

# Deformation of an elastic cylindrical tube filled with water and a dry granular material

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Received 12 June 2015; accepted 17 November 2015

The deformation of a vertical thin-walled tube filled with a liquid or a non-cohesive granular material is investigated theoretically and experimentally. Experiments using latex tubes filled with water or spherical glass beads have been conducted and the results were compared to theoretical results derived from the Janssen model. The results obtained suggest that soft elastic tubes could provide a simple and convenient means to investigate the forces that arise within different materials.

*Keywords:* Granular systems; application of continuum mechanics to structures; theoretical studies (static elasticity).

PACS: 45.70.-n; 46.70.-p; 46.25.Cc

## 1. Introduction

In 1895, H. A. Janssen, a German engineer concerned with the structural design of silos for corn storage, found the laws giving the pressures of dry, non-cohesive granular media on the vertical walls and the base of a vertical bin [1]. These pressures are not the same, contrary to the case a normal liquid. The model put forward by Janssen, and extensions thereof, is widely used today for the design of silos containing granular materials [2-5].

Pressures in quiescent liquids and in granular materials at rest are quite different. Consider a vertical cylindrical container filled with a granular material. The vertical pressure on an horizontal plane at a certain depth below the surface of the material does not increase linearly with depth, as it would do for a normal liquid. Instead, the Janssen's model predicts that this pressure tends to a constant value independent of depth. The origin of this behavior is in the static friction between the grains and the walls of the container. Due to this friction, the container walls can support part of the weight of the material.

The validity and possible shortcomings of the simple theory underlying the original Janssen model have been extensively tested experimentally and numerically, but, essentially, in the context of the pressure and force measurements [2,4-6].

The purpose of this work is to explore the possibility of using the elastic deformation of the wall of a container to probe the distribution of pressure in the material filling the

container. Specifically, we analyze how an elastic, latex-walled vertical tube is deformed by the pressure of a column of liquid or by the pressure of a column of granular material that fills the tube up to a certain height  $H$ .

Latex is well suited for our purposes because it is a soft elastic material with a very wide regime of linear elastic deformation [7]. This property allows visualizing, in a non-destructive testing, the deformations originated by the hydrostatic pressure of a liquid or the stresses of a granular material. Comparison between both deformations clarifies the way the pressure distributes in each material.

The determination of deformations is also a preliminary step toward an analysis of the dynamic pressures and wall deformations arising during the flow of a granular material under the action of gravity. This is a very difficult problem due to the strong interaction between the silo structure and the moving granular material [8] and it will not be tackled here.

The paper is organized as follows. In next section the Janssen's original description of the pressures in a vertical tube is presented. Section 3 is devoted to the analysis of the elastic deformation of a thin-walled tube made of an elastic material due to a hydrostatic or granular load. Section 4 reports a series of experiments where the deformation of elastic tubes filled with water or with spherical glass beads is measured and compared with the theoretical results. Finally, the main conclusions of the work are summarized in Sec. 5.

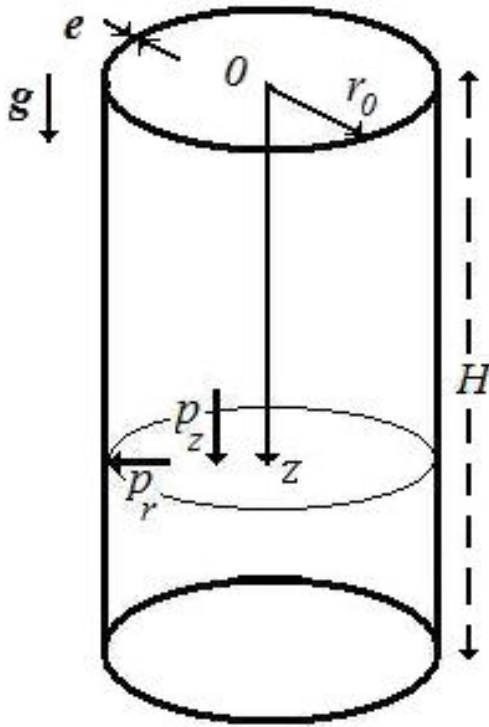


FIGURE 1. Schematic of the tube filled with a granular material. The dimensions of the tube and the vertical ( $p_z(z)$ ) and horizontal ( $p_r(z)$ ) pressures are indicated.

### 2. Pressures of a granular medium in a vertical container

Consider a cylindrical vertical tube of radius  $r_0$  filled with a dry granular material to a certain height  $H$  (Fig. 1). Take a horizontal section of the tube at a depth  $z$  in the granular material. According to Janssen’s model, the force exerted per unit area of this section by the material above it on the material below is a vertical pressure  $p_z(z)$ . This pressure is not equal to the horizontal pressure  $p_r(z)$  of the material on the wall of the tube at the same depth, but the two pressures are linearly related [1-5]:

$$p_r(z) = Kp_z(z), \tag{1}$$

where  $K$ , named the Janssen parameter, is a constant independent of  $z$  that characterizes the conversion of vertical stress into horizontal stress within the granulate.

The horizontal pressure acting on the wall of the tube causes a friction stress  $\tau$  between the wall and the granular material. A simple balance of the forces acting on a slab of the granular material reads

$$\pi r_0^2 \frac{dp_z}{dz} = \pi r_0^2 \rho g - 2\pi r_0 \tau, \tag{2}$$

where  $\rho$  is the bulk density of the granular material, assumed to be independent of  $z$ , and  $g$  is the acceleration due to gravity.

In the Janssen’s model, the vertical friction stress is assumed to be given by Coulomb’s law,  $\tau = \mu_w p_r$ , with a constant static friction coefficient  $\mu_w$ . The solution of Eq. (2) with the condition  $p_z(0) = 0$  is then [1-4]

$$p_z(z) = P_J(z) \tag{3}$$

with

$$P_J(z) = \rho g \lambda \left[ 1 - \exp\left(-\frac{z}{\lambda}\right) \right] \quad \text{and} \quad \lambda = \frac{r_0}{2\mu_w K}. \tag{4}$$

Equation (4) shows that the vertical pressure first increases linearly with  $z$ , as  $p_z(z) \approx \rho g z$  for  $z/\lambda \ll 1$ , and tends to the limiting value  $\lambda \rho g$  for  $z/\lambda \gg 1$ . The length  $\lambda$  is the characteristic size of the region where the pressure undergoes this transition. In laboratory columns  $\lambda \cong 1$  cm [9], so that the wall of the container supports most of the weight of the grains when  $H \gg 1$  cm.

### 3. Deformation of the elastic wall

The deformations caused by the horizontal pressure  $p_r(z)$  and the vertical shear stress  $\tau$  depend on the mechanical properties of the wall of the tube. These deformations are computed here for a thin-walled open tube in the framework of the linear theory of elasticity. Let  $e \ll r_0$  be the thickness of the wall. The radial pressure  $p_r$  produces an approximately uniaxial state of stress in which the only nonzero stress component is the circumferential traction  $p_r r_0 / e$ ; see, e.g., Ref. 10. Hence, by Hooke’s law,  $E \delta r / r_0 = p_r r_0 / e$ , where  $E$  is the elasticity modulus of the material of the wall and  $\delta r = r(z) - r_0$  is the variation of the tube radius. Thus,

$$r(z) = r_0 \left( 1 + \frac{p_r(z)}{eE/r_0} \right). \tag{5}$$

When the tube is filled with a liquid, the hydrostatic pressure referred to the pressure of the gas outside the tube is  $p_r(z) = \rho g z$  [11], which leads to a linear variation of the dimensionless radius,  $r(z)/r_0$ , of the tube with depth,

$$\frac{r(z)}{r_0} = \left( 1 + \frac{\rho g r_0 z}{eE} \right). \tag{6}$$

The volume of the deformed tube filled with liquid is

$$V = \int_0^H \pi r^2 dz = \frac{\pi r_0 e E}{3 \rho g} \left[ \left( 1 + \frac{\rho g r_0 H}{eE} \right)^3 - 1 \right], \tag{7}$$

which is a relation between the volume of liquid and its height.

When the tube is filled with a granular medium, the Janssen model (1) and (3) predicts the exponentially varying dimensionless radius

$$\begin{aligned} \frac{r(z)}{r_0} &= \left[ 1 + \frac{K r_0}{eE} P_J(z) \right] \\ &= \left( 1 + \frac{K r_0 \rho g \lambda}{eE} [1 - \exp(-z/\lambda)] \right), \end{aligned} \tag{8}$$

For large values of  $z/\lambda$ , the non dimensional radius tends to the constant value

$$\frac{r(z)}{r_0} \rightarrow \left[ 1 + \frac{\rho g r_0^2}{2\mu_w E e} \right]. \quad (9)$$

The volume of the deformed tube can be easily computed but its analytical expression is rather cumbersome and will not be reproduced here.

#### 4. Experiments

In this section we describe and analyze some experiments performed to characterize the deformation of soft elastic tubes filled with water or with spherical glass beads, and to measure the compression force acting on the wall in the second case. The experimental results are compared with the theoretical results of the previous sections. Latex tubes were used because of the very wide range of linear deformation displayed by this material [7]. The tubes used in the experiments are 0.20 m long with wall thickness  $e = 7 \times 10^{-5}$  m and internal radius  $r_0 = 1.48 \times 10^{-2}$  m. The elasticity modulus is  $E = 874.5$  kPa [7]. The tubes are closed at the bottom and open at the top. They were filled with water or with spherical glass beads of mean diameter  $d_g = 1$  mm by gently pouring these materials from top. Solid toroidal disks of radius slightly larger than  $r_0$  were fitted at the two ends of the tubes to allow clamping them to external supports and stretching them (see Fig. 2). These disks cause deformations near the ends of the tubes that are not accounted for in the previous section. The region where the experimental results can be meaningfully compared to the theoretical predictions

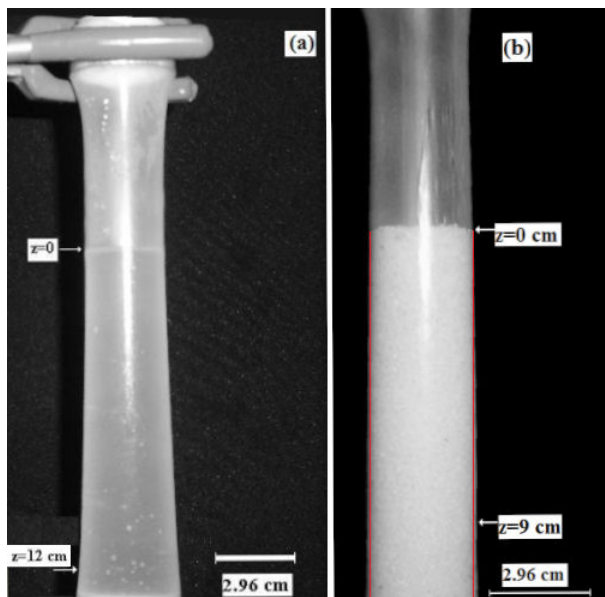


FIGURE 2. Pictures of deformed latex tubes filled with (a) water and (b) spherical glass beads. Red-coloured vertical lines were depicted to show the approximately symmetric shape of the pipe and its relatively small deformation.

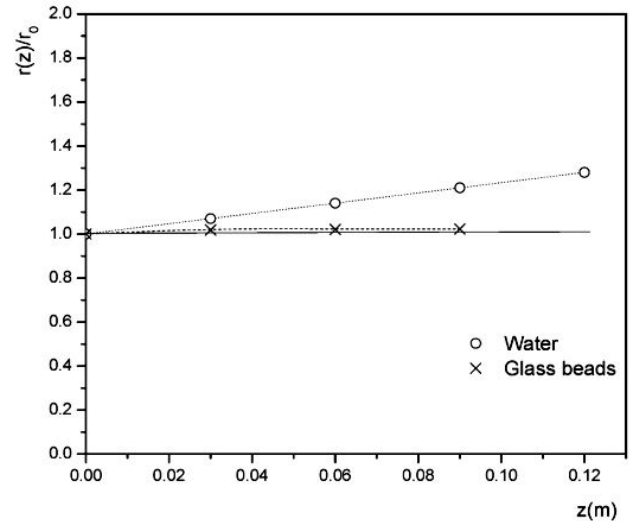


FIGURE 3. Theoretical deformation profiles,  $r(z)/r_0$  as a function of  $z$ , for a latex tube filled with water (dotted straight line) obtained from Eq. (6) and glass beads (dashed curve) obtained from Eq. (8). Symbols  $\circ$  correspond to the mean values of measurements of the shape of tubes filled with water and symbols  $\times$  correspond to mean values of measurements of the shape of tubes filled with glass beads. The line  $(r(z)/r_0) = 1$  corresponds to the undeformed tube. Errors are of the size symbols.

extends from the upper level of the material ( $z = 0$ ) to about  $z = 0.12$  m in the case of water and  $z = 0.09$  m in the case of glass beads. These points are marked by arrows in Fig. 2.

Typical pictures of tubes filled with water and with glass beads are shown in Figs. 2(a) and 2(b), where clear differences between the two cases are apparent. Water causes a radial deformation that increases approximately linearly with depth, while the profile of a tube filled with glass beads is nearly uniform over most of its length. The explanation of this difference lies in the different manners in which the two material distribute the pressures.

Figure 3 shows the experimental profiles of the deformed tubes extracted from Fig. 2(a) and compares them with the theoretical predictions (6) and (8). In the case of water ( $\rho = 1000$  kg/m<sup>3</sup>; circles in Fig. 3), the agreement between the theoretical (Eq. (6)) and the experimental results is excellent (we consider the fit as valid up to  $z = 12.00$  cm because no far from this point the deformation induced by the clamp may affect the pipes shape), showing that the linear theory of elasticity gives a good approximation to the deformation induced by the hydrostatic pressure.

In the case of glass beads, some properties of the granular material must be determined or be specified before Eq. (8) can be used. The beads are made of glass of density  $\rho_g = 2500$  kg/m<sup>3</sup>. The bulk density is  $\rho = \eta\rho_g$ , where  $\eta$  is a packing factor which for a granular column has been reported to be about  $\eta = 0.64$  for the case of dense packing of monodisperse spherical grains poured from top [12]. Consequently, the bulk density used in the calculation is  $\rho = 1600$  kg/m<sup>3</sup>. The friction coefficient between the glass beads and the latex wall was measured with the tilting plate

method, *i.e.*, by laid a thin layer of the glass beads onto a horizontal plate covered with latex and after the plate is slowly tilted and the angle recorded at which the layer of grains slides off [13]. It was found that  $\mu_w = 1.27$ . Finally, the value of the Janssen parameter is determined from the correlation  $K = 1.1(1 - \sin \phi)$ , where  $\phi$  is the angle of internal friction [5]. In our experiment we found that  $\phi = 32^\circ$  [14] which was measured with the heap poured on flat surface method [13], therefore  $K = 0.51$  and thus  $\lambda = 1.14$  cm.

Figure 3 shows that the experimental deformation measured (by using a commercial software of digital pictures to compute the shape) on Fig. 2(b) compares reasonably with the theoretical prediction (8), when the aforementioned values of the properties of the medium are used. In Fig. 2(b) the red-coloured vertical lines were depicted to show visually the approximately symmetric shape and the small deformation of the elastic pipe. In this latest case a slight pixelation (individual pixels are clearly discernible along the vertical direction) is observed on the latex pipe image, however, on average, mean values of measurements fit reasonably the theoretical profile (dashed line), which we consider as valid up to  $z = 9.00$  cm due to presence of the clamp no far from

this point. Other possible origin of the apparent non uniform deformation of the tube could be due to the inhomogeneity of the thickness along its length, which is inherent to the manufacture process of this type of commercial material (latex condom).

## 5. Summary

An experimental and theoretical study has been presented of the deformation of a vertical tube made of a soft elastic material and filled with a liquid or a dry granular material. When the tube is filled with a liquid, the measured deformation increases linearly with depth in the liquid, in good agreement with the prediction based on Hooke's law. When the tube is filled with a granular material, the deformation rapidly levels to a constant value with increasing depth. This result is in line with the prediction of Janssen's model.

## Acknowledgments

Authors acknowledge to the anonymous referee for helpful comments on an earlier version of the manuscript.

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