, since they commute with the Hamiltonian, are constants of motion.

As we shall show below, there is no reason to restrict ourselves to time-independent operators; we can find solutions of the Schrödinger equation that are common eigenfunctions of a set of, possibly time-dependent, mutually commuting operators that are constants of motion (and, therefore, may not commute with the Hamiltonian). This method is analogous to that given by the Liouville theorem on the solutions of the Hamilton–Jacobi equation (see, *e.g.*, Ref. [1]).

In Sec. 2 we establish the basic results of this paper and in Sec. 3 we present several examples, exposing the advantages of the method. We show that making use of the operators that represent the initial position of a particle one readily obtains the Green function of the corresponding Hamiltonian.

2. Basic results

Let A be a, possibly time-dependent, Hermitean operator that is a constant of motion, *i.e.*,

$$i\hbar \frac{\partial A}{\partial t} + [A, H] = 0, \quad (1)$$

where H is the Hamiltonian of the system under consideration. If ψ is an eigenfunction of A with eigenvalue λ ,

$$A\psi = \lambda\psi \quad (2)$$

and ψ satisfies the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi, \quad (3)$$

then λ is a constant. Indeed, taking the derivative with respect to the time of both sides of (2), we obtain

$$\left(i\hbar \frac{\partial A}{\partial t}\right)\psi + A\left(i\hbar \frac{\partial \psi}{\partial t}\right) = i\hbar \frac{d\lambda}{dt}\psi + \lambda i\hbar \frac{\partial \psi}{\partial t}$$

and, making use of Eqs. (1) and (3),

$$-[A, H]\psi + AH\psi = i\hbar \frac{d\lambda}{dt}\psi + \lambda H\psi,$$

that is,

$$HA\psi = i\hbar \frac{d\lambda}{dt}\psi + \lambda H\psi$$

and using Eq. (2) again, it follows that $d\lambda/dt = 0$.

Conversely, assuming that A is a constant of motion and that its eigenvalues are constant, from (2) we have

$$\left(i\hbar \frac{\partial A}{\partial t}\right)\psi + A\left(i\hbar \frac{\partial \psi}{\partial t}\right) = \lambda i\hbar \frac{\partial \psi}{\partial t}$$

or, by virtue of (1),

$$-[A, H]\psi + A\left(i\hbar \frac{\partial \psi}{\partial t}\right) = \lambda i\hbar \frac{\partial \psi}{\partial t},$$

which amounts to

$$-AH\psi + H\lambda\psi + A\left(i\hbar \frac{\partial \psi}{\partial t}\right) = \lambda i\hbar \frac{\partial \psi}{\partial t},$$

or

$$A \left(i\hbar \frac{\partial \psi}{\partial t} - H\psi \right) = \lambda \left(i\hbar \frac{\partial \psi}{\partial t} - H\psi \right).$$

If the spectrum of A is non-degenerate, it follows that

$$i\hbar \frac{\partial \psi}{\partial t} - H\psi = \mu\psi,$$

where μ is some complex-valued function of t only. Then, if $F(t)$ is a solution of

$$\mu F + i\hbar \frac{dF}{dt} = 0, \tag{4}$$

we find that $F\psi$ satisfies the Schrödinger equation. In fact,

$$\begin{aligned} i\hbar \frac{\partial(F\psi)}{\partial t} &= F(H\psi + \mu\psi) + \left(i\hbar \frac{dF}{dt} \right) \psi \\ &= H(F\psi) + \left(\mu F + i\hbar \frac{dF}{dt} \right) \psi. \end{aligned}$$

(Note that, if ψ is an eigenfunction of A with eigenvalue λ , then, if $F(t)$ is a function of t only, $F(t)\psi$ is also an eigenfunction of A with eigenvalue λ .) When the spectrum is degenerate, the proof is similar, considering linear combinations of the eigenfunctions of A with the same eigenvalue λ and, in place of (4), one would have to solve a linear system of first-order differential equations, which may be as complicated as the original Schrödinger equation. However, if we find a complete set of mutually commuting operators that are constants of motion (complete in the sense that, up to a factor, there is only one common eigenvector of these operators for a given set of the eigenvalues), the common eigenfunctions of such a set of operators can be chosen in such a way that they are solutions of the Schrödinger equation (assuming that the eigenvalues are constant).

This result is analogous to the Liouville theorem of the Hamiltonian mechanics, according to which if we have a set of constants of motion such that their Poisson brackets are equal to zero, they can be used to find complete solutions of the Hamilton–Jacobi equation.

3. Examples

In this section we give several examples, finding solutions of the Schrödinger equation starting from the eigenfunctions of constants of motion that depend explicitly on the time.

3.1. The one-dimensional harmonic oscillator

One can readily verify that the operator

$$X_0 \equiv x \cos \omega t - \frac{p}{m\omega} \sin \omega t \tag{5}$$

is a constant of motion if the Hamiltonian is given by

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2. \tag{6}$$

The eigenvalue equation $X_0\psi = x_0\psi$ amounts to the *first-order* differential equation

$$\left(x \cos \omega t - \frac{\hbar \sin \omega t}{im\omega} \frac{\partial}{\partial x} \right) \psi = x_0\psi$$

whose solution is readily found to be

$$\psi = F(t) \exp \left[\frac{im\omega}{2\hbar \sin \omega t} (x^2 \cos \omega t - 2x_0x) \right], \tag{7}$$

where $F(t)$ is a function of t only.

Assuming that x_0 is a constant, one finds that the wavefunction (7) satisfies the Schrödinger equation, $i\hbar\partial\psi/\partial t = H\psi$, if and only if [cf. Eq. (4)]

$$\frac{d \ln F}{dt} = -\frac{\omega \cos \omega t}{2 \sin \omega t} + \frac{m\omega^2 x_0^2}{2i\hbar \sin^2 \omega t},$$

hence

$$F = \frac{C}{\sqrt{\sin \omega t}} \exp \left(\frac{im\omega x_0^2 \cos \omega t}{2\hbar \sin \omega t} \right),$$

where C is a constant, and substituting this expression into (7) we obtain

$$\begin{aligned} \psi_{x_0}(x, t) &= \frac{C}{\sqrt{\sin \omega t}} \\ &\times \exp \left\{ \frac{im\omega}{2\hbar \sin \omega t} [(x^2 + x_0^2) \cos \omega t - 2x_0x] \right\}. \end{aligned} \tag{8}$$

As usual, the wavefunctions ψ_{x_0} are orthogonal for different values of the eigenvalue x_0 . Note also that the functions (8) are not separable.

In the present example, H is time-independent and, therefore, there is a set of stationary states, $\phi_n(x) \exp(-iE_n t/\hbar)$, where the E_n are the eigenvalues of H . Hence, the wavefunction (8) must be expressible in the form

$$\psi_{x_0}(x, t) = \sum_n c_n(x_0) \phi_n(x) \exp \left(-\frac{iE_n t}{\hbar} \right)$$

and the symmetry of the right-hand side of (8) under the interchange of x and x_0 implies that

$$\psi_{x_0}(x, t) = \sum_n \phi_n(x_0) \phi_n(x) \exp \left(-\frac{iE_n t}{\hbar} \right)$$

i.e., with the appropriate value of the constant C , $\psi_{x_0}(x, t)$ is the time-dependent *Green function* of the one-dimensional harmonic oscillator (*cf.*, for instance, Refs. 2 to 4). This was to be expected since the operator X_0 is, in the Heisenberg picture, the position operator at $t = 0$ and, therefore, ψ_{x_0} corresponds to the state of the particle with a well defined value (equal to x_0) of the position at $t = 0$. The value of the normalization constant C can be determined by the condition

$$\lim_{t \rightarrow 0} \int_{-\infty}^{\infty} \psi_0(x, t) dx = 1, \tag{9}$$

which gives

$$C = \sqrt{\frac{m\omega}{2\pi i\hbar}}$$

As is well known [3], from the Green function (8) one can obtain the expression of the stationary states $\phi_n(x) \exp(-iE_n t/\hbar)$ in terms of the Hermite polynomials.

3.2. A time-dependent Hamiltonian

Now we shall consider the time-dependent Hamiltonian

$$H = \frac{p^2}{2m} - kt x, \tag{10}$$

where k is a constant. The operator

$$X_0 \equiv x - \frac{tp}{m} + \frac{kt^3}{3m} \tag{11}$$

is a constant of motion and, in the Heisenberg picture, corresponds to the position of the particle at $t = 0$. The eigenfunctions of this operator are determined by the first-order differential equation

$$\left(x - \frac{\hbar t}{im} \frac{\partial}{\partial x} + \frac{kt^3}{3m}\right) \psi = x_0 \psi,$$

where x_0 is the eigenvalue of X_0 . The solutions of this equation have the form

$$\psi = F(t) \exp \left[\frac{im}{\hbar t} \left(\frac{x^2}{2} - x_0 x + \frac{kt^3 x}{3m} \right) \right],$$

where $F(t)$ is a function of t only. Substituting this expression into the Schrödinger equation, assuming that x_0 is constant, we obtain the equation

$$\frac{d \ln F}{dt} = -\frac{1}{2t} + \frac{m}{2i\hbar t^2} \left(x_0^2 + \frac{4kt^3 x_0}{3m} + \frac{4k^2 t^6}{9m^2} \right),$$

thus,

$$\psi_{x_0}(x, t) = \frac{C}{\sqrt{t}} \times \exp \left\{ \frac{im}{2\hbar t} \left[(x-x_0)^2 + \frac{2kt^3(x-x_0)}{3m} - \frac{4k^2 t^6}{45m^2} \right] \right\}, \tag{12}$$

where C is a constant. With the appropriate value of C ($C = \sqrt{m/2\pi i\hbar}$), (12) is the Green function for the Hamiltonian (10).

It may be remarked that, since the Hamiltonian (10) depends explicitly on the time, the corresponding Schrödinger equation cannot be solved by separation of variables.

3.3. Another standard example

Another example usually considered in connection with the Green functions is that of a particle in a uniform field. If

$$H = \frac{p^2}{2m} - eEx, \tag{13}$$

where e and E are constants, one readily finds that

$$X_0 \equiv x - \frac{tp}{m} + \frac{eEt^2}{2m} \tag{14}$$

is a constant of motion that corresponds to the position of the particle at $t = 0$. Thus, the eigenvalue equation $X_0 \psi = x_0 \psi$ is equivalent to

$$\left(x - \frac{\hbar t}{im} \frac{\partial}{\partial x} + \frac{eEt^2}{2m}\right) \psi = x_0 \psi,$$

with the solution

$$\psi = F(t) \exp \left[\frac{im}{2\hbar t} \left(x^2 - 2x_0 x + \frac{eEt^2 x}{m} \right) \right],$$

where $F(t)$ is a function of t only. Substituting this last expression into the Schrödinger equation one obtains the condition

$$\frac{d \ln F}{dt} = -\frac{1}{2t} + \frac{m}{2i\hbar t^2} \left(x_0^2 - \frac{eEt^2 x_0}{m} + \frac{e^2 E^2 t^4}{4m^2} \right)$$

and, therefore,

$$\psi_{x_0}(x, t) = \frac{C}{\sqrt{t}} \times \exp \left\{ \frac{im}{2\hbar t} \left[(x-x_0)^2 + \frac{eEt^2(x+x_0)}{m} - \frac{e^2 E^2 t^4}{12m^2} \right] \right\},$$

where C is a constant.

3.4. A two-dimensional system

The Hamiltonian

$$H = \frac{1}{2m} \left[\left(p_x + \frac{eB}{2c} y \right)^2 + \left(p_y - \frac{eB}{2c} x \right)^2 \right] \tag{15}$$

corresponds to a charged particle of mass m and electric charge e in a uniform magnetic field B . The operators

$$X_0 \equiv \frac{1}{2}(1 + \cos \omega t)x - \frac{1}{2}y \sin \omega t - \frac{1}{m\omega} p_x \sin \omega t + \frac{1}{m\omega} (1 - \cos \omega t)p_y, \tag{16}$$

$$Y_0 \equiv \frac{1}{2}(1 + \cos \omega t)y + \frac{1}{2}x \sin \omega t - \frac{1}{m\omega} p_y \sin \omega t - \frac{1}{m\omega} (1 - \cos \omega t)p_x, \tag{17}$$

where $\omega \equiv eB/mc$, are two constants of motion that commute with each other. The operators (16) and (17) correspond to the Cartesian coordinates of the particle at $t = 0$. (It may be remarked that, for any system, the operators that represent the initial values of the Cartesian coordinates, or of the Cartesian components of the momentum, are constants of motion that commute with each other.)

Since the operators (16) and (17) are linear in the momenta, one readily finds their eigenfunctions. In fact, the eigenfunctions of X_0 and Y_0 with eigenvalues x_0 and y_0 , respectively, are

$$\begin{aligned}\psi &= F(t) \exp \left\{ \frac{im\omega}{4\hbar(1 - \cos \omega t)} [(x^2 + y^2 - 2x_0x - 2y_0y) \sin \omega t + 2(x_0y - y_0x)(1 - \cos \omega t)] \right\} \\ &= F(t) \exp \left\{ \frac{im\omega}{4\hbar} \left[(x^2 + y^2 - 2x_0x - 2y_0y) \cot \frac{1}{2}\omega t + 2(x_0y - y_0x) \right] \right\},\end{aligned}\quad (18)$$

where $F(t)$ is a function of t only. Substituting (18) into the Schrödinger equation we obtain the single condition

$$\frac{d \ln F}{dt} = -\frac{\omega}{2} \cot \frac{1}{2}\omega t - \frac{im\omega^2}{8\hbar} (x_0^2 + y_0^2) \csc^2 \frac{1}{2}\omega t,$$

therefore,

$$\psi = C \csc \frac{1}{2}\omega t \exp \left[\frac{im\omega}{4\hbar} \left\{ [(x - x_0)^2 + (y - y_0)^2] \cot \frac{1}{2}\omega t + 2(x_0y - y_0x) \right\} \right],\quad (19)$$

where C is a constant.

4. Concluding remarks

The procedure followed here to find the Green functions highly contrasts with the methods usually employed in the literature (see, *e.g.*, Refs. 2 to 4). For instance, the fact that the operators X_0 , defined by Eqs. (5), (11), and (14), are linear in p implies that, in order to find the Green functions, one only has to solve two first-order ordinary differential equations. Even though the results presented in Sec. 2 also apply in the usual case of time-independent operators that commute

with the Hamiltonian, the difference is that there are not many useful time-independent constants of motion linear in the momentum.

After the submission of this work, Ref. 5 was brought to the attention of the present author. In that article, the constants of motion that represent the initial conditions are specifically employed to find the Green functions. Furthermore, in Ref. 5 such constants of motion are obtained for the most general Hamiltonian quadratic in the coordinates and momenta with coefficients that may depend on the time.

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