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Estimation of the space charge limited current with quadratic damping

D. González
Departamento de Matemáticas y Física,
Instituto Tecnológico y de Estudios Superiores de Occidente,
Periférico Sur Manuel Gómez Morín 8585 C.P. 45604, Tlaquepaque, Jal., MEXICO.

V. H. Romero
Universidad de Guadalajara, Centro Universitario de Tonalá,
Div. de Ciencias, Depto. de Ing.,
Tonalá Jalisco C.P. 48525, MEXICO.

G. González
Cátedras CONACYT, Universidad Autónoma de San Luis Potosí,
San Luis Potosí, 78000 México.
Coordinación para la Innovación y la Aplicación de la Ciencia y la Tecnología,
Universidad Autónoma de San Luis Potosí, San Luis Potosí, 78000 México.
e-mail: gabriel.gonzalez@uaslp.mx

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In this paper we analyze the motion of charged particles under the influence of a quadratic velocity dependent frictional force inside a vacuum tube diode. Our study is performed by using the adiabatic approximation that allows for exact analytic solutions for the case of weak damping. The expressions obtained for the space charge limited current reduce to the well known Child-Langmuir law under the adiabatic approximation when the dissipation parameter goes to zero.

Keywords: Child-Langmuir law; Mott-Gurney Law; space charge current.

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1. Introduction

The motion of charged particles accelerated across a gap is of wide interest in fields such as high power diodes and vacuum microelectronics. Child and Langmuir [1,2] first studied the space charge limited emission for two infinite parallel plane electrodes at fixed voltage φ_0 in vacuum separated by a distance D (See Fig. 1). Child and Langmuir worked with ions moving without collisions inside a vacuum tube diode and found an analytical solution for the electric potential, the electric field and the ion density between the plates.

The main result of Child and Langmuir states that the behavior of the current density is proportional to the three-halves power of the bias potential and inversely proportional to the square of the gap distance between the electrodes. Since the derivation of this fundamental law many important and useful variations on the classical Child-Langmuir law have been investigated to account for relativistic electron energies [3-5], non zero initial electron velocities [6-8], quantum mechanical effects [9-11], nonzero electric field at the cathode surface [12], and slow varying charge density [13].

The Mott-Gurney law represents an analog of the Child-Langmuir law for the collision-dominated case for the case when the vacuum tube diode contains particles that produce collisions when electrons pass by [14,15]. The Mott-Gurney law, can be in a natural way applied to a collision-dominated near-cathode parallel plate, just in the same way as the Child-Langmuir law applies to a collision-free near-cathode parallel plate. The equation that describes the behavior of this collission-dominated system is given by

$$mv\frac{dv}{dz} = -eE(z) - \lambda v \tag{1}$$

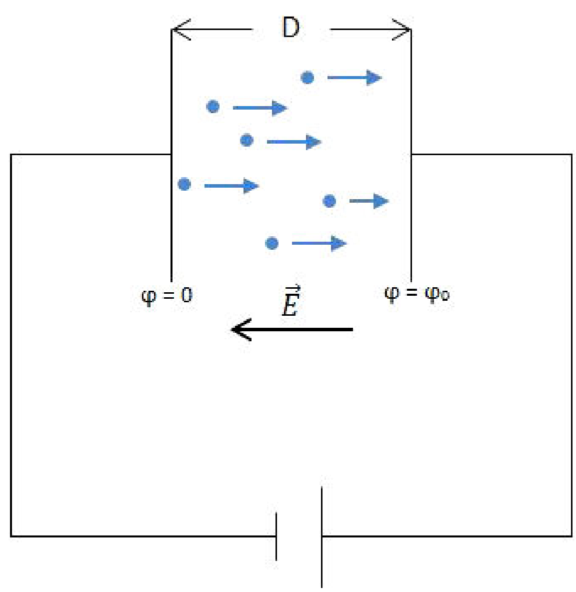

FIGURE 1. Vacuum Tube that consists of two parallel plates subjected to an electrostatic potential between its terminals. The negatively charged electrons moving to the right constitute a steady electric current.

where m is the electron mass, v is the particle velocity and λ is the collision coefficient. The term on the left-hand side of this equation describes the inertia force. If electrons experience several collisions, the velocity lowers abruptly and this term is minor, this becomes identical to the corresponding equation of the Mott-Gurney model. The terms on the right-hand side describe the electric field force and a frictional force resulting from elastic collisions of the electrons with neutral particles. If the system is collision-free, the second term on the right-hand side is zero and becomes identical to the corresponding equation of the Child-Langmuir model.

Although the solution to the problem of charged particles under the influence of a linear velocity dependent frictional force is now well understood [16-20], the problem of the motion of charged particles under the influence of a quadratic velocity dependent frictional force inside a vacuum tube diode has never been solved before within the adiabatic approximation. The physical justification of choosing a frictional force proportional to the particle's velocity is by regarding the medium inside the diode as two interpenetrating fluids composed of neutral gas molecules and singly charged ions, respectively. Phenomenologically, one can express the dynamic frictional force as an average over all possible collisions. One can show that the frictional force may be taken proportional to the square of the ion velocity [21,22].

To tackle this problem, we use a simple one-dimensional dynamic equation in which collisions are assumed to be proportional to the square of the velocity, where collisional effects are controlled by a friction parameter λ, which is assumed to be constant. It is the adiabatic approximation given in Ref. 13 that will allow us to solve the Poisson equation and to analytically solve the problem in the form of an ordinary linear differential equation for the case of weak damping, *i.e.* $\lambda \ll 1$. We will recover in the limit of $\lambda \to 0$ the ordinary Child-Langmuir law for ballistic electron flow.

2. Electron dynamics with quadratic damping

We will follow the spirit of Ref. 13 by expressing the volume charge density as a function of the current density and coordinates only, *i.e.* $\rho = \rho(J, z)$, by first solving the equation of motion and using the following relation $J = \rho v$, where J is the charge current density, ρ is the volume charge density and v is the electron's velocity.

Suppose that during the ions motion inside the diode there is a frictional force which is proportional to the square of the ion velocity. The equation of motion which describes the dynamics of the ion is given by

$$mv\frac{dv}{dz} = -eE(z) - \lambda v^2 \tag{2}$$

Integrating both parts of the equation from 0 to any particular z we get

$$\int_0^z \frac{d}{dz}\left(\frac{mv^2}{2}\right)dz = -e\int_0^z E dz - \lambda\int_0^z v^2 dz \tag{3}$$

Assuming that the ions leave with zero initial velocity, *i.e.* $v(0) = 0$, we obtain

$$\frac{mv^2}{2} = -e\int_0^z E dz - \lambda\int_0^z v^2 dz \tag{4}$$

Integrating by parts the last term of the above equation we have

$$\frac{mv^2}{2} = -e\int_0^z E dz - \lambda z v^2 + \lambda\int_0^z z\frac{dv^2}{dz}dz \tag{5}$$

substituting the equation of motion (2) into the last part of Eq. (5) we get

$$\begin{aligned}\frac{mv^2}{2} &= -e\int_0^z E dz - \lambda z v^2 \\ &\quad + \lambda\int_0^z z\left[-\frac{2\lambda v^2}{m} - \frac{2e}{m}E(z)\right]dz\end{aligned} \tag{6}$$

Assuming that $\lambda \ll 1$ we can neglect the quadratic term in λ to obtain to first order in the friction parameter the following equation

$$\frac{mv^2}{2} = -e\int_0^z E dz - \lambda z v^2 - \frac{2e\lambda}{m}\int_0^z zE(z)dz \tag{7}$$

Integrating once again by parts and collecting terms we get the following expression

$$\begin{aligned}v^2\left(\frac{m}{2} + \lambda z\right) &= -\left(z + \frac{\lambda}{m}z^2\right)eE \\ &\quad + e\int_0^z \left(z + \frac{\lambda}{m}z^2\right)\frac{dE}{dz}dz\end{aligned} \tag{8}$$

Integrating by parts the last term of the equation and using Gauss law, *i.e.* $dE/dz = \rho/\epsilon_0$ we get

$$\begin{aligned}v^2\left(\frac{m}{2} + \lambda z\right) &= -\left(z + \frac{\lambda}{m}z^2\right)eE + e\frac{\rho}{\epsilon_0}\left(\frac{z^2}{2} + \frac{\lambda z^3}{3m}\right) \\ &\quad - \frac{e}{\epsilon_0}\int_0^z \left(\frac{z^2}{2} + \frac{\lambda z^3}{3m}\right)\frac{d\rho}{dz}dz\end{aligned} \tag{9}$$

Using the adiabatic condition over the charge density, *i.e.* $(d\rho/dz) \approx 0$, and substituting $E \approx (\rho z/\epsilon_0)$ in Eq. (9) we get

$$v^2\left(\frac{m}{2} + \lambda z\right) \approx -e\frac{\rho}{\epsilon_0}\left(\frac{z^2}{2} + \frac{2\lambda z^3}{3m}\right) \tag{10}$$

Using the following relation $J = \rho v$ in Eq. (10) we get

$$\frac{J^2}{\rho^2}\left(\frac{m}{2} + \lambda z\right) \approx -e\frac{\rho}{\epsilon_0}\left(\frac{z^2}{2} + \frac{2\lambda z^3}{3m}\right) \tag{11}$$

solving for ρ in terms of J and z we get the following expression for the charge density

$$\rho(J,z) = -\left(\frac{\epsilon_0 J^2 m}{ez^2}\right)^{\frac{1}{3}} \left(\frac{1+\frac{2\lambda z}{m}}{1+\frac{4\lambda z}{3m}}\right)^{1/3} \tag{12}$$

substituting the charge density from Eq. (12) to the Poisson equation we get the following second order linear differential equation

$$\frac{d^2V}{dz^2} = \frac{1}{\epsilon_0}\left(\frac{\epsilon_0 J^2 m}{ez^2}\right)^{\frac{1}{3}} \left(\frac{1+\frac{2\lambda z}{m}}{1+\frac{4\lambda z}{3m}}\right)^{\frac{1}{3}} \tag{13}$$

The Poisson equation given in Eq. (13) can be solved using direct integration. Integrating with Mathematica we obtain the following result

$$\frac{dV}{dz} = \frac{1}{\epsilon_0}\left(\frac{\epsilon_0 J^2 m}{e}\right)^{\frac{1}{3}} 3z^{\frac{1}{3}} \times F_1\left(\frac{1}{3},\frac{-1}{3},\frac{1}{3},\frac{4}{3};\frac{-2\lambda z}{m},\frac{-4\lambda z}{m}\right) \tag{14}$$

where F_1 is the Appel Hypergeometric function [23]. Expanding the Appell Hypergeometric function into series we obtain the first order approximation in terms of λ, which simplifies the previous differential equation into an ordinary differential equation (ODE)

$$\frac{dV}{dz} \approx \frac{1}{\epsilon_0}\left(\frac{\epsilon_0 J^2 m}{e}\right)^{\frac{1}{3}} \left(3z^{\frac{1}{3}} - \frac{6\lambda}{12m} z^{\frac{4}{3}}\right) \tag{15}$$

Integrating Eq. (15) we get the following analytic result for the electric potential

$$V(z) \approx \frac{1}{\epsilon_0}\left(\frac{\epsilon_0 J^2 m}{e}\right)^{\frac{1}{3}} \left(\frac{3^2}{4} z^{\frac{4}{3}} - \frac{18\lambda}{84m} z^{\frac{7}{3}}\right) \tag{16}$$

By setting $V(z=D) = V_0$ in Eq. (16) we get the following result

$$V_0 = \frac{1}{\epsilon_0}\left(\frac{\epsilon_0 J^2 m}{e}\right)^{\frac{1}{3}} \left(\frac{3^2}{4} D^{\frac{4}{3}} - \frac{18\lambda}{84m} D^{\frac{7}{3}}\right) \tag{17}$$

Solving for the current density J in Eq. (17) we get

$$J = \frac{V_0^{\frac{3}{2}} \epsilon_0 e^{\frac{1}{2}}}{m^{\frac{1}{2}} \left[\frac{9D^{\frac{4}{3}}}{4} - \frac{18\lambda D^{\frac{7}{3}}}{84m}\right]^{\frac{3}{2}}} \tag{18}$$

From Eq. (18) we see that the space charged current density J is proportional to the three-halves power of the bias potential but is not inversely proportional to the square of the gap distance between the electrodes. We can express this current density as a function of a non-colliding particles current density given by the Child-Langmuir law under the adiabatic approximation, *i.e.*

$$J_0 = \frac{8Vo^{\frac{3}{2}}\epsilon_0}{27D^2}\sqrt{\frac{e}{m}} = 0.47 J_{CL},$$

to obtain

$$J = \frac{J_0}{\left[1 - \frac{2\lambda D}{21m}\right]^{\frac{3}{2}}}; \tag{19}$$

3. Results

In this section we compare our model with the original Child-Langmuir model. The analysis includes the system's behavior as a function of the values given to $lambda$, which must be $0 \le \lambda < 21m/2D$.

We can see in the Charge Density figure and the Voltage figure that our model behaves identically as the Child-Langmuir model when lambda goes to zero. On the other hand, for large values of lambda, $\lambda = 21m/2D$, we can see the quadratic behavior of our model.

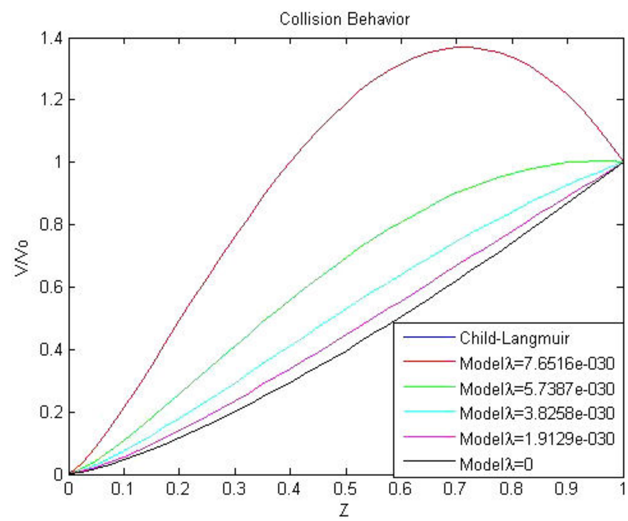

FIGURE 2. Plot showing the electrostatic potential in the adiabatic approximation with D = 1. Note how the functions for the electrostatic potential have the same value at z = 0 and z = D such that the applied bias in the vacuum tube diode is V_0 and when lambda tends to zero the curves are really close to each other for $0 < z < D$.

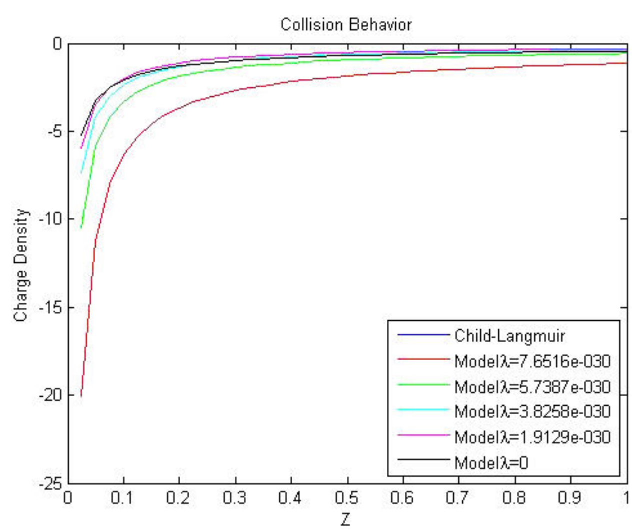

FIGURE 3. Volume charge density for our model compared to the Child-Langmuir model.

4. Conclusions

In this paper we analyze the motion of ions inside a planar diode subjected to a frictional force which is proportional to the square of the ion velocity. We obtain an analytic solution up to first order in the friction paramenter and under adiabatic conditions. The solution to our problem using the adiabatic condition can be readily determined by simply integrating Poisson's equation and applying the boundary conditions at the cathode and anode, respectively.

Acknowledgments

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