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Entropy production: evolution criteria, robustness and fractal dimension

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It was proved through Rössler model, where the funnel case is more robust tan spiral chaos, the entropy production per unit time is a Lyapunov's function on the space of the control system parameters. It was established the conjecture of entropy production fractal dimension. The current theoretical framework will hopefully provide a better understanding of the relationship between thermodynamics and nonlinear dynamics and contribute to unify theses through complex systems theory.

Keywords: Irreversible thermodynamics; complex systems; fractal dimension.

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1. Introduction

The unification of nonlinear dynamics and complexity through thermodynamics is already a challenge despite the many efforts trying to reach this goal [1,2]. Presently, it is needed to make an effort to develop formalism for thermodynamics of complex processes.

The aim of this work is to extend the thermodynamics formalism previously developed [3,4] and offers an approximation to the unification of nonlinear dynamics and complexity through thermodynamics. The manuscript is organized as follow: Section 2 we propose the relation of entropy production with the Lyapunov exponent spectra is a Lyapunov function. Moreover, it is a type of measure of dynamical system robustness. Section 3, a conjecture analogous to Lyapunov dimension is proposed to define a fractal dimension of entropy production as a method to measure the complexity of a dynamical system. Finally, some concluding remarks are presented.

2. Entropy production, Lyapunov exponent spectra and Lyapunov function

The seminal work of Nose-Hoover [5] and of more recent work [6] have showed that entropy production per time unit $\dot{S}_i$ is related with the Lyapunov exponent spectra λ_j through a relation

$$\frac{dS_i}{dt} \equiv \dot{S}_i \approx -\sum_j \lambda_j > 0 \tag{1}$$

The Eq. (1) is per se a natural link between the thermodynamics of irreversible processes formalism [7] and nonlinear dynamics [8], without the need of to know if the dynamical system is far or near the equilibrium.

In previous works [3,4], we showed that entropy production per time unit is a Lyapunov function by its dependence on control parameters. This dependence can be exemplified by numerical experiments with Rössler model (Eq. 2) [9] for some distinct values of control parameters (Table I).

$$\dot{x} = -y - z \quad \dot{y} = x + ay$$
$$\dot{z} = b + (x - c)z \tag{2}$$

As can be seen (Table I) there is a drastic dependence of the entropy production rate on the control parameters. This show our thesis [3,4] that the entropy production per time unit is a Lyapunov function that depends on control parameters. These parameters are constants along all the orbit of the ordinary differential equations system. We calculate Lyapunov spectrum and $\dot{S}_i$ for each orbit with constant parameters.

About the specific case of Rössler model, it is known that its dynamics shows two levels of complexity in its robustness: the spiral chaos and funnel chaos [12]. These chaos types depend on the control parameters values. In another work, we showed that funnel chaos is more robust that spiral chaos [13].

So, it can be showed how the entropy production per time unit, as an extremal criterion, fulfills the necessary and sufficient conditions of a Lyapunov function [14], such that:

$$\dot{S}_i = f(\Omega) > 0 \tag{3}$$

TABLE I. Lyapunov exponents and entropy production per time unit of the Rössler model for some distinct values of control parameters and fixed $b = 0.20$

control parameters	λ_1	λ_2	λ_3	$\dot{S}_i$
$a = 0.1$				
$c = 14$	0.072	0	-13.79	13.718
$c = 18$	0.123	0	-25.79	25.67
$a = 0.15$				
$c = 10$	0.130	0	-14.1	13.967
$c = 14$	0.019	0	-25.5	25.48
$a = 0.2$				
$c = 5.7$	0.064	0	-4.98	4.918
$c = 14$	0.167	0	-25.26	25.1

For the numeric integration of the ordinary differential equations was used the Gear algorithm for stiff equations in Fortran, double precision and tolerance of 10^{-8} [10]. The system was compiled with Open Watcom v1.4 (www.openwatcom.org). The Lyapunov exponents were computed with the Wolf algorithm in Fortran [11].

where Ω is the control parameters vector (a, b, c). The Eulerian derivative of (3) has to fulfill:

$$\frac{d\dot{S}_i}{dt} = \frac{d\dot{S}_i}{d\Omega}\frac{d\Omega}{dt} \leq 0; \tag{4}$$

$\dot{S}_i = f(\Omega)$ is the Lyapunov function of the fixed point Ω_0 of a system of ordinary differential equations $\dot{\Omega} = g(\Omega)$, such as $\Omega \in P$ and $P \subset i^n$, where P is the parameters space of the system of ordinary differential equations $\dot{x} = h(x)$, $x \in i^m$ (as the Rössler system). And we know that $\dot{S}_j = -\sum_{j=1}^{m} \lambda_j$, where λ_j is the jth Lyapunov exponent of $\dot{x} = h(x)$.

If we fix $b = 0.1$ and $c = 18$ and let a to increase monotonically in time, we have:

$$\frac{d\dot{S}_i}{dt} = \frac{d\dot{S}_i}{da}\frac{da}{dt} \leq 0; \tag{5}$$

The control parameter a is linked with the evolution of the spiral chaotic behavior to a funnel one [12]; as the value of is growing so the robustness of the system is growing too [13].

Because $da/dt > a$ during the evolution of the spiral chaotic behavior to a funnel one, this implies $d\dot{S}_i/da < 0$ as show in Fig. 1,

This way, it can be seen that the entropy production per time unit not only satisfies Lyapunov function conditions; moreover, it is a magnitude to quantify the dynamical system robustness [15].

3. Kaplan-York dimension and entropy production

Fractal dimension is one the most important properties of an attractor [16], and it is a measure of the dynamical system complexity. A simple way to compute fractal dimension is

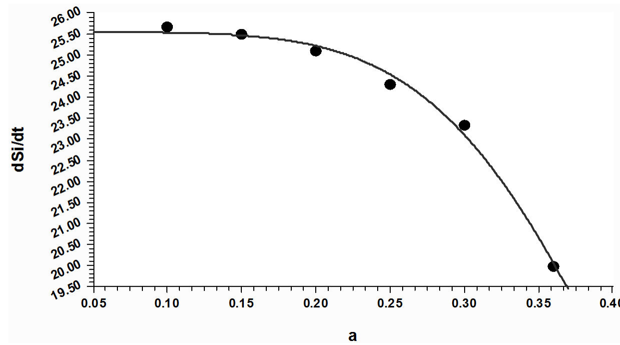


FIGURE 1. The entropy production per time unit vs. the control parameter a in Rössler model [8].

through the Lyapunov D_L or Kaplan-York dimension [17]. It is calculated from the Lyapunov exponents λ_j:

$$D_L = j + \frac{\sum_{i=1}^{j} \lambda_i}{|\lambda_{j+1}|}, \tag{6}$$

Where j is the largest integer number for which

$$\lambda_1 + \lambda_2 + \cdots + \lambda_j \geq 0$$

By analogy with the Eq. (6), we can establish the following conjecture: The fractal dimension of entropy production is defined as:

$$D_{\dot{S}_i} = j + \frac{\dot{S}_i}{(\sum_{i=j+1}^{n} \lambda_i)} \tag{7}$$

where the entropy production per time unit $\dot{S}_i$, is evaluated from Eq. (1), n is the number of all Lyapunov exponents, j is the same as in Eq. (6) (i in $\dot{S}_i$ is not an index, the symbol $\dot{S}_i$ stands for entropy production per time unit).

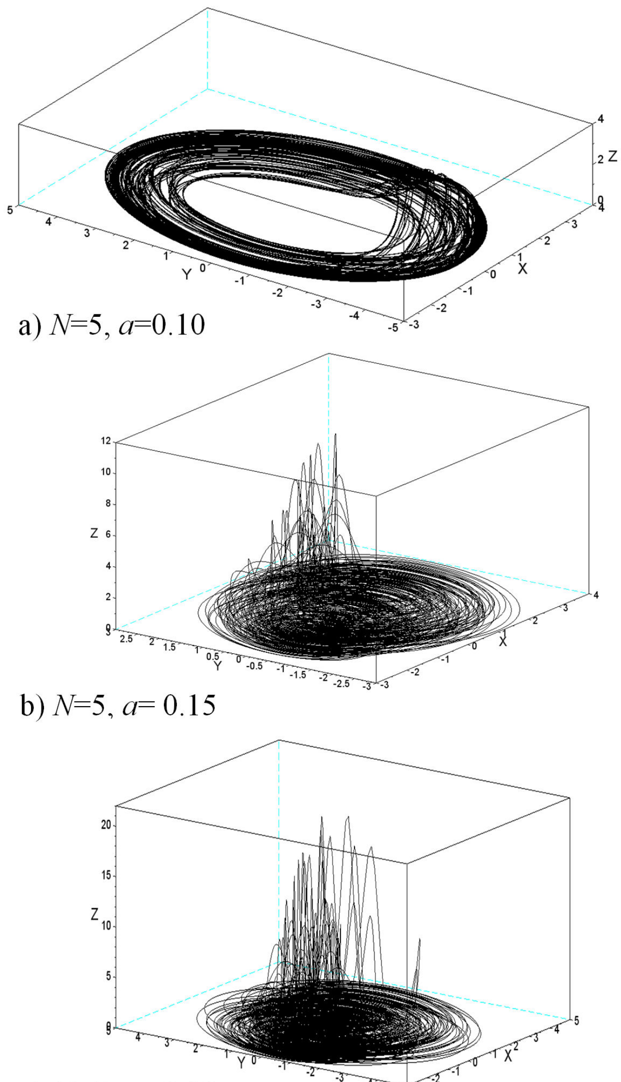
As an example, we used the Baier-Sahle model [18], a N-dimensional model of ordinary differential equations (see Eq. 8). This model is a generalization of the Rössler model. The Baier-Sahle model shows varied levels of complex behavior (see Fig. 2), including chaos and hyperchaos.

$$\dot{x}_1 = -x_2 + ax_1 \qquad \dot{x}_i = -x_{i-1} - x_{i-1} \tag{8}$$
$$\dot{x}_N = e + bx_N(x_{N-1} - d)$$

TABLE II. Lyapunov dimension D_L and entropy production dimension $D_{\dot{S}_i}$, for the N-dimensional de Baier-Sahle model [18] ($b = 4$, $d = 2$, $e = 0.1$).

$N(a)$	$\#(\lambda_i > 0)$*	D_L	$D_{\dot{S}_i}$
5($a = 0.10$)	1	2.704	2.9977
5($a = 0.15$)	2	4.006	4.9937
5($a = 0.20$)	3	4.012	4.9900
7($a = 0.32$)	4	6.026	6.9740
9($a = 0.30$)	6	8.004	8.9959

*number of positive Lyapunov exponents



a) N=5, a=0.10

b) N=5, a= 0.15

c) N=5, a= 0.20

FIGURE 2. Three dimensional projections of five-dimensional Baier-Sahle System, varying parameter a. $X = x_1$, $Y = x_2$ and $Z = x_5$.

Table II shows the values of Lyapunov fractal dimension and those of entropy production fractal dimension for the Baier-Sahle model [18].

As can be seen, both fractal dimensions grows in proportion with the growing of the number of positive Lyapunov exponents $\#(\lambda_i > 0)$. This way, the entropy production fractal dimension is a measure of system complexity [19] and robustness [20]. Figure 2 shows projections of the five-dimensional Baier-Sahle system. It can be seen the apparent increase in complexity.

4. Conclusions and remarks

In summary, in this paper we found:

1. It is shown how the rate of entropy production evaluated through the spectrum of Lyapunov exponents represents a Lyapunov's function depending on the control system parameters. In fact it represents a physical magnitude which measures the robustness [15] of the dynamical system.

2. In the same way of the Lyapunov fractal dimension, it was established a conjecture and it was defined an Lyapunov entropy production fractal dimension which is a measure of complexity and robustness [15,21] of the dynamical systems.

The current theoretical framework will hopefully provide a better understanding of the relationship between thermodynamics and nonlinear dynamics and contribute to unify theses through complex systems theory.

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