Analysis of the adiabatical pulsation of Cepheids

H. Dehnen
Konstanz Universitaet, Fakultaet fuer Physik,
Postfach 55 60, 7750 Konstanz, Germany,
e-mail: Heinz.Dehnen@uni-konstanz.de

M.G. Corona-Galindo
Instituto Nacional de Astrofísica,Óptica y Electrónica,
Apartado Postal 216, 72840 Tonantzintla, Puebla, México,
e-mail: mcorona@inaoep.mx

A.L. Gelover-Santiago
Facultad de Ciencias, Universidad Nacional Autónoma de México,
Circuito Exterior S/N, Ciudad Universitaria, México D.F. 04510,
e-mail: gelover@unam.mx

A model for the adiabatical pulsation of a Cepheid based on primary physical principles such as energy conservation, mechanical adiabaticity and hydrostatic equilibrium is presented. The adiabatical treatment of the pulsation has been already studied by other authors but the novelty of our model is that it allows to obtain physical parameters of the star, namely the amplitude of the oscillation, the mass, maximal velocity of the pulsation and luminosity.

1. Introduction

In the present work we are principally interested in contributing to understand the way in which Cepheids pulsate. Following Baade's proposition [5], we present a model based on primary physical principles such as energy conservation, mechanical adiabaticity and hydrostatic equilibrium. The adiabatic limit is relevant because Cepheids are weakly nonadiabatic pulsators [7,47]. Additionally, we assert that nature does not work in a very complicated way but obeying the least action principle and Cepheids cannot be an exception. Thus, the adiabatic and hydrostatic equilibrium assumptions during pulsation and contraction processes are the primary fulfillment of this imperative. With such assumptions, we get a differential equation for the pulsation velocity of the star, which can be analytically integrated. Unlike other models, this solution depends on a free parameter, (a), which accounts for the anharmonicity of the pulsation of the star. The solution of this equation states that a Cepheid pulsates either as an anharmonic oscillator if $a \approx 1$ or harmonic oscillator when $a \approx 0$. Instead of doing computer simulations that require the introduction of *ad hoc* parameters and constants, the aim of the present paper is to treat the Cepheid problem by a strictly analytic method which can be step by step well understood and fitted with astronomical observations. In the theory of similarity and modeling, the key role is played by dimensionless parameters that characterize the phenomena under consideration. So that the physical system tied with the phenomena obeys scaling which can also be presented as similarity rules [37]. In this context, we have applied similarity and dimensional techniques for describing a Cepheid during its pulsation as a *totum* but not as a layered structure where the variabilities of the opacity and of the adiabatic exponents are relevant for the compensation of the energy loss as well as for the description of the oscillations during the pulsation. In addition, it is important to note that the adiabatic assumption does not affect the physical features of the oscillation, such as the frequency and anharmonicity, which are determined mainly by the elastic properties of the star matter as in every vibrational physical process.

On the basis of adiabaticity and anharmonicity we were able to obtain a theoretical period-luminosity relation. On the other hand, it must be emphasized that for the authors the matching of the method developed in the present paper with observational data is very important. In this context, as input for the calculations, the period, the radius and the the radial velocity curve are introduced as observable parameters and as output the amplitude of the oscillations, the surface gravity, the mass, and the luminosity are obtainable; and since the amplitude of the oscillations, the mass, the surface gravity and the luminosity of any star are not easily observable features, a way to predict their values is very helpful and consequently new.

Of course, the opacity and consequently convection and turbulence can also be taken into account in our model but then the concomitant differential equation is more compli-

cated and cannot be analytically integrated. Consequently its integration can be the scope of future work.

The outline of the paper is as follows. Section 2 presents the model, the stability of the method is analyzed in Sec. 3, the adaptation of the method to Cepheids is presented in Sec. 4 and its application to δ Cephei, Polaris, FM Cas, SY Cas and RR Lac with the corresponding results are provided in Sec. 5. The discussion of the results are contained in Sec. 6 and we outline our conclusions in Sec. 7.

2. Our Hydrodynamical model

2.1. Fundamental equations

We begin by considering the energy conservation law for an oscillating homogeneous star, $i.e.$, a star with spatially constant mass-density, temperature and averaged chemical composition. Using the mass conservation law for an oscillating star and the continuity equation, the kinetic energy for such an oscillating star is given by

$$E_{\text{kin}} = \frac{3}{10} M \dot{R}^2 \tag{1}$$

where M is the mass and R the radius of the star, respectively. The factor $3/5$ has not been introduced $ad\ hoc$, but it emerges directly from the calculation of the kinetic energy with the constrain that the mass density is spatially constant during the oscillation. However, if only a small fraction of the star is actually displaced during the pulsation, this hypothesis is actually rather strong and then the geometric, kinematic or rather dynamic similarity must be done carefully otherwise the matching between the proposal of the paper and the physical system drops dramatically. Additionally, the potential energy can be written as the sum of the gravitational and thermal energies, namely

$$E_{\text{pot}} = -\frac{3}{5} \frac{M^2 G}{R} + \frac{3}{2} kT \frac{1+Z}{A} \frac{M}{m_{\text{H}}} \tag{2}$$

where G is the gravitational constant, m_{H} the hydrogen atom mass, Z the mean atomic number, A the mean atomic weight of the matter of the star, and T its temperature. The mass of a star consists principally of hydrogen and a helium mixture and there is no less of generality, leaving the chemical composition of the star in relation to A and Z so that to take a mean value for the chemical composition is anyway possible. Assuming an adiabatic oscillation for the matter of the star, the temperature obeys the relation

$$TR^2 = T_0 R_0^2 = \text{const.} \tag{3}$$

where T_0 and R_0 are the equilibrium values of the temperature and the radius, respectively.

In hydrostatic equilibrium, the gravitational pressure, by the derivative of the gravitational potential with respect to the volume, equals the gas pressure, $i.e.$,

$$\frac{3}{20\pi} \frac{M^2 G}{R_0^4} = \varrho_0 \frac{1+Z}{Am_{\text{H}}} kT_0 \tag{4}$$

where ϱ_0 is the mass density. By considering $\varrho_0 = MR_0^{-3}/(4\pi/3)$ in Eq. (4), it follows immediately that

$$kT_0 = \frac{1}{5} \frac{A}{1+Z} m_{\text{H}} \frac{MG}{R_0}. \tag{5}$$

By substituting (3) and (5) in (2) one gets

$$E_{\text{pot}} = -\frac{3}{5} \frac{M^2 G}{R} + \frac{3}{10} M^2 G \frac{R_0}{R^2} \tag{6}$$

and with the aid of Eqs. (1) and (6), the energy conservation law takes the form

$$\frac{3}{10} M \dot{R}^2 - \frac{3}{5} \frac{M^2 G}{R} + \frac{3}{10} M^2 G \frac{R_0}{R^2} = \text{const.}, \tag{7}$$

which is a differential equation for the radius of the star as a function of time. Another form of the same equation can be obtained by using the substitution

$$y = R/R_0. \tag{8}$$

Thus, the differential equation (7) transforms into

$$\dot{y}^2 = \frac{2MG}{R_0^3} \left(\frac{1}{y} - \frac{1}{2} \frac{1}{y^2} \right) + E. \tag{9}$$

Since we have a binding system where the potential energy is greater than the kinetic energy due to the pulsation plus that due to the gas pressure, one has that $E = \text{const.} < 0$ with E describing the total energy in mass units. $E = \text{const.}$ is true whereas the star does not loss mass during the pulsation and it is assumed in the present work. The second derivation of Eq. (9) leads to the associated equation of motion

$$\ddot{y} = -\frac{MG}{R_0^3} \left(\frac{1}{y^2} - \frac{1}{y^3} \right) \tag{10}$$

where $\ddot{y} > 0$ for $y < 1$ and $\ddot{y} < 0$ for $y > 1$. Eq. (10) describes the pulsation of a star in hydrostatic equilibrium when the gravitational energy and gas pressure giving rise to a kinetic energy are taken into account only. Such pulsation is anharmonic because it is described as a non-linear oscillating system. Despite of the non-linearity of the differential equation (10), it can be integrated analytically.

2.2. Solution of the equation of motion

Let us consider Eq. (9) with the substitution

$$B = \frac{2MG}{R_0^3} \tag{11}$$

so that, separating the variables, the differential equation transforms into

$$\frac{y\,dy}{\sqrt{B(y - \frac{1}{2}) + Ey^2}} = dt$$

which integral is

$$\frac{\sqrt{B(y - \frac{1}{2}) + Ey^2}}{E} + \frac{B}{2E}$$

$$\times \frac{1}{\sqrt{-E}} \arcsin \frac{2Ey + B}{\sqrt{B(B + 2E)}} = t - t_0 \quad (12)$$

where $t_0 = $ const. Eq. (12) can also be written as

$$\frac{2Ey + B}{\sqrt{B(B + 2E)}} = \sin\left[\frac{2E\sqrt{-E}}{B}\right.$$

$$\times \left(t - t_0 - \frac{\sqrt{B(y - \frac{1}{2}) + Ey^2}}{E}\right)\right].$$

Due to the periodicity of the sinus function, the value of y for a time given by $t = t_1$ and $t = t_2$ will be the same if the period of the pulsation satisfies

$$\tau = t_2 - t_1 = \frac{2\pi B}{2(-E)^{3/2}}. \quad (13)$$

With the aid of the energy conservation law given by (9), and by calculating the maximum (or minimum) value of the amplitude of the oscillation at the surface ($\dot{R} = 0$) one gets the following relation for the energy

$$E = -\frac{1}{2}B\left(\frac{2}{y_{\max}} - \frac{1}{y_{\max}^2}\right). \quad (14)$$

So that, by substituting (14) and (11) into (13), the period becomes

$$\tau = 2\pi\sqrt{\frac{R_0^3}{MG}}\left[\frac{2}{y_{\max}} - \frac{1}{y_{\max}^2}\right]^{-3/2}$$

which means that τ represents the period of an harmonic oscillation times a correction term which is a function of the amplitude (square bracket).

In order to compare our results with those obtained by observations, let us define

$$f = \frac{2}{y_{\max}} - \frac{1}{y_{\max}^2} = 1 - a^2 \quad (15)$$

By considering the middle part of the equality, f is a functional which provides negative values without physical meaning, whereas the right side is the parametrization of the functional in the range interval where it provides positive values and consequently with physical importance; thus a contains the dependence on the real amplitude of every point of the pulsation. With the aid of Eq. (15) the expression for the period can be written as

$$\tau = 2\pi\sqrt{\frac{R_0^3}{MG}}(1 - a^2)^{-3/2} \quad (16)$$

and the oscillation described by y takes the form

$$(1 - a^2)y = 1 + a\sin\left\{(1 - a^2)^{3/2}\left[(t - t_0)\right.\right.$$

$$\left.\left.\times\sqrt{\frac{MG}{R_0^3}} + \frac{\sqrt{2(y - \frac{1}{2}) - (1 - a^2)y^2}}{1 - a^2}\right]\right\} \quad (17)$$

which depends on a as a parameter also.

The oscillation range of y is

$$y_{\min} = \frac{1}{1 + a} \leq y \leq y_{\max} = \frac{1}{1 - a} \quad (0 \leq a \leq 1) \quad (18)$$

corresponding to the roots of Eq. (9) for $\dot{y} = 0$. Evidently, a (for $a \ll 1$) is the amplitude of a quasi-harmonic oscillation.

Using Eq. (16) and Eq. (17), the oscillation described by y as a function of the time t, in units of the period τ, can be written as

$$(1 - a^2)y = 1 + a\sin\left[2\pi x\right.$$

$$\left. + \sqrt{(1 - a^2)[2(y - \frac{1}{2}) - (1 - a^2)y^2]}\right]$$

where

$$x = (t - t_0)/\tau. \quad (19)$$

This equation can be solved for x where the domain of y given by (18) must be taken into account:

$$2\pi x = \arcsin\left[\frac{1}{a}[(1 - a^2)y - 1]\right]$$

$$- \sqrt{(1 - a^2)\left[2\left(y - \frac{1}{2}\right) - (1 - a^2)y^2\right]}. \quad (20)$$

An alternative solution of Eq. (10) as a function of the maximum and minimum amplitudes can be found in the ancient review about variable stars done by Ledoux *et al.* [16], Rosseland [34], Kovács [23] and an exhaustive analysis of the nonlinear adiabatic oscillations of the homogeneous model has been done by Cox [10]. However, in both works the authors have taken no notice of the advantages to express the solution as function of the maximal amplitude only through the parameter a which allows to obtain expressions for the following parameters of a Cepheid: $\Delta R = R_{\max} - R_{\min}$, the maximal pulsation velocity, the equilibrium radius, the mass, the luminosity and the surface gravity, such is being proposed in the present paper.

2.3. Physical considerations for obtaining the stellar parameters

To determine a, as well as M and R_0, in order to obtain $y(x)$ and consequently $y(t)$, the following must be considered: from a practical point of view by measuring the Doppler shift of the spectral lines, the radial velocity $\dot{R}$ (or $\dot{y}$) of the star

oscillation can be calculated and its integration with respect to the time provides, $\Delta R = R_{\max} - R_{\min}$, but according to (18) the relation

$$\frac{\Delta R}{R_0} = \frac{2a}{1 - a^2} \qquad (21)$$

must also be satisfied so that a can be known. Additionally, according to Eq. (10), the maximal radial velocity ($\ddot{y} = 0$) occurs for $y = 1$, so that Eq. (9), (14) and (15) provide also a relation between the maximal velocity of the oscillation and the parameter a:

$$\dot{R}_{\max}^2 = \frac{MG}{R_0} a^2. \qquad (22)$$

Another quantity that can be measured is the period of the pulsation and it is related with the parameter a via Eq. (16). Firstly, solving Eq. (21) for R_0 one obtains

$$R_0 = \frac{1 - a^2}{2a} \Delta R \qquad (23)$$

and substituting this relation into Eq. (16), the following expression for MG is obtained

$$MG = \left(\frac{\Delta R}{2a}\right)^3 \left(\frac{2\pi}{\tau}\right)^2. \qquad (24)$$

Using Eq. (22) and Eq. (24), one obtains another way to determine a:

$$a^2 = 1 - \left(\frac{\Delta R}{\dot{R}_{\max}}\right)^2 \left(\frac{\pi}{\tau}\right)^2. \qquad (25)$$

It is important to note that a is very sensitive to changes in ΔR, $\dot{R}_{\max}$ and τ, and viceversa. According to (25), the value of a can be determined via the observed values of ΔR, $\dot{R}_{\max}$ and τ and using (23) and (24), the values of R_0 and M can be known immediately. But can also be determined via the method proposed in the Sec. 4 of this paper.

2.4. The luminosity

Assuming that the constituent of the star is a plasma, the luminosity can be calculated assuming energy transport by radiation through

$$L_0 = \left(\frac{4\pi}{3}\right)^2 \frac{\sigma c (Am_H)^5}{\sigma_{Th} Z (1 + Z)^4} \left(\frac{G}{5k}\right)^4 M^3 \qquad (26)$$

here σ_{Th} is the Thomson cross section and σ the radiation density constant ($\sigma = 4\hat{\sigma}/c$ where $\hat{\sigma} = 2\pi^5 k^4/15c^2 h^3$ is the Stefan-Boltzmann constant). In thermal equilibrium, and using

$$L_0 = \frac{c}{4} \sigma T_{\text{eff}0}^4 4\pi R_0^2 \qquad (27)$$

the effective temperature, $T_{\text{eff}0}$, can also be determined.

Up to now, the empirical determination of a has been established, in the following we will give a theoretical interpretation of a: since a is associated with the amplitude of the oscillations, a must be related to the forcing mechanism and to the initial conditions of the motion. In this context, one can

assume that the change in the potential energy due to an expansion up to the maximal radius is produced by the radiation pressure (similar to a κ-mechanism) so that

$$E_{\text{pot}}(R_{\max}) - E_{\text{pot}}(R_0) = \int_{R_0}^{R_{\max}} \frac{1}{3} \sigma T_r^4 4\pi R^2 dR \qquad (28)$$

where T_r is the temperature of the radiation, which can be determined through the radiation-matter interaction, given by the adiabatic principle

$$T_r R^2 = T_0 R_0^2. \qquad (29)$$

By substituting (29) on the right hand side of (28) and using (5) and (6) one gets

$$1 - \frac{2}{y_{\max}} + \frac{1}{y_{\max}^2} = \alpha \left[1 - \left(\frac{1}{y_{\max}}\right)^5\right], \qquad (30)$$

where

$$\alpha = \frac{8\pi\sigma}{9} \left(\frac{1}{5k} \frac{Am_H}{1 + Z}\right)^4 M^2 G^3 \qquad (31)$$

then, substituting $y_{\max}$ as function of a in (30), according with (15) or (18), one obtains

$$\alpha = \frac{a^2}{1 - (1 - a)^5}. \qquad (32)$$

The determination of a from α, *i.e.* from the mass M and the chemical composition can be done only numerically. Otherwise, knowing a, one can determine α and therefore, in accordance with (31), also the average of the chemical composition since the mass M is known via another way. The relationship between a and α is a consequence of the assumption implicit in Eq. (29) and as a result of such assumption different values of a would correspond to the same value of the chemical composition expressed in this paper as a function of $(1 + Z)/Am_H$. Now, instead of Eq. (29) we suppose that the adiabatic principle is satisfied for the radiation only, according to

$$T_r R = T_0 R_0 \qquad (33)$$

so that, after evaluating the integral in Eq. (28) and using Eq. (5) and Eq. (6), one gets

$$\frac{1}{y_{\max}} = 1 - 5\alpha,$$

instead of Eq. (30). By substituting $y_{\max}$ from Eq. (18), one obtains the linear relation

$$a = 5\alpha \qquad (34)$$

in place of Eq. (32). Then, knowing the mass and the chemical composition, a is immediately determined. It is important to note that for $a \ll 1$, Eq. (32) reduces to Eq. (34), so that in the limit of $a \ll 1$, both assumptions give the same result.

In Eq. (29) and Eq. (33) a mean value of the temperature in the interior of the star is assumed, according to Eq. (5).

However, keeping in mind that the $\kappa-$mechanism can be present, it seems to be more appropriate to use the radiation temperature on the surface of the star (T_{eff0}) via the relation

$$T_{\text{eff0}}^4 R_0^2 = \left(\frac{4\pi}{3}\right)^2 \frac{(Am_{\text{H}})^5}{\pi\sigma_{\text{Th}}Z(1+Z)^4}\left(\frac{G}{5k}\right)^4 M^3. \quad (35)$$

In accordance with the adiabatic assumption given by Eq. (29) and the evaluation of Eq. (28), one gets from Eq. (35)

$$\frac{a^2}{1-(1-a)^5} = \frac{8}{9}\left(\frac{4\pi}{3}\right)^2$$
$$\times \frac{\sigma(Am_{\text{H}})^5}{\sigma_{\text{Th}}Z(1+Z)^4}\frac{G^3}{(5k)^4}MR_0^2. \quad (36)$$

Otherwise, for the adiabatic assumption given by Eq. (33), one gets

$$a = \frac{40}{9}\left(\frac{4\pi}{3}\right)^2 \frac{\sigma(Am_{\text{H}})^5}{\sigma_{\text{Th}}Z(1+Z)^4}\frac{G^3}{(5k)^4}MR_0^2. \quad (37)$$

The experimental measurements of ΔR, $\dot{R}$ and τ allow the calculation of a, M and R_0 in accordance with Eq. (21), (22) and (25) respectively. After that, the chemical composition can be determined via Eq. (36) or (37).

The temporal behavior of the luminosity can be determined through the Stefan-Boltzmann law given by Eq. (27) and considering once again the adiabatic assumption given by Eq. (33) for the radiation temperature on the surface of the star (thin atmosphere), one obtains for the luminosity

$$L = \left(\frac{c}{4}\sigma T_{\text{eff}}^4 4\pi R_0^2\right)\frac{R_0^2}{R^2} \quad (38)$$

here the quantity in brackets is the equilibrium luminosity, so that considering (8) one gets that

$$L = \frac{L_0}{y^2}. \quad$$

Equation (38) shows that the luminosity takes great values for a small radii and viceversa. This behavior is not in accordance with the observations since a $\pi/2$ phase shift is present in our model: according with the observations, the maximum and the minimum of the luminosity correspond to the same value of the radius. This phase shift can be due to the fact that the $\kappa-$mechanism has not been explicitly taken into account, because in our adiabatic model without damping no permanent excitations of the oscillations are necessary. Hitherto, the relation period-luminosity is given in an implicit way. Indeed, knowing the mass and the mean value of the chemical composition, which can be calculated using the period of the oscillation, the luminosity can be determined in accordance with Eq. (26).

Assuming that the chemical composition is always determined via anyone of the equations given above, then a period-luminosity relation can be approximated for $a^2 \ll 1$. Firstly, solving Eq. (16) for R_0 one gets

$$R_0^3 = \frac{\tau^2(1-a^2)^3}{4\pi^2}MG, \quad (39)$$

and the solution for M of Eq. (26) yields to

$$M = \left[\left(\frac{3}{4\pi}\right)^2 \frac{\sigma_{\text{Th}}Z(1+Z)^4}{\sigma cAm_H A^4}\frac{(5k)^4}{G^4}L_0\right]^{1/3}. \quad (40)$$

By substituting Eq. (40) into Eq. (39) one obtains

$$R_0^2 = \frac{\tau^{4/3}}{4^{2/3}\pi^{4/3}}(1-a^2)^2 G^{2/3}$$
$$\times \left[\left(\frac{3}{4\pi}\right)^2 \frac{\sigma_{Th}Z(1+Z)^4}{\sigma cAm_H A^4}\left(\frac{5k}{G}\right)^4\right]^{2/9} L_0^{2/9}$$

with this expression for the radius of the star, one returns to Eq. (27) and the solution for the luminosity takes the form

$$L_0^{7/12} = \frac{1}{2}T_{\text{eff0}}^3\tau\left(\frac{\sigma c}{\pi}\right)^{7/12}$$
$$\times \left[\left(\frac{3}{4}\right)^2 \frac{\sigma_{\text{Th}}Z(1+Z)^4}{m_H^5 A^5}\frac{(5k)^4}{G}\right]^{1/6}\left(1-a^2\right)^{3/2}. \quad (41)$$

The period-luminosity relation given by Eq. (41) explicitly depends on the surface temperature T_{eff0}, on the chemical composition and on a. However when $a^2 \ll 1$ the luminosity does not depend on a any more and as we show later, it is a very important result. Alternatively, Eq. (41) can be also written as the P-L relation

$$M^* = -4.29log\tau - 4.29log\Lambda + 4.74 \quad (42)$$

where M^* means absolute magnitude and

$$\Lambda = \frac{1}{2L_\odot^{7/12}}T_{\text{eff0}}^3(\frac{\sigma c}{\pi})^{7/12}$$
$$\times \left[\left(\frac{3}{4}\right)^2 \frac{\sigma_{\text{Th}}Z(1+Z)^4}{m_H^5 A^5}\frac{(5k)^4}{G}\right]^{1/6}(1-a^2)^{3/2}. \quad (43)$$

By comparing the slope of the P-L relation given by (42) with that provided by Sandage *et al.* [35] for the galactic calibrators when $[fe/H] = 0.0$, namely $\langle M_B^0\rangle$, $\langle M_V^0\rangle$, $\langle M_I^0\rangle$, the following relative differences between the slopes in degrees units can be found 7.8%, 5.9%, 2.2%.

3. Stability of the method

Following Ripepi *et al.* [33], in this section we want to study the sensitivity of the model to the parameters and data. For convenience and keeping in mind a general application of the model, a normalization of the physical quantities must be introduced: $\mu = M/M_\odot$ describes the mass of the star in units of the solar mass; the normalization of the radius deserves more care, firstly $\xi(t) = R(t)/R_{\text{obs}}$ describes the temporal evolution of the radius of the star $R(t)$ in units of R_{obs} which
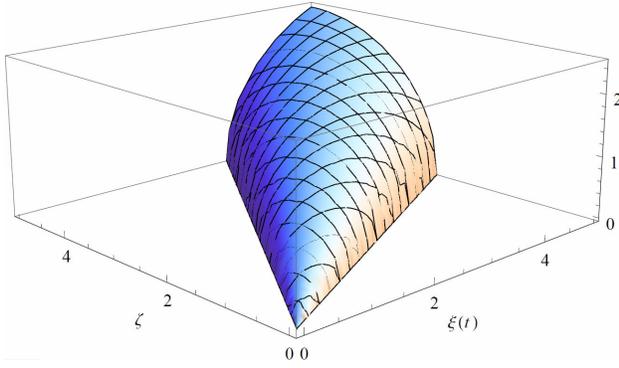
FIGURE 1. Graphics of the Function F(ξ, ζ).

is the radius of the star determined on the basis of observational data *e.g.* via the Baade-Wesselink [46] method or else proposed on the basis of physical considerations. $\chi(t) = R_{\text{obs}}/R_\odot$ characterizes the observed radius of the Cepheid in units of the solar radius. It is relevant to note that R_{obs} is a mean radius. $\zeta = R_0/R_{\text{obs}}$, gives an account of the choice of the hydrodynamical equilibrium radius R_0 in terms of the observed radius R_{obs}. So that ζ as well as χ are changing parameters because of the pulsation of the star.

With the aid of

$$E = -\frac{MG(1-a^2)}{R_0^3}$$

and using the suitable parameters, the differential equation given by Eq. (9) can be written as

$$\dot{\xi}(t) = \left\{\frac{M_\odot G\mu}{R_{\text{obs}}^3\zeta}\right\}^{1/2}\frac{1}{\xi(t)}F(\xi(t),\zeta) \qquad (44)$$

with

$$F(\xi,\zeta) = \left\{2\zeta\xi(t) - (1-a^2)\xi(t)^2 - \zeta^2\right\}^{1/2}. \qquad (45)$$

Figure 1 shows the behavior of $F(\xi(t),\zeta)$ as a function of $\xi(t)$ and ζ with null points given by $\xi_1(t) = \zeta/(1-a)$ and $\xi_2(t) = \zeta/(1+a)$. According to the variational calculus [24,32], the stationary value of the function $F(\xi(t),\zeta)$, considering the variations of $\xi(t)$ because of the pulsation and the possible variations of the parameter ζ due to changes in the empirical determination of the observable radius, appears when $\xi(t) = \zeta = 1/(1-a^2)$ and this allows to linearize the function $F(\xi(t),\zeta)$. Finally, it is important to note that Eq. (45) is very sensitive to changes in a, μ, ζ, $\xi(t)$ and that in our model the parameters ζ and a play a relevant role on the determination of the physical properties of the star.

4. Adaptation of the equations containing the physical stellar quantities for Cepheids

Equations (21) to (25) contains the physical quantities of the Cepheid, namely the amplitude of the pulsation, the surface gravity and mass, whereas the luminosity can be calculated with (41). Taking into account the normalization conditions,

the adiabaticity requirement and using the variational principle the equation of motion for Cepheids can be written as

$$\dot{R}^2 = \frac{M_\odot Gg_{\text{cep}}}{\zeta(\zeta^2 - \zeta)^2\chi R_\odot g_\odot}[2\zeta\xi(t) - (1-a^2)\xi^2(t) - \zeta^2]. \quad (46)$$

Here, one has substituted $\mu/\xi^2(t) = \chi^2 g_{\text{cep}}/g_\odot$ obtained from

$$g_{\text{cep}} = g_\odot \frac{\dfrac{M}{M_\odot}}{\left(\dfrac{R(t)}{R_\odot}\right)^2}$$

where g_{cep} denotes the surface gravity of the Cepheid star and $\log g_\odot = 4.437$ is the surface gravity of the sun [39]. Equation (46) works very well and its deduction is not immediate and trivial (cf. Appendix A). The pulsation amplitudes and masses are also very sensitive to the changes of ζ. Moreover, $\zeta = 1$ corresponds to a very special case, when $R(t) = R_0 = R_{\text{obs}}$ so that the function $F(\xi(t),\zeta)$ linearizes and the perturbations $\delta(\xi(t))$ and $\delta(\zeta)$ remain arbitrary. For the calibration of the model presented in this paper, the adoptions of the observational radial velocity curve and the observational radius R_{obs} are crucial. In the last case, the R_{obs} obtained via an experimental method, *e.g.* the Baade-Wesselink (BW) method and its variants for which the projection factor (p) is a key quantity would be desirable, but to do that is not always possible. The projection factor defined as $v_{\text{puls}} = pv_{\text{obs}}$ [36] is used to convert the observed velocity into the pulsation velocity. Because of the troubles for verifying theoretical results with observations, *e.g.* the observed velocity is often referred to as radial velocity [19] and since we match our pulsation velocity curve with the observational radial velocity curve a mismatch of the results is not excluded but for values of p close to one, one obtains better fits so when $p = 1.27$ [27,20,28] the reliability of the matching is 79%, whereas $p = 1.44$ [19] provides an accuracy of the matching of 69%. However, the matching must be done with the pulsation velocity curve. There are still inconsistencies about p, on one hand lies the fact that p is not constant with respect to the phase and therefore with respect to the time [19,36] and on the other emerges the suggestion that the projection factor is constant and equal to 1.27 ± 0.05 [20]. In any case the basis of our model is not affected and for its application in the present paper one has select 1.27 for δ Cep, FM Cas, SY Cas, RR Lac and 1.31 for Polaris [4].

Even though the selection of R_0 also is not arbitrary and constraining the selection to the domain of feasible values of $R(t)$ the maximal velocity of the pulsation can be calculated from

$$\dot{R}_{\text{max}}^2 = \frac{M_\odot Gg_{\text{cep}}a^2}{\zeta(\zeta^2 - \zeta)^2\chi R_\odot g_\odot}. \qquad (47)$$

Estimations for the amplitudes and the mass can also be elucidated. By substituting R_0 into Eq. (21), one gets

$$\Delta R = \frac{2a\zeta R_{\text{obs}}}{1-a^2} \qquad (48)$$

TABLE I. Measured parameters [row 1 to 5], masses using the $M - \tau$ relations provided by Gieren [row 6 to 9] and physical parameters calculated with the method proposed in the present paper [row 10 to 15] for a sample of five Cepheids.

Stars Parameters	δCep	Polaris	FM Cas	SY Cas	RR Lac
τ(days)	5.3663	3.9729	5.809284	4.07110	6.41629
χ	42.7	46	49	40	52
$Log\, g_{\mathrm{Cep}}$	2.2	2.2	†	†	†
$\Delta R/R_\odot$	†	†	†	†	†
$Log\,(L/L_\odot)$	3.29	3.34	3.45	3.31	3.49
$M_{\mathrm{ev}}/M_\odot$	5.72 ± 0.5	5.30 ± 0.5	5.86 ± 0.5	5.37 ± 0.5	6.03 ± 0.5
$M_{\mathrm{th}}/M_\odot$	6.32 ± 3.3	5.77 ± 3.3	6.47 ± 3.3	5.81 ± 3.3	6.68 ± 3.3
$M_{\mathrm{pul}}/M_\odot$	4.47 ± 2	4.12 ± 2	4.60 ± 2	4.14 ± 2	4.78 ± 2
$M_{\mathrm{wess}}/M_\odot$	5.22 ± 0.9	4.92 ± 0.9	5.34 ± 0.9	4.94 ± 0.9	5.5 ± 0.9
a	0.5	0.05	0.5	0.5	0.5
ζ	0.435	0.435	0.435	0.435	0.435
$M/M_\odot$	7.07	6.87	9.29	10.1	9.07
$\Delta R/R_\odot$	24.8	2.01	28.6	23.1	30.3
$Log\, g_{\mathrm{Cep}}$	2.66	2.97	3.37	3.62	3.56
$Log\,(L/L_\odot)$	3.25	3.82	3.51	3.23	3.54

† Means no available observed data.

with such expression, one can obtain the amplitude of pulsation normalized to the radius of the sun ($\Delta R/R_\odot$) and with that, the mass can be calculated via Eq. (24) but written as

$$M = \frac{1}{G}\left(\frac{\Delta R}{2a}\right)^3 \left(\frac{2\pi}{\tau}\right)^2. \qquad (49)$$

5. Application of the model and results

As illustrative examples of the functionality of the method, one have selected δ Cep, FM Cas, SY Cas, RR Lac and Polaris. The first step is to match Eq. (46) containing the solution (20) with the observational pulsation velocity curve in order to determine the value of the parameter a. For the case of δ Cep, one has chosen the Shane's [38] radial velocity curve and the Wallerstein's [44] over other options because on the one hand, Shane examined 116 blue lines and he did not find definitive evidence of line level effects on the radial velocity curve; besides, his curve compiles the data obtained in three different observation series. Incidentally, the Wallerstein's data for the metallic absorption lines are also in agreement with that of the Shane's curve. More recently, Butler [8] measured iodine velocities and his data matched well with Shane's curve within a standard deviation of 0.289 km/sec . It is important to point out that both curves were obtained with metallic absorption lines only [9]. More recent results published by Bersier *et al.* [6] and Mérand *et al.* [26] deserve special attention because both of them provide more recent pulsation velocity curves, which fit very well with that of Shane and consequently with ours. On the other hand, we have selected the Wallerstein's results because they contain H_α absorption lines which provide changes in the amplitude

of the oscillation which are relevant for the application of our model. Here must be indicated that the Wallerstein's curve is selected only as a control upper bound of the maximal velocity for calculating the surface gravity. Indeed, the maximal radial velocities inferred from the Shane's and Wallerstein's observational radial velocity curves for the metallic absorption lines multiplied by $p = 1.27$ are a little less than the maximal radial velocity provided by the H_α absorption line. Another different thing is to select ad hoc the H_α absorption lines for applying the method because it is known that such lines behave significantly different from the metallic lines due to the large extension of the line forming zone and other physical effects described by Taylor *et al.* [40] and Nardetto *et al.* [30].

For Polaris we have select as radial velocity curve that provided by Dinshaw *et al.* [12] because it has more points than that reported by Hatzes *et al.* [21] and Arellano Ferro [2] although the three curves are simple sine waves. In both cases our pulsation velocity curve matches very well with those obtained experimentally: for δ Cep when $a = 0.5$ and for Polaris with $a = 0.05$. These situations are illustrated in Fig. 2 and 3, where the pulsation velocity $dR(t)/dt$ versus the normalized period x given by Eq. (34) is plotted. For obtaining the curve for Polaris, one has used that of Dinshaw *et al.* [12] as trail. Additionally, the physical parameters obtained in the present paper either for δ Cep FM Cas, SY Cas, RR Lac as for polaris with our proposition are shown in in Table I.

6. Discussion

In the present paper we have outlined a method to determine some physical properties of a pulsating star, namely the mass,
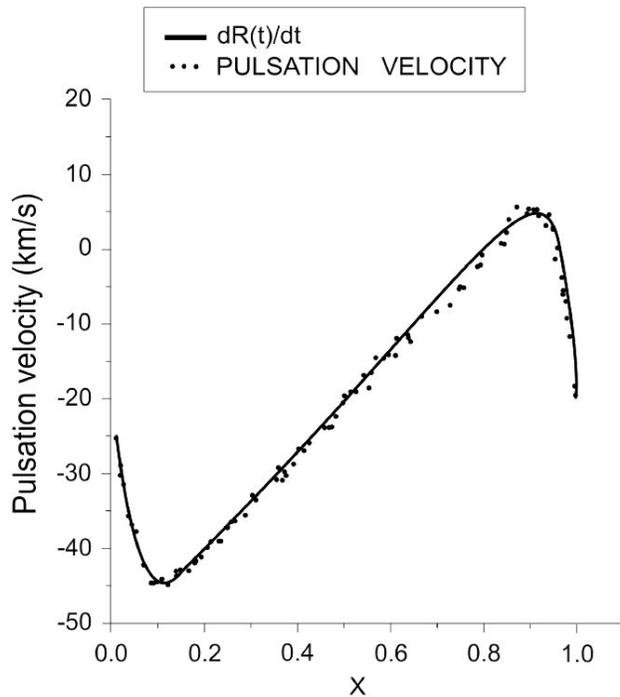
FIGURE 2. Pulsation velocity curve of δ Cep as function of a normalized period x.

FIGURE 3. Pulsation velocity curve of Polaris as function of a normalized period x.

the amplitude of the oscillation, the surface gravity and the luminosity (row 12 to 15 of Table I). As input, one needs experimental data for the period, the radius and the pulsation velocity curve of he star. Our method has properly two free parameters namely a and ζ. The first one is fixed by matching the solution (20) of Eq. (9) with the observational pulsation velocity curve. This is the starting point because a is present in the relations to determine the mass, the amplitude of the oscillation, the surface gravity and the luminosity. To begin with, we have matched our solution with the experimental pulsation velocity curves of δ Cep, and Polaris. In both cases the fitting is successful showing that the pulsation of δ Cep is anharmonic since $a = 0.5$ whereas the pulsation of Polaris is harmonic because $a = 0.05$ (Cf. Fig. 2 and 3). Here, it must be pointed out that the following values of the projection factor 1.27 and 1.31 have been taken into account to obtain the pulsation velocity of the observed radial velocities (the points in Fig. 2 and Fig. 3) for δ Cep and Polaris respectively. Once the value of the parameter a is obtained, the values for the physical parameters can be calculated. Because the pulsation radial velocity curve for δ Cep is similar to that of FM Cas, SY Cas, RR Lac [6], the study is extended to such Cepheids and Table I also shows the results for such sample of Cepheids. We have matched our theoretical pulsation velocity curve with the experimental data provided by Shane for δ Cep, which range from -5 km/sec up 45 km/sec, however an adjustment with points obtained via theoretical models $e.g.$ that supplied by Natale $et\ al.$ [29] is not excluded but the matching must be carried out carefully because a vertical translation is present and the translation constant must be known.
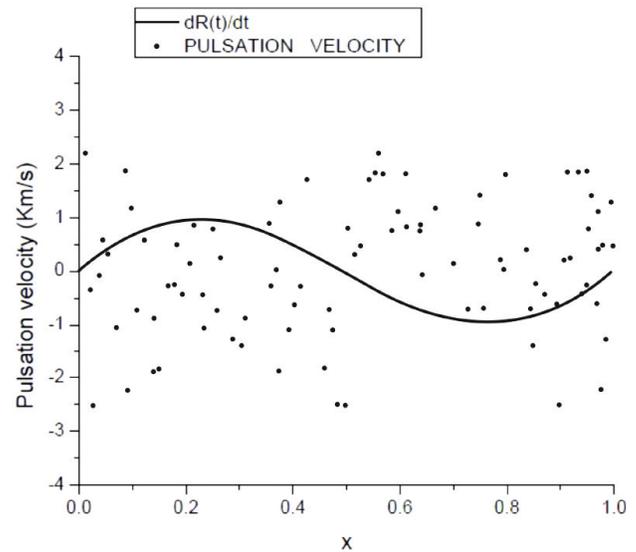
It is desirable to compare our results with observational data or with results obtained by other methods. However, for the sample of Cepheids we are interested on, there are no direct observational measurements and just a few data obtained by other methods. Thus, Table I gives account for either experimental data and those obtained by our method. For knowing the mass we resort to the $M - \tau$ relations provided by Gieren [17] to calculate the values that also show the well known mass discrepancy for Cepheids. For δ Cep, the radius provided by Turner [41], the surface gravity proposed by Andrievsky $et\ al.$ [1] and the luminosity claimed by Kervela $et\ al.$ [22] are included in Table I. For Polaris the experimental quantities are taken from Nordgren $et\ al.$ [31] (the radius), Usenko $et\ al.$ [43] $Log\ g_{\text{Cep}} = 2.2$ (surface gravity) and Turner $et\ al.$ [42] $Log\ L/L_\odot = 3.34$ (luminosity). For the Cepheids FM Cas, SY Cas, RR Lac, the periods provided by Bersier $et\ al.$ [6] have been used and their radii have been calculated via $\log R = 1.244 + 0.587 \log \tau$ [15]. The temperatures have been estimated with $\log T = 3.886 - 0.175\,(B - V)$ where $(B - V) = -0.101 \log^2 \tau + 0.5385 \log \tau + 0.2644$ [25] and the luminosities were obtained through $M_{\text{bol}} = 96^m.57 - 5 \log R - 10 \log T$ [25]. Finally it must be pointed out that the † means that there are not observational data for such parameter.

Besides to the Table I quantities, let us mention for Polaris the mass proposed by Arellano Ferro [3] of $6M_\odot$, and that of Evans $et\ al.$ [13] of $5 \pm 1.5 M_\odot$.

Additionally, Table I contains the mass, the amplitude of the pulsation, the surface gravity and the luminosity of the same sample of stars, obtained by the method proposed in the present paper. Our mass must be properly compared with the observational mass $M_{\text{wess}}/M_\odot$ and with the theoretical mass $M_{th}/M_\odot$. With regard to the last one, the relative differences of the masses are 12% 20% for δ Cep and Polaris respectively, whereas for FM Cas, SY Cas, RR Lac such relative

differences are much bigger. It means that the radii of the last three stars have been measured with large errors. The same evidence can be inferred from the most bigger relative differences between our result and the $M_{\text{wess}}/M_\odot$ which is directly based on the radii measures. In the case of Polaris a direct comparison of our results with that of Arellano Ferro [3] and that of Evans *et al.* [13], one obtains a relative differences of 12% and 5% respectively, which means that our model is well calibrated with the most well observed Cepheids namely δ Cep and polaris. With respect to the surface gravity in a logarithmic scale one can notice that the relative differences for δ Cep and polaris are 17% and 25% respectively.

Owing to the fact that our Eq. (41) for calculating the luminosity does not depend on the radius but on the period, our estimations of the luminosity are in very good agreement with the observational data. The relative differences lay between 1.2% and 2.4% in the logarithmic scale but for Polaris it is 13%. It means that the period has been measured with a very good precision. The amplitudes of the oscillation provided in the present work would represent an estimation that should be experimentally corroborated, although at the present it will not be easy to do it because the instruments for such observation must be extremely refined.

Finally, it must be pointed out that despite of the simple treatment of the pulsation, our model predicts acceptable values for the physical properties of a Cepheid, namely, the mass, the amplitude of the oscillation, the surface gravity and specially the luminosity. However, such predictions must be taken as reference data for observations and more realistic pulsations models but never as a deterministic algorithm because our model does not take into account pulsations in the overtone modes. Moreover, peculiarities as time dependence of the radial velocities, secular decreases or increases of the pulsations amplitudes [14] and mass loss have also been put aside because our primary purpose was to treat the Cepheid problem by a strictly analytic method which could be step by step well understood and fitted with astronomical observations. However, in this framework in a coming paper the evolutive and pulsative masses will bee conciliated.

7. Conclusions

In order to understand the pulsation mechanism of cepheids, we have presented a model based on primary physical principles, namely energy conservation, mechanical adiabaticity and hydrodynamical equilibrium, and we have found that the star pulsates anharmonically if the parameter a takes values close to 1 or harmonically when $a \approx 0$. The model has only two free parameters a and ζ but relies on three observables: period, radius and pulsation velocity. a is coupled with the pulsation velocity curve and ζ with a suitable selection of the hydrodynamic normalization radius. Moreover, the model is presented so that a matching with the observational data is possible, then knowing the oscillation period and the radius, one can predict the amplitude of the oscillation, the mass, the superficial gravity and the luminosity of a pulsating star. In-

deed the adiabatical approximation has been already studied by by Ledoux et al. [16] and Cox [10] but not in the way presented in this paper, which allows the prediction of the stellar parameters mentioned above. For calibrating the method, we have applied it to δ Cep and Polaris and we were able to reproduce the experimental pulsation velocity curves reported by Shane [38] and Wallerstein [44], Bersier et al. [6], Mérand et al. [26] in the first case and that from Dinshaw [38] in the second one. For δ Cep, we get $a = 0.5$ and for Polaris $a = 0.05$, consequently the pulsation of δ Cep is anharmonic and that of Polaris is harmonic. After the matching the treatment is extended to FM Cas, SY Cas, RR Lac and Table 1 gives account for he results which are in very good agreement with the observational data for the case of δ Cep and Polaris $a = 0.05$, and for FM Cas, SY Cas, RR Lac remain as predicted values.

Appendix

A. Constrictions into the pulsation of Cepheids

In this paragraph we focus on the physical constrictions of the method for its application to Cepheids. One begins by writing Eq. (44) as

$$Q(\xi, \dot{\xi}(t), t) = \xi(t)\dot{\xi}(t) - \left\{ \frac{M_\odot G \mu}{R_{\text{obs}}^3 \zeta} \right\}^{1/2}$$
$$\times \left\{ 2\zeta\xi(t) - (1 - a^2)\xi(t)^2 - \zeta^2 \right\}^{1/2} \quad \text{(A.1)}$$

Then a necessary condition for the functional $J(\xi(t))$ of the form

$$J(\xi(t)) = \int\limits_{v_{\min}}^{v_{\max}} Q(\xi, \dot{\xi}(t), t)dt \quad \text{(A.2)}$$

to have an extremum for a given function $\xi(t)$, is that $\xi(t)$ satisfies the Eulers equation

$$Q_{\xi(t)} - \frac{d}{dt}Q_{\dot{\xi}(t)} = 0 \quad \text{(A.3)}$$

where the subscripts denote partial derivatives with respect to the corresponding arguments [18]. Firstly, for $\xi(t)$ Eq. (A.3) provides

$$\xi(t) = \frac{\zeta}{(1 - a^2)} \quad \text{(A.4)}$$

which is in accordance with the stability condition obtained before in paragraph 3. For $\zeta(t)$ and $\chi(t)$, the Eulers condition reduces to the differential equation

$$\frac{M_\odot G \mu}{2}\left(\frac{M_\odot G \mu}{\zeta(t)R_{\text{obs}}^3} \right)^{-\frac{1}{2}} \frac{1}{\zeta(t)R_{\text{obs}}^3}\left(\frac{1}{R_{\text{obs}}}\frac{dR_{\text{obs}}}{d\zeta(t)} + \frac{1}{\zeta(t)} \right)$$
$$\times F^2(\zeta(t), \xi(t)) - \left(\frac{M_\odot G \mu}{\zeta(t)R_{\text{obs}}^3} \right)^{\frac{1}{2}} (\xi(t) - \zeta(t)) = 0 \quad \text{(A.5)}$$

Using the condition given by Eq. (A.4) and solving for $\chi(t)$ one obtains

$$\chi(t) = \zeta^2(t) - \zeta(t) + \text{const} \qquad (A.6)$$

In view of the fact that the star is never motionless during the pulsation and during the observation, the assumption that ζ as well as χ must be a function of time is the most reasonable, so that for $const = 0$ and since ζ is always ≤ 1 to first approximation the dominant term in the last equation is $-\zeta(t)$ but in order to recover the physical meaning of the equation of motion, the expression $\chi^2(t) = (\zeta^2(t) - \zeta(t))^2$ must be taken into account, which is also in accordance with the adiabatical assumption (cf. Eq. 3). Therefore Eq. (46) describes the pulsation of a Cepheid satisfying the stability condition for the time functions $\xi(t)$, $\zeta(t)$ and $\chi(t)$ as well as the fact that the adiabatical radial pulsation of the star comply with the least action principle.

Acknowledgments

1. S. M. Andrievsky, R. E. Luck, and V.V. Kovtyukh, *AJ* **130** (2005) 1880.

2. A. Arellano Ferro, *ApJ* **274** (1983) 755.

3. A. Arellano Ferro, *MNRAS* **209** (1984) 481.

4. A. Arellano Ferro and P. Rosenweig, *MNRAS* **315** (2000) 296.

5. W. Baade *Astronomishe Nachrichten* **228** (1926) 359.

6. D. Bersier, G. Burki, M. Mayor, A. Duquennoy, *A&A* **108** (1994) 25.

7. J.R. Buchler, P.A. Yecko, and Z. Kolláth, *A& A* **326** (1997) 669.

8. R.P. Butler, *ApJ* **415** (1993) 323.

9. R.P. Butler, *ApJ* **480** (1997) 767.

10. J.P Cox,. *Rep. Prog. Phys.* **137** (1974) 563.

11. A. Cox, *Ann. Rev. Astron. Astrophys.* **18** (1980) 15.

12. N. Dinshaw Matthews, J. M. Gordon, A. Walker, H. Hill, M. Grant, *AJ* **98** (1989) 2249.

13. N.R. Evans *et al.*, Proceedings IAU Symposium No. 240, W.I. Hartkopf, E.F. Guinan & P. Harmanec, eds. (2006).

14. S. G. Engle, E. F. Guinan, G. M. Harper, H. R. Neilson, and N. R. Evans, *ApJ* **794:80** (2014) 1.

15. J. D. Fernie, *ApJ* **282** (1984) 641.

16. S. Fluegge, Handbuch der Physik Band LI, Atrophysik II: Sternaufbau, Sringer-Verlag, Berlin, Goettingen, Heidelberg,(ed), (1958) pp 353.

17. W. P. Gieren, *A&A* **225** (1989) 381.

18. I.M. Gelfand and S.V. Fomin, *Calculus of variation*, Dover Publication (2000) Inc., Mineoola, N. Y.

19. G. Guiglion *et al.*, *A&A* **550** (2013) L10.

20. M. A. T. Groenewegen, *A&A* **474** (2007) 975.

21. A.P. Hatzes W. D. Cochran *AJ* **120** (2000) 979.

22. P. Kervela, D. Bersier, D. Mourand, N. Nardetto, and V. Coudé du Foresto, *A&A* **423** (2004) 327.

23. G. Kovács, *ASPC* **11** (1990) 141K.

24. C. Lanczos, *The variational principles of mechanics*, Dover Publications, Inc., (New york 1986).

25. I. N. Latyshev, *Astrophysika* **4** (1968) 451.

26. A. Mérand, *et al.*, *A&A* **438** (2005) L9.

27. N. Nardetto, A. Fokin, D. Pourand, P. Mathias, Ph. Kervella, and D. Bersier, *ApJ.* **428** (2004) 131.

28. A. Mérand, *et al.*, *A&A*. **438** (2005) L9.

29. G. Natale, M. Marconi, J. Bono, *ApJ.* **674** (2008) L93.

30. N. Nardetto, J. H. Groh, S. Kraus, F. Millour, and D. Gillet, *A&A*. **489** (2008) 1263.

31. T. E. Nordgren, J. T. Armstrong, M. E. Germain, R. B. Hindsley, A. R. Hajian, J. J. Sudol, C. A. Hummel, *ApJ* **543** (2000) 972.

32. M. J. D. Powell, *The computer Journal* **5** (1962) 147.

33. V. Ripepi, F. Barone, L. Milano, and G. Russo, *A&A* **318** (1997) 797.

34. S. Rosseland, *The pulsation theory of variable stars 1949*, Oxford at the Clarendon Press, Oxford, Great Britain.

35. A. Sandage, R. A. Bell, M. J. Tripicco, *ApJ.* **522** (1999) 250.

36. C. N. Sabbey, D. D. Sasselov, M. S. Fieldus, J. B. Lester, K. A. Venn and R. P. Butler, *ApJ* **446** (1995) 250.

37. L.I. Sedov, *Similarity and dimensional methods in mechanics*, 10th edition, CRC Press, Boca Raton, Ann Arbor, London, Tokyo 1993.

38. W. W. Shane, *ApJ* **127** (1958) 573

39. B. Smalley, *Mem. S.A. It. Suppl.* **8** (2005) 130.

40. M. M. Taylor, M. D. Albrow, A. J. Booth, and P. L. Cottrell, *MNRAS* **292** (1997) 662.

41. D. G. Turner, *AJ* **96** (1988) 1565.

42. D. G. Turner, V. V. Kovtyukh, I. A. Usenko, and N. I. Gorlova, *ApJ* **762** (2013) L8.

43. I. A. Usenko, A. S Miroshnichenko., V. G. Klochkova, M. V. Yushkin, **362** (2005) 1219.

44. G. Wallerstein, *PASP*. **91** (1979) 772.

45. A.J. Wesselink,, *Bull. Asrt. Inst. Netherlands*, **10** (1946) 91.

46. P.A. Yecko, Z. Kolláth, and J.R. Buchler, *A&A* **336** (1998) 553.

47. P.A. Yecko, Z. Kolláth, *ASPC* **135** (1998) 94.