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The Jacobi elliptic functions and their applications in the advance of mercury's perihelion

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In this paper we find the exact solution of the nonlinear differential equation describing the trajectory of mercury considering relativistic effects. Instead of the classical perturbation method we use the Jacobi elliptic functions to obtain the exact solution. The deviation of the perihelion of Mercury and other planets is calculated. The results may be of great interest to astronomers and for those interested in the study of nonlinear differential equations.

Keywords: Perihelion; aphelion; analytic solution; Jacobi elliptic functions.

PACS: 02.30Gp, 03.65Fd, 05.45 Yv

1. Introduction

While the consequences of the special theory of relativity have been verified to a very high degree of accuracy by numerous experiments, the experimental verification of the general theory has so far been limited to three cases only.The reason for this is obvious and is connected with the fact that the Newtonian gravitational theory represents a very good approximation for all gravitational phenomena inside the solar system. Mercury is the inner most of the four terrestrial planets in the Solar system, moving with high velocity in the Sun's gravitational field. Only comets and asteroids approach the Sun closer at perihelion. This is why Mercury offers unique possibilities for testing general Relativity [1,2] and exploring the limits of alternative theories of gravitation with an interesting accuracy [3].

There are many works in which the perihelion of Mercury has been studied [3,4] , however, in most of them the nonlinear differential equation describing the trajectory of Mercury has been solved using the perturbation method. Instead of the classical method we use the Jacobi elliptic functions to obtain the exact solution for a nonlinear differential equation and calculate the perihelion of Mercury.

The material of this paper is organized as follows. In the first part we find the nonlinear differential equation that describes the trajectory of mercury considering relativistic effects; this aims to be a compact work. In the second part we study the nonlinear differential equation of interest. Getting away from the classic procedures we find the exact solution of the nonlinear differential equation by means of Jacobi elliptic functions. This section is novel because it allows us to calculate the perihelion of Mercury and other planets. In the third part of the perihelion of Mercury and other planets is calculated. The obtained theoretical results are compared with experimental results.

Let us consider the motion of a planet in the gravitational field of a much heavier body (the sun). The gravitational field of the center in spherical coordinates is given by the linear differential element

$$ds^2 = -\alpha dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 + \beta c^2 dt^2, \tag{1}$$

where α, β are functions of r which are determined from field's equation. According to the Schwarzschild's exterior solution [5] we have that

$$\alpha = \left(1 - \frac{2m}{r}\right)^{-1}, \qquad \beta = \left(1 - \frac{2m}{r}\right), \tag{2}$$

where $m = GM/c^2$ is the relativistic mass of the planet, M is the mass of the sun, G is the constant of gravitational constant and c is the light speed. With a second order approximation [6], we have that

$$\alpha = 1 + \frac{am}{r^2} + \frac{bm^2}{r^2}, \qquad \beta = 1 - \frac{2m}{r} - \frac{cm^2}{r^2}, \tag{3}$$

where the coefficients a, b, c are undetermined. Consider the motion of a particle in field (1). Using the equations of geodesic lines gives

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{\alpha\beta}\frac{dx^\sigma}{ds}\frac{dx^\beta}{ds} = 0, \tag{4}$$

which can lead to the following form

$$\frac{d}{dt}\left(\frac{g_{\sigma\sigma}}{g_{44}}\frac{dx^{\sigma}}{dt}\right)-\frac{1}{2g_{44}}\frac{\partial g_{\alpha\alpha}}{\partial x^{\sigma}}\left(\frac{dx^{\alpha}}{dt}\right)^2=0,$$
$$\sigma=1,2,3 \tag{5}$$

Assuming that $x^1=r$, $x^2=\theta$, $x^3=\varphi$, find the first integrals of motion. For $\sigma=3$ and taking into account (5)

$$\frac{d\varphi}{dt}=\frac{C_1\beta}{r^2\sin^2\theta}, \tag{6}$$

where C_1 is the constant of integration. For $\sigma=2$ we have that

$$\frac{d}{dt}\left(\frac{r^2}{\beta}\frac{d\theta}{dt}\right)-\frac{r^2}{\beta}\sin\theta\cos\theta\left(\frac{d\varphi}{dt}\right)^2=0 \tag{7}$$

From (6) and multiplying by $(2r^2/\beta)(d\theta/dt)$, Eq. (7) may be easily integrated giving

$$\left(\frac{d\theta}{dt}\right)^2=\frac{\beta^2}{r^4}\left(C_2^2-\frac{C_1^2}{\sin^2\theta}\right), \tag{8}$$

where C_2 is a new constant of integration. On account of the central symmetry of our problem, any plane through the center may however be chosen as the plane $\theta=\pi/2$, *i.e.* the orbit can lie in any plane through the center.

From the first integral of motion $\sigma=1$ we find the differential element (1) taking into account the previous results and the geodesic equation (4). For the time coordinate we have.

$$\frac{d^2t}{ds^2}+\Gamma^4_{\alpha\beta}\frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds}=0 \tag{9}$$

Bearing in mind the Christoffel symbols, the Eq. (9) leads to.

$$\frac{d^2t}{ds^2}+\frac{1}{g_{44}}\frac{\partial g_{\alpha 4}}{\partial x^{\beta}}\frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds}=0 \tag{10}$$

or

$$g_{44}\frac{d^2t}{ds^2}+\frac{dg_{44}}{ds}\frac{dt}{ds}=0. \tag{11}$$

Thus,

$$\frac{dt}{ds}=\frac{h}{\beta}, \tag{12}$$

where h is constant of integration. Dividing the differential element (1) by dt^2 and taking into account the results of the first integrals we find that

$$\left(\frac{dr}{dt}\right)^2=\frac{\beta}{\alpha}-\frac{C_1^2\beta^2}{r^2\alpha}-\frac{\beta^2}{h^2\alpha}. \tag{13}$$

To find the orbit trajectory let us divide (13) by (6) squared. Introducing $u=1/r$ we have that.

$$\left(\frac{du}{d\varphi}\right)^2=\frac{1}{C_1^2\alpha\beta}-\frac{u^2}{\alpha}-\frac{1}{C_1^2h^2\alpha} \tag{14}$$

Replacing (3) in (14) and conserving the terms of second order with respect to m gives

$$\left(\frac{du}{d\varphi}\right)^2+u^2-\frac{h^2-1}{C_1^2h^2}-\frac{m}{C_1^2}\left(2-a+\frac{a}{h^2}\right)u$$
$$=\frac{m^2}{C_1^2}\left(4+a^2-2a-b-c-\frac{a^2-b}{h^2}\right)u^2$$
$$+amu^3-m^2(a^2-b)u^4 \tag{15}$$

For distances far enough from the center all the terms on the right side in Eq. (14) are very small and can be neglected. In this case Eq. (14) is consistent with the Newtonian case.

$$\left(\frac{du}{d\varphi}\right)^2+u^2-\frac{2}{p}u-\frac{e^2-1}{p^2}=0,$$

where p is the focal parameter and e the eccentricity. From (14)-(15) equating coefficients gives

$$\frac{h^2-1}{C_1^2h^2}=\frac{e^2-1}{p^2},\qquad \frac{m}{C_1^2}\left(2-a+\frac{a}{h^2}\right)=\frac{2}{p}. \tag{16}$$

Therefore

$$C_1^2=\frac{2mp}{2+(e^2-1)\frac{am}{p}}$$
$$h^2=\frac{2+(e^2-1)\frac{am}{p}}{2-(e^2-1)\left(\frac{2-a}{p}\right)m}. \tag{17}$$

Taking into account the previous expressions and conserving the linear terms with respect to m/p with a good approximation for our purposes we have that

$$\left(\frac{du}{d\varphi}\right)^2+u^2-\frac{2u}{p}-\frac{e^2-1}{p^2}$$
$$=\frac{m}{p}(4-2a-c)u^2+amu^3 \tag{18}$$

Differentiating with respect to φ and simplifying we obtain

$$\frac{d^2u}{d\varphi^2}+u-\frac{1}{p}=\frac{m}{p}(4-2a-c)u+\frac{3}{2}amu^2. \tag{19}$$

According to the Schwarzschild's exterior solution [5] $a=2$, $b=4$, $c=0$. Therefore

$$\frac{d^2u}{d\varphi^2}+u=\frac{1}{p}+3\frac{GM}{c^2}u^2. \tag{20}$$

2. Analytic Solution to Planetary Motion Equation.

In this section we will give the exact solution to initial value problem

$$\frac{d^2u}{d\varphi^2}+u=\frac{1}{p}+3\frac{GM}{c^2}u^2,$$
$$u(0)=u_0,\ u'(0)=0 \tag{21}$$

Instead of solving problem (21), we will solve a more general one

$$\frac{d^2y}{dx^2}+y=\frac{\lambda}{2}+\frac{3}{2}\rho y^2,$$
$$y(0)=y_0>0\,,\quad y'(0)=0,\quad \text{where}\quad \rho>0. \tag{22}$$

Multyplying Eq. (22) by $y'(x)$ and integrating once gives

$$\frac{1}{2}\left(\frac{dy}{dx}\right)^2+\frac{1}{2}y^2=\frac{\lambda}{2}y+\frac{\rho}{2}y^3+C, \tag{23}$$

where C is the constant of integration. Its value may be obtained form (23) setting $x=0$:

$$C=\frac{1}{2}y_0^2-\frac{\lambda}{2}y_0-\frac{\rho}{2}y_0^3. \tag{24}$$

From (23) and (24) we obtain

$$\left(\frac{dy}{dx}\right)^2=\rho y^3-y^2+\lambda y+y_0^2-\lambda y_0-\rho y_0^3$$

or, equivalently,

$$\left(\frac{dy}{dx}\right)^2=(y-y_0)$$
$$\times\left(\rho y^2+(\rho y_0-1)y+\rho y_0^2+\lambda-y_0\right) \tag{25}$$

We will suppose that all three $y-$roots of equation

$$(y-y_0)\left(\rho y^2+(\rho y_0-1)y+\rho y_0^2+\lambda-y_0\right)=0 \tag{26}$$

are real and distinct. These roots are

$$y_0\quad \text{and}\quad y_\pm=\frac{1}{2\rho}\left(1-\rho y_0\pm\sqrt{\Delta}\right),$$
$$\Delta=-3\rho^2y_0^2+2\rho y_0-4\lambda\rho+1>0 \tag{27}$$

Equation (25) may be written in the form

$$\left(\frac{dy}{dx}\right)^2=\rho\,(y-y_0)\,(y-y_1)\,(y-y_2), \tag{28}$$

where

$$y_1=y_-<y_+=y_2. \tag{29}$$

We seek a solution to Eq. (28) in the form

$$y(x)=\lambda+(y_0-\lambda)\,\mathrm{cn}^2(\sqrt{w}x\,,\sqrt{k}) \tag{30}$$

Putting (30) into (28) gives the following equation :

$$(\lambda-y_0)\,(\lambda\rho-\rho y_0-4kw)\,\mathrm{cn}^6\left(\sqrt{w}x\,,\sqrt{k}\right)$$
$$+\,(y_0-\lambda)\,(3\lambda\rho-\rho y_0-\rho y_1-\rho y_2-8kw+4w)\,\mathrm{cn}^4$$
$$\times\left(\sqrt{w}x\,,\sqrt{k}\right)+\left(3\lambda^2\rho-2\lambda\rho y_0-2\lambda\rho y_1-2\lambda\rho y_2\right.$$
$$-\,4\lambda kw+4\lambda w+\rho y_0y_1+\rho y_0y_2+\rho y_1y_2+4kwy_0$$
$$\left.-\,4wy_0\right)\mathrm{cn}^2\left(\sqrt{w}x\,,\sqrt{k}\right)+\rho\,(y_1-\lambda)\,(\lambda-y_2)=0.$$

Equating the cooefficients of this polynomial in $\mathrm{cn}\big(\sqrt{w}x\,,\sqrt{k}\big)$ to zero and solving the resulting algebraic system gives

$$\lambda=y_1,\quad w=\frac{\rho}{4}(y_2-y_0),\quad k=\frac{y_1-y_0}{y_2-y_0}$$
$$\lambda=y_2,\quad w=\frac{\rho}{4}(y_1-y_0),\quad k=\frac{y_2-y_0}{y_1-y_0} \tag{31}$$

Making use of the identities

$$\mathrm{cn}\left(\sqrt{w}x,\sqrt{k}\right)=\mathrm{nc}\left(\sqrt{-w}x,\sqrt{1-k}\right),$$
$$0<k<1\quad \text{and}\quad w<0. \tag{32}$$

and

$$\mathrm{cn}\left(\sqrt{w}x,\sqrt{k}\right)=\mathrm{cd}\left(\sqrt{1-k}\sqrt{w}x,\sqrt{\frac{k}{k-1}}\right),$$
$$k<0\quad \text{and}\quad w>0. \tag{33}$$

$$\mathrm{cn}\left(\sqrt{w}x,\sqrt{k}\right)=\mathrm{nd}\left(\sqrt{w(k-1)}x,\frac{1}{\sqrt{1-k}}\right),$$
$$k<0\quad \text{and}\quad w<0. \tag{34}$$

we choose λ, and w from the solution set (31) to obtain a periodic solution (30) as follows :
First Case : $y_0>y_2>y_1$. We set $\lambda=y_2$, $w=(\rho/4)(y_1-y_0)<0$ and $k=(y_2-y_0)/(y_1-y_0)>0$. By formula (32) the solution to i.v.p. problem (22) is

$$y(x)=\frac{1}{2\rho}\left(1-\rho y_0+\sqrt{\Delta}\right)$$
$$+\,\frac{1}{2\rho}\left(3\rho y_0-1-\sqrt{\Delta}\right)\,\mathrm{nc}^2(\omega\,x\,,m), \tag{35}$$

where $\Delta=1+2(y_0-2\lambda)\rho-3\rho^2y_0^2>0$ and

$$0<m=\sqrt{\frac{y_2-y_1}{y_0-y_1}}=\sqrt{\frac{2\sqrt{\Delta}}{3y_0\rho-1+\sqrt{\Delta}}}<1. \tag{36}$$

$$\omega=\frac{\sqrt{\rho}}{2}\sqrt{y_0-y_1}=\sqrt{\frac{3y_0\rho-1+\sqrt{\Delta}}{8}}>0. \tag{37}$$

Solution (35) is $2T-$periodic where $T=K(m)/\omega$ and $K(m)$ is the complete elliptic integral of the first kind given by

$$K(m)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-m^2\sin^2\theta}}. \tag{38}$$

thus,

$$T=\frac{K(m)}{\omega}=\sqrt{\frac{8}{3y_0\rho-1+\sqrt{\Delta}}}K$$
$$\times\left(\sqrt{\frac{2\sqrt{\Delta}}{3y_0\rho-1+\sqrt{\Delta}}}\right). \tag{39}$$

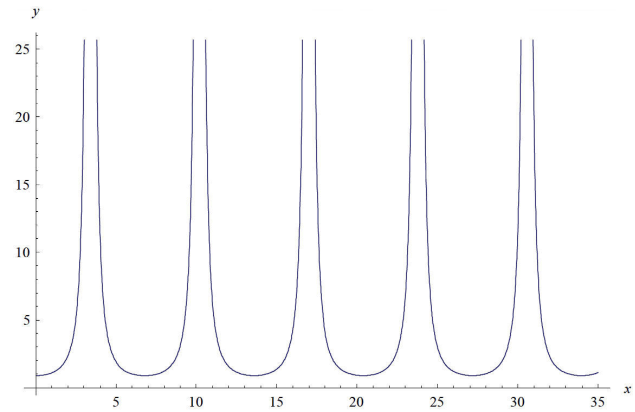

FIGURE 1.

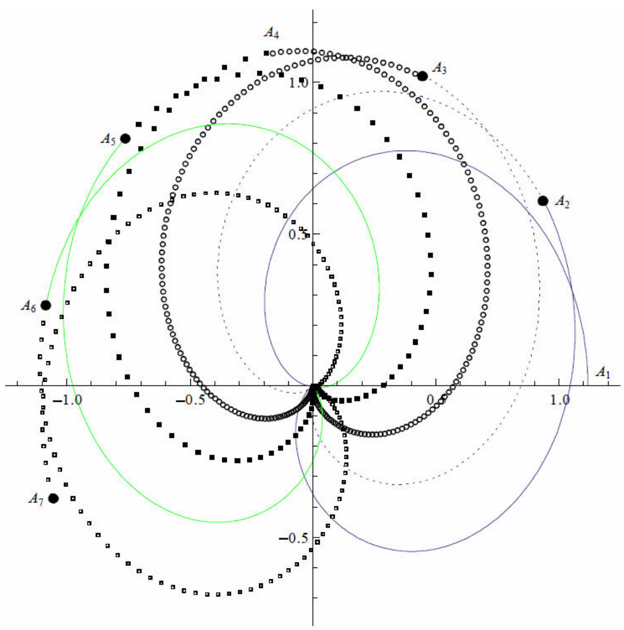

FIGURE 2.

Example 1. Let $\lambda = 0.087$, $\rho = 1$ and $y_0 = 0.895$. We have

$$y_0 = 0.895 > y_2 = 0.151147 > y_1 = -0.0461471.$$

The solution to i.v.p. problem

$$\frac{d^2y}{dx^2} + y = 0.043\,5 + \frac{3}{2}y^2,$$
$$y(0) = 0.895, \qquad y'(0) = 0$$

is

$$y(x) = 0.151147 + 0.743853\text{nc}^2(0.485064x, 0.209632)$$

This solution is unbounded and periodic with period $\tau = 6.86306789474$.Its graph is shown in Fig. 1.
Let

$$r(x) = \frac{1}{y(x)} = \frac{1}{0.151147 + 0.743853\ \text{nc}^2(0.485064x, 0.209632)}.$$

If we interpret $r(x)$ as the radius of the orbit of a certain particle moving in a certain stellar system of planets moving around the origin and $x = \theta$ is the polar angle, then the polar plot of planet's trajectory is shown in Fig. 2. The A_i's are interpreted as "aphelions".

This case shows us, from a mathematical point of view, the inability to have a planet with zero angular speed, since it would fall to the sun; why we must be ruled out this solution for the physical system proposed.

Second Case : $y_2 > y_1 > y_0$. We set $\lambda = y_1$, $w = (\rho/4)(y_2 - y_0) > 0$ and $k = (y_1 - y_0/y_2 - y_0) > 0$. Solution to i.v.p. problem (22) is

$$y(x) = \frac{1}{2\rho}\left(1 - \rho y_0 - \sqrt{\Delta}\right) + \frac{1}{2\rho}\left(3\rho y_0 - 1 + \sqrt{\Delta}\right)\ \text{cn}^2(\omega\, x\,, m), \tag{40}$$

where $\Delta = 1 + 2(y_0 - 2\lambda)\rho - 3\rho^2 y_0^2 > 0$ and

$$0 < m = \sqrt{\frac{y_1 - y_0}{y_2 - y_0}} = \sqrt{\frac{1 - 3y_0\rho - \sqrt{\Delta}}{1 - 3y_0\rho + \sqrt{\Delta}}} < 1. \tag{41}$$

$$\omega = \frac{\sqrt{\rho}}{2}\sqrt{y_2 - y_0} = \sqrt{\frac{1 - 3y_0\rho + \sqrt{\Delta}}{8}} > 0. \tag{42}$$

Solution (40) is $2T-$periodic where

$$T = \frac{K(m)}{\omega} = \sqrt{\frac{8}{1 - 3y_0\rho + \sqrt{\Delta}}}K \times \left(\sqrt{\frac{1 - 3y_0\rho - \sqrt{\Delta}}{1 - 3y_0\rho + \sqrt{\Delta}}}\right). \tag{43}$$

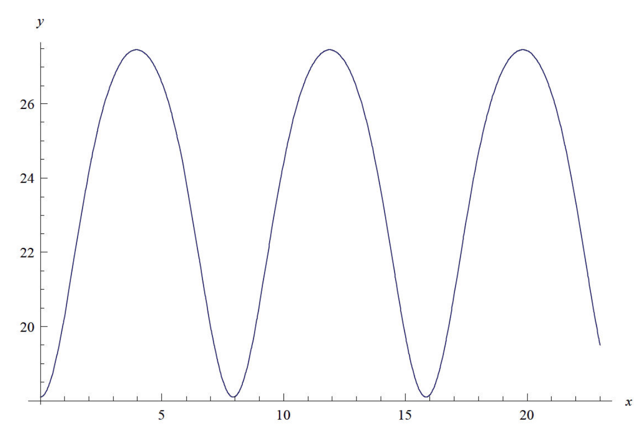

FIGURE 3.

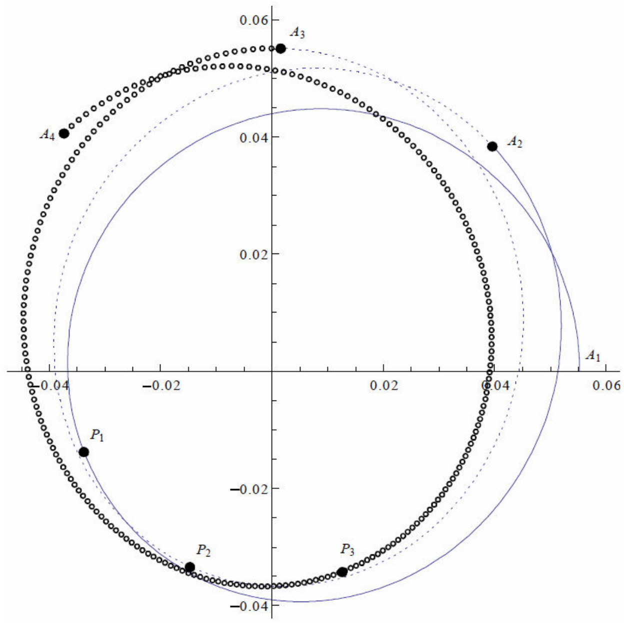

FIGURE 4.

Example 2. Let $\lambda = 0.012$, $\rho = 0.055$ and $y_0 = 18.1$. We have $y_2 = 37.7704 > y_1 = 27.4629 > y_0 = 18.1$. Then the solution to i.v.p. problem

$$\frac{d^2y}{dx^2} + y = 0.06 + 0.0825y^2,\, y(0) = 18.1\ ,\, y'(0) = 0$$

$$y(x) = 27.462911 - 9.362911\mathrm{cn}^2(0.520066\,x, 0.6899197)$$

This solution is periodic with period 7.053916. Its graph is shown in Fig. 3.

Let

$$r(x) = \frac{1}{y(x)} = \frac{1}{27.462911 - 9.362911\mathrm{cn}^2(0.520066\,x, 0.6899197)}.$$

If we interpret $r(x)$ as the radius of the orbit of a certain particle moving in a certain stellar system of planets moving around the origin and $x = \theta$ is the polar angle, then the polar plot of planet's trajectory is shown in Fig. 4. The P_i's are interpreted as "perihelions" and the A_i's are interpreted as "aphelions".

Third Case : $y_2 > y_0 > y_1$. We set $\lambda = y_2$, $w = (\rho/4)(y_1 - y_0) < 0$ and $k = y_1 - y_0/y_2 - y_0 < 0$. By formula (34) the solution to i.v.p. problem (22) is

$$y(x) = \frac{1}{2\rho}\left(1 - \rho y_0 + \sqrt{\Delta}\right) + \frac{1}{2\rho}\left(3\rho y_0 - 1 - \sqrt{\Delta}\right)\ \mathrm{nd}^2(\omega\, x\ , m), \qquad (44)$$

where $\Delta = 1 + 2(y_0 - 2\lambda)\rho - 3\rho^2 y_0^2 > 0$ and

$$0 < m = \sqrt{\frac{y_2 - y_0}{y_2 - y_1}} = \sqrt{\frac{1 - 3y_0\rho + \sqrt{\Delta}}{2\sqrt{\Delta}}} < 1. \qquad (45)$$

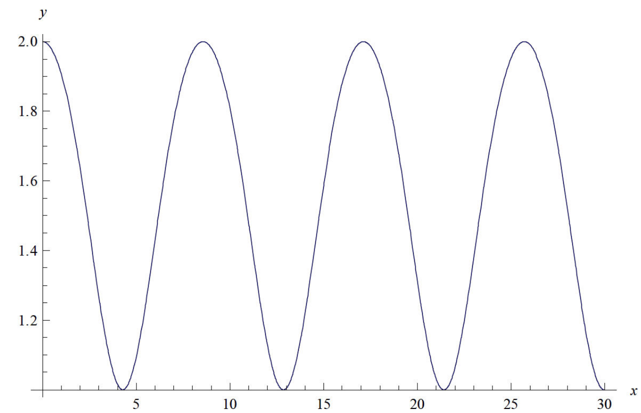

FIGURE 5.

and

$$\omega = \frac{\sqrt{\rho}}{2}\sqrt{y_0 - y_1} = \sqrt{\frac{3y_0\rho - 1 + \sqrt{\Delta}}{8}} > 0 \qquad (46)$$

Solution (44) is $2T-$periodic where

$$T = \frac{K(m)}{\omega} = \sqrt{\frac{8}{3y_0\rho - 1 + \sqrt{\Delta}}}K \times \left(\sqrt{\frac{1 - 3\rho y_0 + \sqrt{\Delta}}{2\sqrt{\Delta}}}\right). \qquad (47)$$

Example 3. Let $\lambda = \rho = 1/4$ and $y_0 = 2$. We have $y_2 = 3 > y_0 = 2 > y_1 = -1$. The solution to i.v.p. problem

$$\frac{d^2y}{dx^2} + y = \frac{1}{8} + \frac{3}{8}y^2, \quad y(0) = 2, \quad y'(0) = 0$$

is

$$y(x) = 3 - \mathrm{nd}^2\left(\frac{\sqrt{3}}{4}x, \frac{1}{2}\right)$$

This solution is bounded and periodic with period 7.7861474. Its graph is shown in Fig. 5.

Let

$$r(x) = \frac{1}{y(x)} = \frac{1}{3 - \mathrm{nd}^2\left(\frac{\sqrt{3}}{4}x, \frac{1}{2}\right)}.$$

If we interpret $r(x)$ as the radius of the orbit of a certain particle moving in a certain stella system of planets moving around the origin and $x = \theta$ is the polar angle, then the polar plot of planet's trajectory is shown in Fig. 6. The P_i's are interpreted as "perihelions" and the A_i's are interpreted as "aphelions"..

Thus, we have succesfully solved the initial value problem (22). We obtained the solution to this problem in terms of Jacobi elliptic functions cn, nc and nd with positive frequency and module on the interval $(0, 1)$. This is important

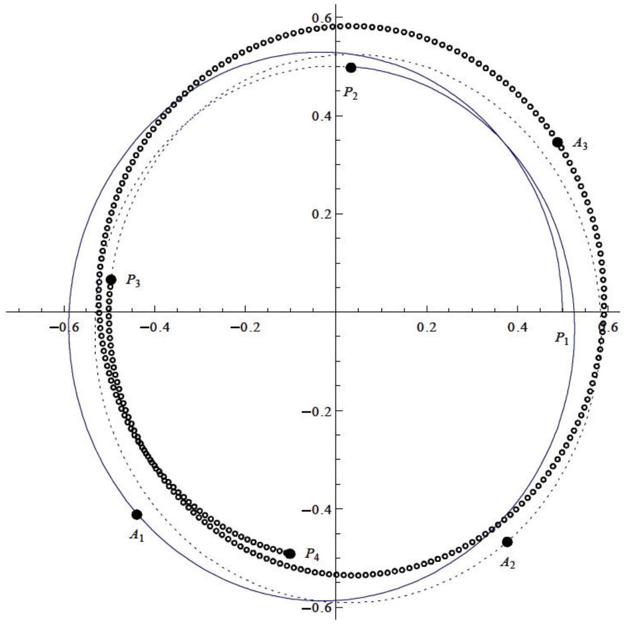

FIGURE 6.

for physical interpretations. These solutions are the key to give an exact value not only for the Mercury's perihelion advance but also for the rest of planets in our solar system. This is the main purpose of next section.

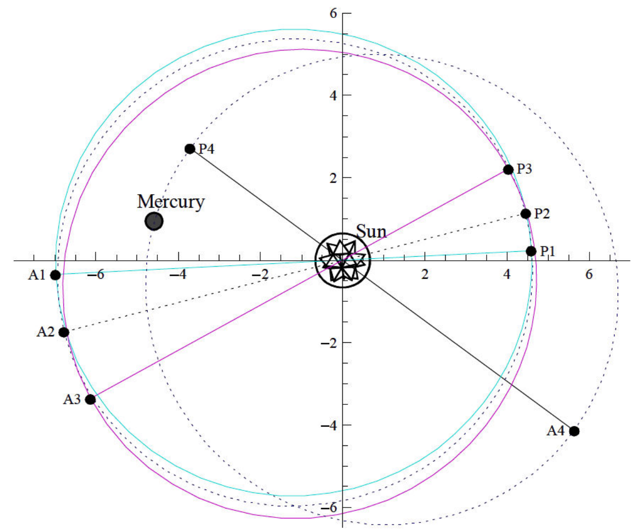

FIGURE 7.

The trajectory of all planets in our solar system is similar to the trajectory in Fig. 6 where the sun is located at the origin. Figure 7 illustrates the way Mercury's perihelion advances.

The polar equation for the planet's trajectory is given by $r(\theta) = 1/u(\theta)$, where $u(x)$ is the solution to initial value problem (21) and it is

$$r(\theta) = \frac{1}{\frac{1}{2\rho}\left(1-\rho y_0+\sqrt{\Delta}\right)+\frac{1}{2\rho}\left(3\rho y_0-1-\sqrt{\Delta}\right)\ \mathrm{nd}^2(\omega\theta, m)}, \tag{48}$$

where $\rho = 2GM/c^2$, $y_0 = 1/P_{er}$, P_{er} is the perihelion distance of the planet from the sun, c is the light speed in the vacuum, $\Delta = 1 + 2(y_0 - 2\lambda)\rho - 3\rho^2 y_0^2 > 0$, $\lambda = 2/p = 2L^2/GM$, L is the angular momentum of the planet and ω and m are given by (45)-(46). The formula for planet's perihelion advance is

$$\delta_{\text{rad}} = 2T - 2\pi, \tag{49}$$

where T is given by (47). Thus

$$\delta_{\text{rad}} = 2\frac{K(m)}{\omega} - 2\pi, \tag{50}$$

where

$$\omega = 2\sqrt{\frac{2}{\frac{6\mu}{c^2 Per} - 1 + \sqrt{\Delta}}},$$

$$m = \sqrt{\frac{1}{2} - \frac{6\mu - c^2 P_{er}}{2c^2 P_{er}\sqrt{\Delta}}}, \mu = GM, \tag{51}$$

$$\Delta = 1 + \frac{1}{c^2}\left(\frac{4\mu}{P_{er}} - 16L^2\right) - 12\left(\frac{\mu}{c^2 P_{er}}\right)^2$$

$$\text{and} \qquad L = \sqrt{\mu P_{er}(1+e)}, \tag{52}$$

being e the planet's orbit eccentricity, M the sun's mass and G the gravitational constant. Usually, the precession is given in arcseconds per century, that is

$$\delta_{\text{arcsec}} = \delta_{\text{rad}} \times \frac{180}{\pi}\frac{\text{deg}}{\text{rad}} \times 60^2\frac{\text{arcsec}}{\text{deg}} \times \frac{365.25636}{S_{\text{id}}}\frac{\text{rev}}{\text{yr}} \times 100\frac{\text{yr}}{\text{century}}, \tag{53}$$

or

$$\delta_{\text{arcsec}} = \delta_{\text{rad}} \times \frac{23668612128}{\pi S_{\text{id}}}\frac{\text{yr}}{\text{century}}, \tag{54}$$

where S_{id} is the sidereal period of the planet [9]. The equation of motion that describes planet's trajectory is

$$r(\theta) = \frac{P_{er}}{D + (1-D)\mathrm{nd}^2\left(\frac{1}{2}\sqrt[4]{\Delta}\theta, \sqrt{\frac{1}{2} - \frac{6\mu - c^2 P_{er}}{2c^2 P_{er}\sqrt{\Delta}}}\right)},$$

$$D = \frac{c^2 P_{er}(1+\sqrt{\Delta}) - 2\mu}{4\mu}, \tag{55}$$

where Δ and μ are given by (52).

TABLE I. Perihelion Precessions.

Planet	P_{er} (meters)	e	S_{id}	δ_{rad}	δ_{arcsec}
Mercury	46001009000	0.20563593	87.969	$5.018133082401732\times10^{-7}$	42.97693500862488
Venus	107476170000	0.00677672	224.701	$2.572055430860587\times10^{-7}$	8.623791317233945
Earth	147098291000	0.01671123	365.25636	$1.860888749760647\times10^{-7}$	3.838358574167813
Mars	206655215000	0.09339410	686.98	$1.231693227410346\times10^{-7}$	1.350769916476435
Jupiter	740679835000	0.04838624	4332.589	$3.584047814086944\times10^{-8}$	0.062323125075340
Saturn	1349823615000	0.05386179	10759.22	$1.956432882366244\times10^{-8}$	0.013699574726097
Uranus	2734998229000	0.04725744	30685.4	$9.716618620814188\times10^{-9}$	0.002385647580547
Neptune	4459753056000	0.00859048	60189	$6.187282686198614\times10^{-9}$	0.000774472052943
Pluto	4436820000000	0.24880766	90465	$5.022942950461129\times10^{-9}$	0.000418312245384

TABLE II. Equations of motions in Polar Form

Planet	Equation of Motion in PolarForm
Mercury	$r(\theta)=\frac{46001.009}{1.557803988877954-1.557803888877954\mathrm{nd}^2(0.49999996280418970\theta,0.0001479789442513079)}$
Venus	$r(\theta)=\frac{10747.617}{3.639633535560302-3.639633435560301\mathrm{nd}^2(0.4999999795784685\theta,0.00001923216058165254)}$
Earth	$r(\theta)=\frac{14709.8291}{4.981419425598379-4.981419325598379\mathrm{nd}^2(0.4999999852740075\theta,0.00002568877974426872)}$
Mars	$r(\theta)=\frac{20665.5215}{6.998288757462074-6.998288657462074\mathrm{nd}^2(0.4999999905036310\theta,0.00004940721657982459)}$
Jupiter	$r(\theta)=\frac{74067.9835}{25.08280015202405-25.08280005202404\mathrm{nd}^2(0.4999999971939060\theta,0.00001918347301137213)}$
Saturn	$r(\theta)=\frac{134982.3615}{45.71118922451532-45.71118912451532\mathrm{nd}^2(0.4999999984710723\theta,0.00001495382384989734)}$
Uranus	$r(\theta)=\frac{273499.8229}{92.61952483754239-92.61952473754238\mathrm{nd}^2(0.4999999992389563\theta,9.871246084989281\times10^{-6})}$
Neptune	$r(\theta)=\frac{445975.3056}{151.0275966064833-151.0275965064833\mathrm{nd}^2(0.4999999995090416\theta,3.358446164452811\times10^{-6})}$
Pluto	$r(\theta)=\frac{443682}{150.2509786813500-150.2509785813500\mathrm{nd}^2(0.4999999996334375\theta,0.00001628509997473153)}$

3. Motion equation for the planets and their precessions

Formulas (50)-(55) allow us to obtain the polar equation for the planet's trajectory and to calculate perihelion's advance for each planet in our solar system. Following values will be used:

$M=1.9891\times10^{30}$ Kg,

$G=6.67128190396304099151153428 9\times10^{-11}\mathrm{Nm}^2/\mathrm{Kg}^2$

and $c=299792458$ m/s

Table I shows these data for each planet. This is the main result of this work. The values of P_{er} and e were taken from NASA web site [7]. The values of S_{id} appear in the web site [8]. The values presented in Table I for precessions are consistent with NASA data. However, we did not take into account perturbations caused by Jupiter and Saturn. Table II contains the equations of motion in polar form.

4. Conclusions

We solved exactly the nonlinear differential equation that describes the orbit of the inner planets around the sun. The solution is expressed in terms of Jacobi elliptic function dn. Because the elliptic functions are periodic we can calculate the precession of perihelion's advance for each of the planets in the solar system.

The obtained results are also applicable to asteroids , in general, to any body moving aroun the sun. We think that they may be of great interest to astronomers and for those interested in the study of nonlinear differential equations.

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