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FOUNDATIONS FOR QUANTUM MECHANICS. RESULTS AND PROBLEMS†

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ABSTRACT

The conceptual problems of quantum mechanics, which over the last twenty years have led to the development of other approaches, are outlined and briefly analyzed to show why only stochastic theories could be acceptable. The formal stochastic theories, in which no physical basis for the stochastic process is presupposed, are discussed; some of their weaknesses are adduced to justify the need for physical stochastic theories. Of these, stochastic electrodynamics is chosen for more detailed discussion as by far the most successful. Its achievements so far are outlined, as are the difficulties it still faces. It is concluded that its mathematical formalism is still not adequate to permit the full exploitation of its conceptual framework, which appears to be sound.

RESUMEN

Se delinean y analizan brevemente los problemas conceptuales de la mecánica cuántica que en los últimos veinte años han provocado enfoques alternativos, con el fin de mostrar porqué sólo teorías estocásticas pueden ser aceptadas. Se discuten las teorías estocásticas formales que no presuponen ninguna imagen física para el proceso estocástico; se

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exhiben algunas de sus debilidades con el fin de subrayar la necesidad de teorías estocásticas físicas. De éstas se escoge la electrodinámica estocástica para un examen más detallado por ser de lejos la más exitosa. Se bosquejan sus logros hasta la fecha, así como las dificultades que aún ha de enfrentar. Se concluye que su formalismo matemático todavía no permite explotar a fondo su marco conceptual, el cual se ve sólidamente establecido.

1. CONCEPTUAL DIFFICULTIES OF QUANTUM MECHANICS

Almost from the first days of quantum mechanics, the debate concerning its interpretation and correct foundation has raged, and it shows no signs of flagging today. The majority of physicists —and certainly almost all writers of textbooks— adhere in one way or another tc what is known as the Copenhagen or orthodox conception; most textbooks, moreover, present it in rather dogmatic fashion, leaving the young generation of physicists in ignorance of the fact that there is a small but not negligible group actively engaged in developing alternative views. It is the purpose of this paper to present a brief review of what appear to be the most significant achievements and the chief problems still open in this line of research.

Quantum theory offers us a function Ψ, defined over the whole of space and satisfying a wave equation (which here we shall take to be Schrödinger's), to describe corpuscular behaviour. Are the objects described by quantum mechanics particles or waves? To this question Bohr (1935) answers that they are at times particles, at times waves, and the two descriptions complement each other yet never coexist; Mott (1964) insists that only particles have real existence, while waves appear simply as the collective behaviour of many particles; de Broglie (1953) sees a pilot wave carrying along, piggyback, a particle; Bunge (1967) offers the "quanton" concept —objects that are neither particles nor waves but combine aspects of both. Quot homines tot sententiae.

To back up this apparently limitless variety of fundamental concepts, a bewildering range of Gedankenexperiment have been developed to show up one or another conceptual weakness. Even a quantum version of Zeno's paradox has made its appearance in recent years (Misra and Sudar-

shan 1977; for a simplified discussion see Peres 1980).

The most famous of these thought experiments is, of course, due to Einstein (Einstein, Podolsky and Rosen 1935, referred to, according to custom, as EPR). Consider a compound system, e.g., a neutral hydrogen atom, in a well-defined state. After a time it disintegrates, and when the resulting particles are sufficiently far from each other so as no longer to interact, we measure the momentum, say, of the electron; because linear momentum is conserved, the proton must be in the corresponding eigenstate of its momentum operator, $\phi(p)$, let us say. But had we determined the electron's position instead, the proton would have to be in an eigenstate $\Psi(x)$ of its position operator. Yet these two states are incompatible and cannot simultaneously be used to describe the proton. Then how did the proton, beyond any interaction with the electron, know which state to adopt?

In his equally famous rebuttal of the conclusion EPR drew from this odd situation, Niels Bohr (1935) explicitly stated that their arguments were straightforward consequences of the transformation theorems of quantum mechanics; moreover, Schrödinger (1935, 1936), submitting the matter to a very searching examination, so far from removing the problem, generalized and deepened it; he concluded that it indicated a serious deficiency in quantum mechanics. Nevertheless, the EPR result has commonly been treated as a paradox (e.g., Cooper 1950, Mittelstaedt 1974) or even as fallacious (e.g., Sharp 1961, Rosenfeld 1968, Kellett 1977), —so much so that it has been thought worthwhile to prove its validity from an entirely different standpoint (Flores et al. 1981).

In fact, the conceptual confusion is so great that one philosopher of science openly speaks of "the great quantum muddle" (Popper 1967); yet, perhaps one should say unexpectedly, quantum mechanics has achieved remarkable advances in our knowledge of nature and continues to do so. The "muddle" seems to be irrelevant. Even the out-and-out Copenhagen textbooks reveal this: they emphasize Bohrian complementarity and the reduction of the wave packet (von Neumann's projection postulate) in the first few chapters on basic concepts, but later on make no further use of these notions.

This situation is explained when a more detailed analysis shows that opposing the Copenhagen interpretation there is only one genuine alternative, which most physicists use in a quite intuitive fashion, often without being aware of it, at the same time as they declare their firm belief in Copenhagen. This alternative view is what is commonly called the statistical interpretation, though a better name would be the ensemble interpretation. The confusion of the great quantum muddle largely arises from more or less inconsistent mixtures of these two points of view, which in themselves can achieve considerable internal consistency; they have, however, very different implications when confronted with actual practice.

The Copenhagen view takes as its starting point the postulate that the wave function $\Psi(x)$ describes one and only one physical system —a given particle, for instance. When measurement on this particle yields one result (and not another one of the possibilities whose probability as given by Ψ is > 0), this requires explanation. The projection postulate is added for this purpose: the wave function "collapses", in an unpredictable fashion, into one of its components during the measurement process. When a straightforward argument leads to Heisenberg's famous inequality $\Delta x \Delta p \geqslant \frac{1}{2}\hbar$, the two quantities Δx and Δp must be interpreted as limits on experimental precision, for we have only one system to which they could apply and no other plausible meaning is open for them. But if the theory offers an intrinsic limitation on measurement precision, this implies that in quantum mechanics we have reached as far as it is possible for physical research to go: a consistent defender of the Copenhagen view therefore sees quantum mechanics as the ultimate theory, which may be refined and extended but not improved upon. When, finally, we talk of probability as applying to a single system (and therefore not defined by means of an appropriate sample space), there is no consistent way of seeing this as a property of the system by itself, so that it must be interpreted as a degree of uncertainty in our knowledge of the system: thus the way lies open to abandoning the fundamental philosophical principle that physics (like any science) is about this world and not about our knowledge of it: "Quantum mechanics does not describe an objective state

in an independent external world, but the aspect of this world gained by considering it from a certain subjective standpoint" (Born 1949). And one step further leads to this: [When its position is being measured] "the electron has to make a decision. We force it to take up a well-defined position; before that it was not in general either here or there; it has not yet decided on its position... it is we who produce the facts we observe" (Jordan 1934). Even more disastrous conclusions are possible, as we shall see below.

The ensemble interpretation, in contrast, takes the wave function to represent an ensemble (in the Einstein-Gibbs sense), a generally infinite set of theoretical replicas of the physical system under study, all similarly prepared but otherwise not necessarily alike. Quantum-mechanical predictions are therefore expectation values in the statistical sense: the values which we expect the mean of long series of measurements to take. These predictions imply nothing for individual measurement results (except, of course, that they must belong to the spectrum of possible values) and hence no projection postulate is required to explain them; only their relative frequency must be —and is accounted for. Hence no "measurement problem" arises (Moldauer 1972) and the endless theories on this subject become superfluous, —always provided that due account is taken of the distinction between the preparation of a state and a measurement on it (Margenau 1959). Similarly, of course, the projection postulate (which gives rise to the "collapse" mentioned above) is now superfluous: it was only needed to explain the individual measurement, a requirement explicitly renounced here. The Heisenberg inequality is merely a relation between the dispersions of two measurement series carried out on systems described by the same ensemble; the two types of measurement can but need not be carried out on the same set of systems, a conclusion which mirrors the experimental procedure used to verify that the inequality holds; calling it an "uncertainty relation" rather than a "variance relation" is therefore misleading (on this see Popper 1967). And the probability of any event is here well defined within an unproblematic materialist philosophy: while the frequency interpretation of von Mises (1919, 1928) and Reichenbach (1935) is possible, with some difficulties

and awkward unsolved problems, it is much more straightforward to use the ensemble concept itself as the basic notion (Brody 1975, 1979) on which probability can be defined as the expectation value (i.e., the average) of an appropriate indicator function.

The ensemble interpretation was first adumbrated by Slater (1929). By 1935, when he was working on the EPR paper, Einstein had probably accepted it; certainly he contributed considerably towards its development (Einstein 1949, 1953), as also did Blokhintsev (1953), Margenau and his erstwhile pupil Park (Margenau 1958, 1959, 1963; Park and Margenau 1968; Park 1968; Park and Band 1971; Band and Park 1979; Park, Band and Yourgrau 1980) and many others. Detailed accounts of it will be found in Ballentine (1970), Ross-Bonney (1975) or Brody (1983); see also Jammer (1974).

2. THE INCOMPLETENESS OF QUANTUM MECHANICS

The principal achievement of the ensemble interpretation is its conceptual simplicity and clarity: the traditional paradoxes do not appear in it.

In the EPR situation, for instance, the single measurement one imagines according to the Copenhagen view does not determine the ensemble. We need a long series of measurement results; if we let them range freely, we obtain an experimental sample of the ensemble, while if we accept only those that satisfy a particular condition, we obtain a subensemble, with properties conceivably different from those of the complete one. If only those electrons in a whole series are picked out which have the same position as the first one, then we have selected a subensemble whose corresponding protons will be correctly described by a position eigenfunction. But if we pick the electrons with the same momentum as the first one, they will be a different set and hence make an entirely different subensemble; and so will their partner protons.

Similar considerations hold for other so-called paradoxes.

That the ensemble interpretation thus cleanly resolves the paradoxes should make it very attractive. Yet the philosophers, for one,

either ignore it completely (thus Putnam 1965) or dismiss it with a few words (e.g., Fine 1973), even with contempt (Hanson 1959).

One reason for this state of affairs —apart from the bandwagon effect— is, presumably, that the ensemble interpretation has not yet achieved the status of a finished theory. If it is to provide the basis for considering quantum mechanics a fully statistical theory, then it should (among other things) offer an account of the joint probability distribution of two observables, even non-commuting ones, so that expectation values may be computed over phase space, in a way analogous to statistical mechanics. To take the case of the position x and the momentum p of a one-particle system in one dimension, it should be possible to derive from the wave function $\Psi(x)$ a distribution $f(x,p)$ such that the expectation of any function $g(x,p)$ may be calculated as

$$\langle \Psi | \hat{g} | \Psi \rangle = \iint f(x,p)\, g(x,p)\, dx\, dp \quad , \tag{2.1}$$

where $\hat{g}$ is the Hilbert-space operator corresponding to g. Eq. (2.1) includes of course the three special cases:

$$\iint f(x,p)\, dx\, dp = 1 \quad , \tag{2.2a}$$

$$\int f(x,p)\, dp = |\Psi(x)|^2 \quad , \tag{2.2b}$$

$$\int f(x,p)\, dx = |\phi(p)|^2 \quad ; \tag{2.2c}$$

where $\phi(p)$ is the momentum-space wave function corresponding to $\Psi(x)$. Many such functions f have been proposed; the best known was found by Wigner (1932):

$$f_w(x,p) = \frac{1}{2\pi} \int \Psi^*(x + \tfrac{1}{2}\hbar\tau) e^{i\tau p} \Psi(x - \tfrac{1}{2}\hbar\tau)\, d\tau \quad , \tag{2.3}$$

and a complete phase-space formalism for (2.3) has been given by Moyal (1949). But it can be shown (Shewell 1959) that none of these functions is satisfactory over the whole range of quantum problems; for each such function is linked to one particular quantization rule, through the as-

sociation $g(x,p) \longleftrightarrow g$, and no one of these rules will always work; the Wigner function, for instance, is positive-definite only for a restricted class of wave functions (Urbanik 1967, Hudson 1974). However, as Cohen (1966) has shown, the class of possible distribution functions is circumscribed by

$$f(x,p) = \frac{1}{4\pi^2} \iiint \Psi^*(\eta + \tfrac{1}{2}\hbar\tau)\, \gamma(\theta,\tau) e^{i(\theta\eta+\tau p-\theta x)} \Psi(\eta - \tfrac{1}{2}\hbar\tau) d\theta d\eta d\tau \ , \quad (2.4)$$

where $\gamma(\theta,\tau)$ is any function satisfying

$$\gamma(0,\tau) = \gamma(\theta,0) = 1 \quad . \quad (2.5)$$

Cohen (1976) has also taken the first steps towards a complete phase-space formalism with a general distribution of the type (2.4), and has also shown (Cohen and Zaparovanny 1980) that they can be constructed so as to have any desired linear correlation between x and p. For, given any distribution $h(u,v)$ on the unit square $0 \leqslant u,v \leqslant 1$, we can define a function

$$k(x,p) = h(u,v) - h_u(u) - h_v(v) + 1 \quad , \quad (2.6)$$

where

$$h_u(u) = \int_0^1 h(u,v)dv \quad , \quad h_v(v) = \int_0^1 h(u,v)du \quad (2.7)$$

and

$$u = u(x) = \int_{-\infty}^{x} |\Psi(\xi)|^2 d\xi \quad , \quad v = v(p) = \int_{-\infty}^{p} |\phi(\varpi)|^2 d\varpi \quad . \quad (2.8)$$

Then

$$f(x,p) = |\Psi(x)|^2 |\phi(p)|^2 \{1 + ck(x,p)\} \quad (2.9)$$

(where the constant c can be arbitrarily chosen provided the factor in { } is never negative is a perfectly satisfactory joint probability den-

sity, with a connecting function of the form

$$\gamma(\theta,\tau) = \iint f(x,p)e^{i(\theta x+\tau p)}dx\,dp \Big/ \int \Psi^*(\eta + \tfrac{1}{2}\tau\hbar)e^{i\theta p}\Psi(\eta - \tfrac{1}{2}\tau\hbar)d\eta \,. \quad (2.10)$$

The linear and higher-order correlations are (except for a trivial transformation) c times those of h(u,v). Now these results show that constructing a phase-space formalism may need much more information than standard quantum mechanics provides; thus even if all the correlations are 0, i.e., if h(u,v) factorizes, neither its detailed form nor the value of c is unambiguously determined. The question whether the higher-order correlations are experimentally accessible does not seem to have received attention.

It is commonly stated that quantum mechanics cannot be a statistical theory since no joint probability distribution can be written for it. The results just quoted show that this is not the case: rather do we have an infinite set of possible distribution functions, since almost any correlation whatever is compatible with a given wave function Ψ. What is true, however, is that no one choice of the connecting function γ will provide a satisfactory joint distribution for all cases. This is what is usually meant when the existence of joint distributions is denied. Yet such a requirement is patently absurd: not only do we not impose any similar condition on classical statistical theories, but the condition would furthermore imply the choice of a particular quantization rule, thus making quantum mechanics dependent on classical mechanics in a highly undesirable way (for further details see, e.g., Brody 1983).

Accepting, then, the freedom of choice of γ as physically plausible, we observe that in the ensemble interpretation quantum mechanics is unfinished in that finding the form of Ψ, say as the eigenfunction of an appropriate Hamiltonian, still leaves undetermined the correlation needed to define γ and hence (2.4). The usually accepted form for the covariance, $\frac{1}{2}\langle(\hat{x}\hat{p} + \hat{p}\hat{x})\rangle$, is not derived from the basic postulates of quantum mechanics but is merely a plausible analogy with the classical case. This is recovered in the limit $\hbar \longrightarrow 0$. However, for any real λ,

$$\tfrac{1}{2}\langle(\hat{x}\hat{p} + \hat{p}\hat{x})\rangle + i\lambda\langle(\hat{x}\hat{p} - \hat{p}\hat{x})\rangle \quad , \quad (2.11)$$

also goes to the classical limit and satisfies the statistical requirements for this sort of parameter. Should it turn out that quantum-mechanical arguments do not allow us to deduce a value for λ, then the ensemble interpretation will be not so much unfinished as incomplete, in the specific sense of EPR.

We return to their 1935 paper for the conclusions they drew from the seeming paradox sketched above. They started from two postulates which we may rephrase as follows:

(i) The world has real and independent existence, its component parts therefore have real properties, and one way of establishing the reality of a property is that one can measure its value without perturbing the system.

(ii) A physical theory is complete if to each (relevant) real property of the systems it describes, it assigns a theoretical counterpart.

Assuming the validity of the quantum-mechanical formalism, they conclude that both the position and the momentum of the proton are real properties, since they can be determined by measuring the corresponding property of the electron, measurements which do not perturb the proton if it is far enough away. But quantum mechanics cannot predict both values for the proton: it is therefore incomplete.

It is this incompleteness that becomes unpalatably evident in the ensemble interpretation.

One way to overcome it is the addition to quantum theory of so-called "hidden variables"; these would make determinate such quantities as the correlations discussed above. Now if these hidden variables are "deterministic", i.e., reduce the theory to a non-statistical one analogous to Newtonian mechanics, a well-known argument due to von Neumann (1932; for a discussion of the limits to its validity and references to earlier work see Brody 1983) shows the impossibility of introducing them consistently. The von Neumann proof, in other words, implies that hidden variables, if any, must be stochastic in nature.

More recently, Bell (1965) has used a revised version of the EPR Gedankenexperiment to derive an inequality that hidden-variable theo-

ries must satisfy if they are local; if, that is to say, electron and proton no longer interact when they are far apart, so that a hidden variable affects one or the other but not both. Because the quantum predictions, as well as most of the experimental evidence since produced, violate this inequality, Bell concludes that local hidden-variable theories, even if they are statistical in nature, cannot account for quantum phenomena (the extensive literature is reviewed in Clauser and Shimony 1978 or Bruno et al. 1977). Since then the very ingenious experiments of Aspect et al. (1982) have seemingly yielded definitive confirmation even in one extreme case. It is a pity that Bell's inequality can be proved in at least three further ways (Wigner 1970 and also Holt 1973; Eberhard 1977, a simplified version of which is Peres 1978; and in a somewhat different vein, Suppes 1982); and none of these requires the notion of a hidden variable at all. Hence the violation of the inequality in quantum mechanics cannot be laid at the door of the hidden-variable theories, and the Bell argument is irrelevant to the question if quantum mechanics can be completed by adding further dynamical variables. The inequality has, indeed, been shown to require a tacit assumption quite evidently not satisfied by quantum mechanics (de la Peña, Cetto and Brody 1972, Brody and de la Peña 1979, Brody 1980; for a different point of view see Selleri and Tarozzi 1981). This has not prevented several misguided authors from seeking to explain its violation by postulating the transmission of signals at speeds beyond that of light, either macroscopically (Costa de Beauregard 1965) or microscopically (Vigier 1982). The resulting semi-philosophical speculations —that physics has experimentally disproved the reality of the material world (d'Espagnat 1979) or that it has established the possibility of parapsychological phenomena (Le''pinay 1980; Zohar 1980; Costa de Beauregard 1981)— can only bring discredit on physicists.

Those who perpetrate such speculations are apparently unaware that their very assumptions make them unnecessary. For if signals can propagate at superluminal velocities, then the electron and the proton in the EPR situation never become independent, we cannot any longer conclude that our choice of measurement on the electron has no influence on the

proton, and hence we ought not to accept the reality of position or momentum as proved by the EPR argument. In other words, hidden variables are no longer necessary; indeed, the superluminal signals merely provide one specific mechanism to implement the "feature of wholeness" sought by Bohr (1963). We simply fall back into the Copenhagen interpretation.

It might be noted here that quite independently of the Bell inequality it should have been evident that quantum mechanics is inherently a non-local theory. Thus to determine a particle's energy we have to integrate over its configuration space to compute $\langle \Psi | H | \Psi \rangle$, while the same quantity is found directly from the local coordinates in classical mechanics; and the Pauli principle is another obviously non-local element in the theory. That such points (and their implications) should be so generally ignored is only another sign of how badly quantum mechanics is understood.

We conclude that statistical local hidden-variable theories cannot be excluded on these grounds. They are, however, undesirable on quite other grounds. Physical theories possess an organic integrity which is particularly important in the case of a fundamental theory such as quantum mechanics, since as well as a formalism they provide the conceptual background for our understanding; patches on them, such as hidden variables, should therefore be considered at best a last resort. In the present case, this implies that instead of adding stochastic variables we should rather attempt to formulate a complete stochastic theory.

The stochastic nature of such a theory follows already from the von Neumann hidden-variable argument referred to above: this shows that the theory must be statistical, and since it must also give an account of the time evolution of quantum systems, it must be based on one or more stochastic processes. Another way to see this is to observe that quantum-like behaviour is of three kinds: the existence of discrete values for certain dynamical variables, interference phenomena and other wave-like behaviour, and the appearance of fluctuations in experimental results. Now Bohr's "old" quantum theory took the first phenomenon as its starting point; present-day quantum mechanics starts from the second; but the third way has not yet been tried (L. de la Peña, private communication).

A more serious argument is the following: Every physical system is subject to the constant influence of the remainder of the universe (or at least of large parts of it); in order to restrict ourselves to the variables intrinsic to the system, we average over these influences. If this averaging is done before setting up and solving the equations of motion for the system, we arrive at that central concept of classical physics, the closed system; if we average afterwards, we have a stochastic theory for open systems. To the extent that this argument is valid, the peculiar inconsistencies of quantum mechanics (at least in the Copenhagen sense) are explained: it is an attempt to do closed-system physics where the open-system effects are still significant. This matter will be taken up again below.

Several varieties of stochastic theories have been developed. In what follows we shall classify them as formal or physical theories according to whether they postulate the necessary stochastic process or derive it from a physical model. In spite of considerable successes achieved by the formal theories, we shall argue that they have certain fundamental weaknesses, and we therefore devote more attention to that physical stochastic theory which has shown far-reaching results.

3. FORMAL STOCHASTIC THEORIES

The first to suggest a formal stochastic approach was Schrödinger (1931), who observed that the equation which bears his name has the form of a diffusion equation. However, the diffusion coefficient is purely imaginary when Ψ itself is taken as the stochastic process, so that the equation is of hyperbolic type and has time-reversible solutions, while a physically meaningful diffusion process must have a parabolic differential equation.

The first reasonably complete stochastic theory was formulated by Fényes (1952). He begins his development by postulating, without further discussion of his choice, a Markov process for the position vector; for the probability density $\rho(\mathbf{r},t)$ of this process he then establishes the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = -\partial_i (b^i \rho) + \partial_i \partial_j (D_{ij} \rho) \quad , \tag{3.1}$$

where b_i is the drift velocity:

$$b_i(\underline{r},t) = \lim_{\delta t \to 0^+} \frac{1}{\delta t} \langle r_i(t + \delta t) - r_i(t) \rangle \quad ; \tag{3.2}$$

while D_{ij} is the diffusion tensor:

$$D_{ij} = \lim_{\delta t \to 0^+} \frac{1}{\delta t} \langle (r_i(t + \delta t) - r_i(t))(r_j(t + \delta t) - r_j(t)) \rangle \quad . \tag{3.3}$$

In (3.1), ∂_i indicates partial derivation with respect to r_i, repeated indices are summed over, and $\langle\ \rangle$ signals an ensemble average (that is to say, over all possible realizations of the stochastic process). Fényes defines for this process a velocity operator

$$\hat{c}_i = b_i - \partial_j D_{ij} \quad , \tag{3.4}$$

which has a commutator

$$[r_i, \hat{c}_j] = D_{ji} \quad . \tag{3.5}$$

From (3.5) he can derive a very general inequality of the Heisenberg type:

$$\Delta r_i \Delta c_j \geq |\langle r_i \hat{c}_j \rangle - \langle r_i \rangle \langle \hat{c}_j \rangle + \langle D_{ij} \rangle| \quad . \tag{3.6}$$

If the Markov process is a simple diffusion, then

$$D_{ij} = D\delta_{ij} \quad , \tag{3.7}$$

with D a constant; neglecting the covariance between r_i and $\hat{c}_j$ in (3.6) yields in this case the more usual Heisenberg inequality:

$$\Delta r_i \Delta c_j \geq D\delta_{ij} \quad . \tag{3.8}$$

Fényes then supposes that the drift velocity can be derived from a po-

tential s,

$$b_i = \frac{1}{m} \partial_i s \quad , \tag{3.9}$$

where m is the mass of the particle; then ρ can be written in terms of another potential σ:

$$\rho = \alpha^2 \exp\left(-\frac{\sigma}{mD}\right) \quad . \tag{3.10}$$

Now with the definition

$$\Psi = \alpha \exp\left[-\frac{i(s+\sigma)}{2mD}\right] \exp\left(-\frac{\sigma}{2mD}\right) \quad , \tag{3.11}$$

the Fokker-Planck Eq. (3.1) can be rewritten as

$$\frac{i}{D} \frac{\partial}{\partial t} (\Psi^* \Psi) + \operatorname{div} (\Psi \nabla \Psi^* - \Psi^* \nabla \Psi) = 0 \quad . \tag{3.12}$$

This is the quantum-mechanical continuity equation provided we fix

$$D = \frac{\hbar}{2m} \quad . \tag{3.13}$$

Fényes also derived the Schrödinger equation but needed to postulate for this a somewhat artificial Lagrangian. A much cleaner derivation is due to Nelson (1966, 1967). He writes the Markov process in differential form as

$$dr_i(t) = b_i(\underline{r},t)dt + dW_i(t) \quad , \tag{3.14}$$

where b_i is the drift velocity as before and $W_i(t)$ is a Wiener process, that is to say a stochastic process of Gaussian distribution with zero mean and a diffusion tensor

$$D_{ij} = \langle dW_i(t) dW_j(t) \rangle = D\delta_{ij}dt \tag{3.15}$$

(This sort of theory was of course first established for Brownian motion,

where

$$D = \frac{kT}{m\beta}$$

instead of (3.13), is the diffusion coefficient). Nelson now defines the backward drift velocity

$$\tilde{b}_i(\underline{r},t) = \lim_{\delta t \to 0^+} \frac{1}{\delta t} \langle r_i(t) - r_i(t-\delta t) \rangle \quad . \tag{3.16}$$

Note that this does not correspond to a time reversal of the process, but to a change in how the ensemble averaging is done: both in (3.2) and in (3.16) all trajectories going through $r_i(t)$ are selected, but in (3.2) their position a time δt later is averaged over, while in (3.16) it is their positions a time δt earlier that are used. With (3.16) we also have, for the same stochastic process,

$$dr_i(t) = \tilde{b}_i(\underline{r},t)dt + d\tilde{W}_i(t) \quad , \tag{3.17}$$

with $\tilde{W}_i(t)$ an analogous Wiener process. Now (3.14) and (3.17) yield two Fokker-Planck equations:

$$\frac{d\rho}{dt} = -\partial_i(b_i\rho) + D\partial_i\partial_i\rho \quad , \tag{3.18a}$$

$$\frac{d\rho}{dt} = -\partial_i(\tilde{b}_i\rho) - D\partial_i\partial_i\rho \quad , \tag{3.18b}$$

which together give the continuity equation

$$\frac{\partial\rho}{\partial t} = -\partial_i(v_i\rho) \quad , \tag{3.19}$$

where we have defined

$$v_i = \frac{1}{2}(b_i + \tilde{b}_i) \quad . \tag{3.20}$$

If v_i is derived from a potential function, then it is straightforward to

show that for the function Ψ of (3.11):

$$-i\frac{\partial}{\partial t}\Psi = D\nabla^2\Psi - \frac{1}{2mD}\Psi + \beta(t)\Psi \quad , \tag{3.21}$$

where $\beta(t)$ can be made to vanish by a suitable normalization of the velocity potential. With the identification (3.13), this is then the Schrödinger equation.

A third and physically perhaps more plausible version has been given by de la Peña (1969, see also de la Peña and Cetto 1975, 1982). Observing that for functions $g(\underline{r},t)$ of a stochastic variable $\underline{r}$ one may define numberless derivative-like operators, they chose two:

$$Dg^{(o)} = \frac{1}{\delta t}\left\langle \frac{1}{2}(g^{(+)} - g^{(-)})\right\rangle \quad , \tag{3.22}$$

$$\mathcal{Y}g^{(o)} = \frac{1}{\delta t}\left\langle \frac{1}{2}(g^{(+)} + g^{(-)}) - g^{(o)}\right\rangle \tag{3.23}$$

(we have written $g^{(o)} = g(\underline{r}(t))$, $g^{(\pm)} = g(\underline{r}(t \pm \delta t))$ to clarify the notation). When we take $g(\underline{r}(t)) = r_i$, these are evidently related to the two drift velocities by

$$Dr_i = v_i = \frac{1}{2}(b_i + \tilde{b}_i) \quad , \tag{3.24}$$

$$\mathcal{Y}r_i = u_i = \frac{1}{2}(b_i - \tilde{b}_i) = D\partial_i \mathrm{Ln}\rho \quad , \tag{3.25}$$

where in (3.25) we define the stochastic velocity u_i. Now the stochastic generalization of Newton's equation,

$$m\frac{d}{dt}v_i = f_i \quad , \tag{3.26}$$

is plausibly written as a linear combination of the four accelerations Du_i, Dv_i, $\mathcal{Y}u_i$, $\mathcal{Y}v_i$. It should satisfy three physical conditions:

(i) Given the time-reversal properties

$$D(t) = -D(-t) \quad , \qquad \mathcal{Y}(t) = \mathcal{Y}(-t) \quad , \tag{3.27}$$

$$v_i(t) = -v_i(-t) \quad , \qquad u_i(t) = u_i(-t) \quad , \tag{3.28}$$

which follow from the physical meaning of these quantities, the generalization of (3.26) should be invariant under time reversal, just as (3.26) is;

(ii) the probability density should satisfy the continuity equation (3.19);

(iii) in the classical limit ($D \to 0$), Eq. (3.26) should be recovered.

The stochastic generalization of (3.26) then becomes

$$m(Dv_i - \lambda \mathcal{D}u_i) = f_i \quad , \tag{3.29}$$

with λ an as yet determined parameter. If the force f_i also has a potential V, then one can show that so do the velocities:

$$v_i = 2D\partial_i S \quad , \tag{3.30a}$$

$$u_i = 2D\partial_i R \quad . \tag{3.30b}$$

Now (3.29) and (3.19) may be integrated once and the solution rewritten in linear form, to give

$$\pm 2mD\sqrt{\lambda}\,\frac{\partial}{\partial t}\,\psi_\pm = 2mD^2\lambda\partial_i\partial_i\psi_\pm + V\psi_\pm \quad , \tag{3.31}$$

where

$$\psi_\pm = e^{R \pm S/\sqrt{\lambda}} \quad . \tag{3.32}$$

In (3.31) λ is still undefined; its absolute value may be combined with D, so that only its sign is significant. If $\lambda = 1$, Eqs. (3.31) are parabolic and describe the irreversible evolution of the real amplitudes $\psi_\pm$; superposition is always additive so that no interference effects appear, and we have a stochastic process of Brownian type. If on the other hand $\lambda = -1$, Eqs. (3.31) reduce, using (3.13), to Schrödinger's equation, which is hyperbolic and has solutions that allow interference terms to appear and that admit time reversal. It is evident from this formulation that the two stochastic processes, while they have much in

common, must nevertheless be clearly distinguished because they also show essential differences (de la Peña 1968, 1969).

This detailed discussion of the three approaches should make clear their strengths. Among these are a conceptual clarity (shared by the ensemble interpretation) and a simple mathematical structure for the stochastic process. They allow several generalizations beyond quantum mechanics (for instance, extended and sharpened Heisenberg relations) and a clearer discrimination between various sorts of stochastic processes. These advantages have made feasible a long list of applications of very diverse nature. To quote only a few examples: a phenomenological description of spin has been given (Dankel 1970, 1977; de la Peña 1970a, 1971); a path-integral formalism has been developed (Berrondo 1973) and used to study barrier penetration (Weaver 1978; Yasue 1981); relativistic generalizations have been developed (de la Peña 1970; de la Peña and Cetto 1971; Cetto and de la Peña 1971; Lehr and Park 1977; Vigier 1979); and a stochastic field theory has been developed (Moore 1980 and references therein). The theory is restated from a different point of view by Davidson (1979). More detailed accounts and further references will be found in Guerra (1981) and de la Peña and Cetto (1982). One of the most remarkable uses made of the basic conception of these approaches was to develop an ergodic theory of quantum processes —or at least the rudiments of it— (Claverie and Diner 1973, 1975) which has helped to dissipate several of the confusions in the "great quantum muddle".

It should be noted that the correspondence between the stochastic process in a formal and the equivalent quantum-mechanical description is not one-to-one. As Suppes and Zanotti (1976) have shown in general terms and Davidson has (1979) demonstrated, to each wave function there corresponds an infinite set of different Markov processes. In other words, the incompleteness of quantum mechanics is even more profound than was noted by EPR (1935): the quantum mechanical account does not even fully specify the equivalent stochastic process.

In spite of their considerable achievements, the formal stochastic theories have received much criticism (Nicholson 1954; Gilson 1968; Albeverio and Høegh-Krohn 1974; Kracklauer 1974; Ghirardi et al. 1978; Grabner et al. 1979; Mielnik and Tengstrand 1980). Unhappily, many of

these comments either do not take into account the distinction worked out above between Brownian and quantum stochastic processes, or else take (3.16) to be a time-reversed stochastic process. Gilson attempts to show that the diffusion coefficient (3.13) must be zero if quantum mechanics is a stochastic process, and overlooks the fact that he specifies the conditions, not for a quantum process, but for a classical process of the Maxwell-Boltzmann type, where indeed, as already pointed out by Fényes (1952), no diffusion term appears. Some other questions raised by these critics are commented upon by Lavenda (1980).

A more serious point was discussed first by Albeverio and Høegh-Krohn (1974) and later by several of the authors cited. Since the formal stochastic theories lead straightforwardly to the Schrödinger equation, they accept also the excited-state solutions of the latter as possible forms for the density. But these solutions have nodal surfaces at which $\rho = 0$. The stochastic system thus has vanishing probability of crossing these surfaces, which therefore break up configuration space into mutually inaccessible regions. Nelson (1967) suggested reconnecting them by a small perturbation. Though this way of solving the problem is physically not unreasonable, it does tacitly make the theory at best an approximate one. Another solution of the difficulty is mentioned below.

But only the last of the quoted papers shows an understanding of where the real weakness of the formal theories lies, namely in their very formality: they provide us only with a mathematical model of the underlying stochastic process. The physical mechanism behind this process is not elucidated. There cannot then be any explanation beyond that of mathematical convenience of why it should be Markovian; deriving the velocities from as many potential functions, though plausible in general and certainly an important special case, does not receive any adequate physical justification; nor, finally, can one explain the identification of D by means of Eq. (3.13). What these theories provide, in other words, is a mathematical illustration of the ensemble interpretation, which they therefore show to be perfectly viable. But while their considerable contributions should not be underestimated, it must be said that they do not offer us any deeper physical understanding.

The physical stochastic theories pursue the aim of remedying this defect. They propose specific mechanisms for the stochastic processes underlying quantum mechanics. In the next section we discuss the most successful of these theories; section 5 will review some of the difficulties it exhibits, and also some of the suggestions that have been made for resolving them.

4. A PHYSICAL STOCHASTIC THEORY: STOCHASTIC ELECTRODYNAMICS

Stochastic electrodynamics (SED) originated almost in parallel with the formal stochastic theories discussed above. The idea of treating the vacuum fluctuations of quantum electrodynamics (QED) as real rather than virtual lies behind the work of Welton (1948) and Weisskopf (1949), but seems first to have been spelt out explicitly by Kalitsin (1953). It was developed quite independently by Braffort and coworkers (Braffort, Spighel and Tzara 1954, Braffort and Tzara 1954), and also by Marshall (1963, 1965a, 1965b). A detailed account is given by de la Peña (1983), following shorter reviews by Boyer (1975a) and Claverie and Diner (1976).

These and later papers consider real fluctuations of the classical electromagnetic field. The origin of these fluctuations is best understood as a special case of the conception, mentioned above, that a physical system is isolated from the rest of the universe only to first approximation. A charged particle, rather than moving in a null field, must be considered to move in a random background field formed by the uncorrelated emissions of all accelerated charges in the universe. The fundamental question for SED is, then: how does a classical charged particle behave in such a random field? This question may be broken up into several parts.

The first part is the statistical characterization of the random electromagnetic background field. Because we expect this field to play a role in the stability of atomic structures (which would collapse rapidly if electrons were classical particles moving in a null background), even at the absolute zero it must have a zero-point component that is therefore temperature independent. Such a component, of energy $\frac{1}{2}\hbar\omega$ per

normal mode, was postulated already by Planck (1911, 1912), and its cosmological and other consequences were explored by Nernst (1916). In order to find its distribution, arguments of simple physical plausibility suffice. To begin with, even if quite strong correlations between the sources are assumed, Einstein and Hopf (1910) showed that the Fourier components of such a radiation field have amplitudes with independent Gaussian distributions; a fortiori this holds when the sources have no such correlations. The means of these distributions are of course zero, and only the fluctuation amplitudes (i.e., the standard deviations) remain to be determined. A number of quite different arguments converge to show that, as well as

$$\langle E_i(\underline{r},t)\rangle = \langle B_i(\underline{r},t)\rangle = 0 \quad , \tag{4.1a}$$

we have

$$\langle \tilde{E}_i(\omega)\tilde{E}_j(\omega')\rangle = \langle \tilde{B}_i(\omega)\tilde{B}_j(\omega')\rangle = \alpha\omega^3\delta(\omega-\omega')\delta_{ij} \quad , \tag{4.1b}$$

where E_i and B_i are the (instantaneous) components of the electric and magnetic vectors, while the symbol $\tilde{}$ indicates the Fourier transform. Only α remains now to be determined. The main arguments are: Firstly, if a charged particle moves through the radiation field, it should not suffer a frictional drag which would be observable; in other words, the distribution function of the field components should be Lorentz invariant, since otherwise it could provide a privileged reference frame for which there is no evidence. Secondly, if the energy density at the absolute zero does not vanish, it must be given by the low-temperature limit of Wien's law,

$$\rho(\omega,T) = \omega^3\Phi(\omega/T) \quad .$$

Thirdly, the Wheeler-Feynman absorber theory, combined with the very suggestive idea of random boundary conditions, yields a random action on charged particles of precisely the right spectrum (Braffort, Spighel and Tzara 1954; Pegg 1980). Lastly, the fluctuation-dissipation theorem, applied to the radiation reaction, yields the same form of random field

(Milonni 1981). All these arguments yield the form (4.1b); since the number of normal modes per unit frequency interval is proportional to ω^2, this makes the energy per normal mode proportional to ω, as Planck had already anticipated and as is, furthermore, in agreement with QED (even though we are considering real fluctuations, not virtual ones).

The value of α may be found as follows: if we add the hypothesis of a random zero-point field characterized by (4.1a) and (4.1b) to the classical assumptions that lead to the Rayleigh law for the black-body radiation, then instead the Planck distribution (including the zero-point term) is found; comparison with the usual form then fixes α:

$$\alpha = \frac{4\hbar}{3\pi c^2} \tag{4.2}$$

(Jiménez, de la Peña and Brody 1980; Jiménez and del Valle 1982).

A very simple way of seeing how the Planck distribution arises is due to Boyer (1969b) and Theimer (1971). Write the energy of the black-body radiation at a mode of frequency ω as the sum of the zero-point and thermal parts:

$$\varepsilon(\omega) = \varepsilon_o + \varepsilon_T \quad . \tag{4.3}$$

If the amplitudes of the field components have a Gaussian distribution with zero mean, then the variance of this energy is given by

$$\sigma_\varepsilon^2 = \langle \varepsilon^2 \rangle^2 = \sigma_o^2 + \sigma_T^2 \quad . \tag{4.4}$$

Here we suppose the zero-point and thermal components to be uncorrelated: this is reasonable, since they arise from different sources, the one is Lorentz invariant and the other not. Now we know that the zero-point field is also Gaussian:

$$\sigma_o^2 = \langle \varepsilon_o \rangle^2 \quad . \tag{4.5}$$

Combining these three equations, we have

$$\sigma_T^2 = \langle \varepsilon_T \rangle^2 + 2\langle \varepsilon_o \rangle\langle \varepsilon_T \rangle \quad . \tag{4.6}$$

We now employ a well-known result for the variance of the energy of a system in thermal equilibrium with a heat bath at temperature T (see, e.g., Reif 1965):

$$\sigma_\varepsilon^2 = \frac{\partial\langle\varepsilon\rangle}{\partial\beta} , \qquad \beta = \frac{1}{kT} , \tag{4.7}$$

a relation that Einstein (1904) made very effective use of in analyzing Planck's law. Applying (4.7) to the thermal fluctuations (4.6) we get

$$\frac{\partial\langle\varepsilon_T\rangle}{\partial\beta} + \langle\varepsilon_T\rangle^2 + 2\langle\varepsilon_o\rangle\langle\varepsilon_T\rangle = 0 . \tag{4.8}$$

The solution of (4.8) is

$$\langle\varepsilon_T\rangle = \frac{2\langle\varepsilon_o\rangle}{\exp\,(2\beta\langle\varepsilon_o\rangle) - 1} . \tag{4.9}$$

Note that Planck's law appears simply because of the cross term in (4.8). Without it, one obtains the Rayleigh law. This cross term is of course due to the assumption of a random zero-point field —we do not even need to know its spectrum in order to derive (4.9), we could obtain it by comparison with what we know experimentally to hold. No other non-classical assumption enters the argument. Hence we can no longer regard the Planck spectrum as a quantum phenomenon; it is, rather, the result of two special assumptions, that there is a zero-point field, and that equilibrium with it is achieved. But whether these assumptions lead us beyond the framework of classical physics is, perhaps, a terminological rather than a physical question.

The conclusion from such arguments is that the total random background field which is in equilibrium at a temperature has a spectral density which is the sum of the zero-point field and the Planck term (4.9); putting in the form (4.1b) for $\langle\varepsilon_o\rangle$ multiplied by the number of normal modes per frequency interval and using the value (4.2) for α, we find

$$\rho(\omega,T) = \frac{\hbar}{2\pi^2c^3}\,\omega^3 + \frac{\hbar\omega^3}{\pi^2c^3[\exp(\hbar\omega/kT) - 1]} = \frac{\hbar\omega^3}{2\pi^2c^3}\coth\frac{\hbar\omega}{kT} , \tag{4.10}$$

where k is the Boltzmann constant.

It must not be forgotten, however, that if other matter is present in the neighbourhood, perhaps not even distributed isotropically, then (4.10) will not apply: the field will have a distribution that must be worked out anew for each case. This fact has consequences we mention below.

The concept of the zero-point field thus established has one important blemish: with the spectrum (4.10), the zero-point energy density, given by

$$u = \int_0^\infty \rho(\omega,0)\,d\omega \quad , \tag{4.11}$$

diverges. We come back to this problem below. Disregarding it here, we note that the concept of the zero-point field has been very successfully applied in various ways. Part of this success may be understood through the parallelism with QED; this question has been extensively studied in the literature (Marshall 1965a, 1965b; Santos 1974, 1975a). Yet it seems more natural to consider the zero-point fluctuations as real, so that QED mimics them rather than the oter way around. This point of view has significant consequences in thermodynamics (Boyer 1969b).

Historically the first important use made of the zero-point field was in the explanation of the Casimir effect, that is to say the attractive force between neutral plates due to the correlation between the fluctuating dipoles induced by the random field. This effect was calculated (Casimir 1948) on the basis of taking as real the QED fluctuations; it was then derived in a physically much more transparent way on the basis of SED by Marshall (1965b; see also Henry and Marshall 1966) and extensively studied by Boyer (1970, and references cited there) from the same point of view. Both attacks predict an attractive force between the plates of the form

$$F = -\frac{\pi^2}{240}\,\hbar c A d^{-4} \quad , \tag{4.12}$$

where A is the area of the plates and d the distance separating them. Though this force is very small, it has been measured and the theory well

confirmed (Sparnaay 1958).

Later, Casimir (1956) suggested that this attractive force, acting between the infinitesimal charge elements making up the electron, could counterbalance the electrostatic repulsion and so explain not only the stability of the particle but also the magnitude of its charge. This idea is very appealing: the quantization of charge would then be due to the zero-point fluctuations, and a typical feature of the submicroscopic world would find its explanation in very large-scale, indeed cosmological, effects. Unfortunately, detailed calculations for the case of a spherical charged shell suggest that the force is repulsive rather than attractive (Boyer 1968, 1970; Milton, de Raad and Schwinger 1978), though the last word has not yet been said (Milton 1980).

According to SED, the same fluctuating electromagnetic field which on a macroscopic scale gives rise to the Casimir effect gives rise to forces between neutral but polarizable molecules that are well known since the days of van der Waals. This point of view has given results of great conceptual simplicity and excellent accuracy in the hands of Boyer (1969a, 1970, 1972a, 1972b, 1974, 1975c).

Having thus established the nature and statistical behaviour of the zero-point field, we take the next step towards answering the question at the beginning of this section. Now the physical picture is simple: a classical charged particle, moving in the zero-point field, both absorbs energy from it and emits it again (whenever it is accelerated). On the average, stability will be achieved if the emission and absorption rates balance each other. The motion will then be largely that of the same particle were it neither to emit nor to radiate (as was Bohr's assumption in the "old" theory), but modified by the fluctuations due to the background field. This field will be given by (4.10), unless nearby matter perturbs it. The equation of motion should then simply be the Newtonian one, with one added term for emission and another one for absorption:

$$m\ddot{\underline{r}}(t) = \underline{f}(\underline{r},t) + m\tau\dddot{\underline{r}}(t) + e\underline{E}(t) \quad , \qquad \tau = \frac{2e^2}{3mc^3} \quad . \qquad (4.13)$$

In writing (4.13), known as the Braffort-Marshall equation, a number of simplifying assumptions have been made. Firstly, it is assumed

that the magnetic part of the Lorentz force, $\dot{\underline{r}} \times \underline{B}/c$, is negligible because all velocities are small compared to c. Secondly, it is assumed that the random part of the electric field, $\underline{E}(t)$, varies slowly with $\underline{r}$ and may be taken, in the so-called dipole approximation, to be independent of it; this may well be a dangerous approximation, in view of the dominant role dipole interactions play in the Casimir effect. Thirdly, the Abraham-Lorentz approximation has been used for the radiation reaction (whence also the value of τ); this is valid for a point particle (which the electron probably is not, al least in SED, as we shall see below), but is known to be the source of troublesome problems. Fourthly, the influence of the rest of the universe on the electron is represented by a simple additive term. This seems a rather simplistic assumption. If we start from the full notion of an open system, it is clear that we cannot write the equation of motion even for a single electron, let alone solve it; but making the fundamental assumption that this influence may be adequately represented by the stochastic background field, then (analyzing it into its Fourier components in a finite region of size L) we may write a Hamiltonian

$$H = \frac{1}{2m}(\underline{p} - \frac{e}{c}\underline{A})^2 + V(\underline{r}) + \frac{1}{2}\sum_{n\lambda}(p_{n\lambda}^2 + q_{n\lambda}^2\omega_n) \tag{4.14}$$

to find that $\underline{r}$, $\underline{p}$ and $\underline{A}$ satisfy the equations of motion

$$m\dot{\underline{r}} = \underline{p} - \frac{e}{c}\underline{A} \quad , \tag{4.15}$$

$$\dot{\underline{p}} = -\nabla V(\underline{r}) \sqrt{\frac{4\pi}{L^3}}\, e \sum_{n\lambda}(\dot{\underline{r}} \cdot \hat{\varepsilon}_{n\lambda})\underline{k}_n(q_{n\lambda} \sin \underline{k}_n \cdot \underline{r} + \frac{p_{n\lambda}}{\omega_n} \cos \underline{k}_n \cdot \underline{r}) \tag{4.16}$$

and

$$\underline{A} = \sqrt{\frac{4\pi}{L^3}}\, c \sum_{n\lambda} \hat{\varepsilon}_{n\lambda}(q_{n\lambda} \cos \underline{k}_n \cdot \underline{r} - \frac{p_{n\lambda}}{\omega_n} \sin \underline{k}_n \cdot \underline{r}) \quad , \tag{4.17}$$

together with corresponding equations for the $q_{n\lambda}$ and $p_{n\lambda}$. Here the $\hat{\varepsilon}_{n\lambda}$ are the usual polarization vectors and $\omega_n = c|\underline{k}_n|$. We have supposed the

force $\underline{f}$ in (4.13) to be derived from a potential $V(\underline{r})$. These equations should make it clear that the Braffort-Marshall equation (4.13) can be validly derived only under certain special circumstances which do not hold in general. (For further detail see, e.g., Cetto and de la Peña 1978). Among these are the statistical properties of the stochastic field components $q_{n\lambda}$ and $p_{n\lambda}$: if these do not have independent Gaussian distributions, then averaging over (4.14) may not lead to (4.13), even in first approximation.

Fifthly and lastly, all the equations have been written for the non-relativistic case, and this is not consistent with the Maxwell equations the fields are expected to satisfy.

Since the Braffort-Marshall equation contains a stochastic force for which only the statistical properties are known, direct solutions of it (possible in the linear case) are not normally of interest. Rather one requires the calculation of expectation values (in the statistical sense). The usual approach is therefore to derive from it a Fokker-Planck equation for the probability density for finding the particle at a point in the appropriate space. This involves further approximations, in particular approximating the "real" process by a Markovian one, so as to be able to write a Fokker-Planck equation in the usual way. This can be done in various ways. Alternatively (de la Peña and Cetto 1977a, 1977b, 1978) one may write a Liouville equation,

$$\frac{\partial}{\partial t} R + \hat{L} R = 0 \quad , \qquad R = R(\underline{r},\underline{p},\{q_{n\lambda}\},\{p_{n\lambda}\},t) \quad , \tag{4.18}$$

for the distribution of both field and particle variables, and then, using Eqs. (4.14) to (4.17), eliminate the field variables. To first order one finds that

$$Q = Q(\underline{r},\underline{p},t)$$

satisfies

$$\frac{\partial}{\partial t} Q + \frac{1}{m} \underline{p} \cdot \nabla_r Q + \nabla_p \cdot \left[\underline{f} + \tau(\underline{p} \cdot \nabla_r)\underline{f} \right] Q = e^2 \nabla_p \cdot \int_0^t G(t,t') \cdot \nabla_p Q(t') dt' \ , \tag{4.19}$$

where $G(t,t')$ is an operator involving the averaged electric field at times t and t'. The integral term on the right of (4.19), very different from the diffusion term in Brownian motion, implies that here the process is not Markovian and has a much longer "memory". This is why, in the commonest approach, a suitable Markovian approximation is sought for.

For the traditional workhorse of the theoretical physicist, the harmonic oscillator, these and similar methods lead to very satisfactory and complete results. If the oscillator has frequency ω_o and is surrounded by a heat bath at temperature T, then the equilibrium distribution takes the form (de la Peña and Cetto 1979)

$$Q(\underline{r},\underline{p}) = \frac{1}{\pi\hbar}\frac{1-\Theta}{1+\Theta}\exp\left[-\frac{2}{\hbar\omega_o}\frac{1-\Theta}{1+\Theta}\left(\frac{p}{2m}+\frac{1}{2}m\omega_o^2 r\right)\right] , \qquad (4.20)$$

where

$$\Theta = e^{-\hbar\omega_o/kT} .$$

This is the Wigner distribution (2.3) for the harmonic oscillator (Feynman 1972). From it all the usual quantum results may be obtained, including the analysis into discrete levels of energy $(n+\frac{1}{2})\hbar\omega_o$. But these no longer correspond to stable eigenstates; not only are they simply components of the equilibrium state, but the instantaneous energy of the oscillator fluctuates widely around them. Yet because these fluctuations are highly correlated, the width of emission or absorption lines is much narrower and in fact exactly equals the quantum-theoretical prediction. Note here that these conclusions resolve the difficulty the formal theories had with the excited states, since these no longer have an independent existence; we see here one of the strengths of SED as a physical theory.

The same agreement with quantum mechanics is found when a constant magnetic field is added to the harmonic-oscillator force; already in 1963, Marshall was able to show from this that SED predicts precisely the diamagnetic behaviour found quantum-mechanically at all temperatures. Indeed, even the spin may be taken into account (de la Peña and Jáuregui 1982).

It is worth noting that if one further order of correction

terms beyond the Markov approximation is introduced, an integral giving the Lamb shift for the harmonic oscillator is obtained. This integral diverges logarithmically, a difficulty also faced by QED. This divergence is due to that of the background radiation density, Eq. (4.11). Since it subsists even for the free particle, the observed Lamb shift should be given by the difference; this now converges, and the result indeed agrees with QED (de la Peña and Cetto 1979).

We have not discussed the derivation of the Schrödinger equation in SED. Wherever a Markov approximation to the SED process is satisfactory, such a derivation is possible and simply follows along the lines already described in section 3. Indeed, for any Markov process with velocities derivable from potential functions, a Schrödinger equation may be written (de la Peña 1967). But where the non-Markovian aspects are important, the Schrödinger equation would contain modifications and additional terms. This is to be expected: since SED explicitly takes into account the interaction with the electromagnetic field (or at least its random component), the quantum-theoretical equivalent of SED must contain both radiation terms and others possibly reflecting spin effects; and neither appear in the Schrödinger equation.

That spin is intrinsic to SED is strongly suggested by the consideration that in any stochastic motion of a particle (Markovian or not) under the influence only of central forces we must have that $\langle \underline{L} \rangle = 0$, where $\underline{L} = \underline{r} \times \underline{p}$ is the angular momentum, while $\langle L^2 \rangle \neq 0$. Using the harmonic oscillator as a model and letting its frequency go to 0, one may calculate from (4.13) that

$$\langle \underline{L}^2 \rangle = \frac{3}{2} \hbar^2 \quad . \tag{4.21}$$

This is twice the quantum-theoretical value if this is indeed the spin (Marshall 1963; Boyer 1975b; the spin interpretation is due to Jáuregui and de la Peña 1981, de la Peña and Jáuregui 1982). One can also show that this "spin", even if its value is not yet right, does add to the orbital momentum and gives rise to the correct gyromagnetic ratio.

5. PROBLEMS AND PERSPECTIVES IN STOCHASTIC ELECTRODYNAMICS

A large number of other problems have also been tackled from the starting point of SED; several results have been obtained that are entirely satisfactory, in the sense of coinciding with those of the usual quantum approach where this has a basis in experimental data, or else differing in ways not susceptible to experimental confirmation. SED thus appears to be a rather successful theory.

Yet in some problems quite unacceptable results turn up. From among these we here select two situations which we believe shed light on the outstanding questions in SED.

The first one is a study of n harmonic oscillators coupled through the zero-point field (Blanco and Santos 1979). The authors write the equations of motion in the form

$$m_i \ddot{\underline{r}}_i = f_i(\underline{r}_1 \ldots \underline{r}_n) + \frac{2e_i}{3c^3} \sum_{j=1}^{n} e_j \dddot{\underline{r}}_j + e_i \underline{E}(t) \qquad (i = 1 \ldots n) \quad . \qquad (5.1)$$

The radiation-reaction term includes also the interaction (to lowest order) between the particles and is derived in this form by Landau and Lifshitz (1964); the supposition that the field is the same for all particles is justified by the authors provided that the bounds R and Ω of the distances and frequencies satisfy

$$R\Omega << c \quad . \qquad (5.2)$$

This is reasonable if the oscillator energy, which is at most $mR^2\Omega^2$, is of the order of $\hbar\Omega$, for then $R\Omega = (\hbar\Omega/m)^{1/2}$ is a typical velocity. But now the authors show that the system (5.1) can be decoupled in at least one specific case into a single quantum-mechanical oscillator (having radiation-reaction and random-field terms) and n - 1 classical ones (having neither). Taking for simplicity the one-dimensional version of (5.1), with harmonic-oscillator forces, the equations of motion reduce to

$$\left[\frac{d^2}{dt^2} + K - \frac{2}{3c^2} \underline{Q}\,\underline{Q}^T \frac{d^3}{dt^3}\right] \underline{X}(t) = \underline{Q}\, E(t) \quad , \qquad (5.3)$$

where K is the diagonal matrix of force constants, $\underline{Q}$ is the vector of charges and $\underline{X}(t)$ that of positions; we have supposed the masses all = 1. Now if K is a multiple of the unit matrix, the Eqs. (5.3) will decouple if $\underline{Q}\,\underline{Q}^T$ (^{T} indicates the transpose) is diagonal; but this is possible only if all elements of $\underline{Q}$ but one are 0. Let this be the first; we then have

$$\ddot{x}_1 + k_1 x_1 - \frac{2e_1^2}{3c^3}\dddot{x}_1 = e_1 E(t) \quad , \tag{5.4a}$$

$$\ddot{x}_i + k_i x_i = 0 \qquad (i = 2 \ldots n) \quad . \tag{5.4b}$$

Blanco and Santos show that this case is actually more general than here described; but even if there is much less degeneracy in the system than needed for (5.4a,b) to result, they show (by an approximate procedure correct to second order) that the degenerate oscillator modes will have one stochastic mode only, the others becoming classical.

This paradoxical and unexpected result has given rise to some speculation. Further work remains to be done, but as we shall comment below, a main source of the problem turns out to be the dipole approximation used here for all the particles.

The second problematic situation is the hydrogen atom. Simple heuristic arguments (see, e.g., Claverie and Diner 1976) lead one to expect that SED should provide a satisfactory account in the shape of a stable equilibrium state with an ensemble average energy at the well confirmed value of - 13.6 eV. Now a Fokker-Planck equation for the hydrogen atom can be derived in various ways (Marshall and Claverie 1980 , Claverie and Soto 1982); the simplest would appear to be in terms of the relevant integrals of motion. The three classical ones, the energy E, the total angular momentum M and the excentricity ε of the orbit, are related by

$$\varepsilon = (1 + \frac{2M^2E}{mk^2})^{1/2} \quad , \tag{5.5}$$

where

$$V(r) = -\frac{k}{r} \tag{5.6}$$

is the potential and m the electron mass. Choosing E and M as the independent variables, the Fokker-Planck equation can be derived fairly straightforwardly; but it does not seem easily soluble. However, in the limit $\varepsilon \longrightarrow 0$ it simplifies to the form

$$\frac{\hbar m k^2}{2M^2}\frac{\partial W}{\partial E} + \frac{\hbar}{2}\frac{\partial W}{\partial M} + W = 0 \quad , \tag{5.7}$$

where W is the probability density in the (E,M,ε) space, which in the limit of circular orbits depends on only one variable, because of (5.5). Taking this to be M, we find

$$W = \alpha e^{-2M/\hbar} \quad . \tag{5.8}$$

This solution is not satisfactory, however. Its phase-space integral diverges, and hence the expected value $\langle E \rangle$ for the energy becomes 0. The hydrogen atom, in other words, has zero binding energy and will therefore ionize spontaneously. This corresponds very well to another aspect of the matter: the coefficient of W in the general Fokker-Planck equation —not (5.7)— can be shown to vanish as a consequence of the specific form of the radiation-reaction term, so that

$$W = \text{const} \tag{5.9}$$

is also a steady-state solution. Now not only is (5.9) also not integrable over phase space; the existence of two distinct solutions, (5.8) and (5.9), which correspond to $\partial W/\partial t = 0$ means that there is no equilibrium. This is intuitively obvious, for the system might jump from one of these to the other in an uncontrolled fashion, and yet the probability density would not change in time —except discontinuously and perhaps non-causally at the jumps. The conclusion can also be proved rigorously (Khas'minskii 1960): if a generalized diffusion process (such as the present one) is recurrent, then it has a unique invariant measure —a unique steady-state solution, that is to say, which corresponds to equilibrium. Conversely, if it has an integrable invariant measure, then the process is recurrent

and the measure is unique. "Recurrence" here is defined as follows: if with probability 1 a system returns to a given neighbourhood of its starting point within a finite time, it is recurrent. Note that a recurrent system is not necessarily ergodic; but a non-recurrent one is certainly not ergodic, for the probability of its definitely escaping from any finite region in phase space is greater than 0.

For the hydrogen atom, then, the Khas'minskii theorem shows that the SED solution is not ergodic, so that there cannot be an equilibrium state. This prediction is in complete and fundamental contradiction with standard quantum mechanics and with the experimental evidence.

The picture so far presented is this: SED arose from the attempt to give quantum mechanics a sounder conceptual basis than the Copenhagen interpretation allowed, and a sounder physical basis than the ensemble interpretation provided. In this the theory is clearly very successful. It has also achieved considerable success in treating a wide range of phenomena in detail: we have described some of the results above. But it still has to face a number of difficulties. We list here the ones already discussed:

- The stochastic background field has a spectral density proportional to ω^3, which makes the energy density divergent.
- The Braffort-Marshall equation (4.13), starting point for describing the motion of particles, is based on a number of approximations whose validity is not well established.
- Using these same approximations, coupled harmonic oscillators turn out to have only one stochastic mode of motion.
- The description offered by SED for certain systems, notably the hydrogen atom, is wrong.

To this list we might add a last matter, not yet discussed but implicit in what has gone before:

- SED does not explain the quantum behaviour of neutral particles, for which (4.13) degenerates into an ordinary Newtonian equation.

Some possible answers to these problems will be sketched below.

The spectral density

At least two possible sources of modification have been suggested for the unrestricted ω^3 law. Firstly, if $\hbar\omega >> 2mc^2$, the probability of pair creation can no longer be neglected. And secondly, for large ω the energy involved will come from a correspondingly large volume, where a flat space-time is no longer a good approximation; Lorentz invariance must be widened to a general-relativistic invariance. That this could remove the ω^3 divergence seems to be implied in the work of Ford (1976), who showed how to regularize the energy-momentum tensor in a closed Robertson-Walker metric so that the energy density for the electromagnetic vacuum is finite:

$$u = \frac{11\hbar c}{240\pi^2 R^4} \quad , \tag{5.10}$$

where R is the radius of the universe. One may speculate here that a more adequate theory could derive the value of $\hbar$ from cosmological considerations. The scope of SED would thus widen in a remarkable fashion.

It should be noticed that the divergences in SED create problems quite analogous to those in QED. They may be side-stepped, as in QED, by using a cutoff frequency (usually of the order of $mc^2/\hbar$) or by means of a renormalization procedure. But in SED these methods appear to have more physical significance than in QED: thus the argument that using a cutoff corresponds to ignoring all but a relatively near neighbourhood of the system finds a ready interpretation in the physical model underlying SED. Moreover, there is less need to renormalize in SED (Cavalleri 1981).

The Braffort-Marshall equation

The approximate arguments mentioned in connection with the derivation of the Braffort-Marshall equation are not of equal importance; nor are the improvements offered by recent work of equal value. Thus rewriting (4.13) in relativistic terms, though it would represent a significant step forward, remains essentially impossible because no theoretical background for relativistic stochastic processes is available; we do not know how to formulate relativistic ensembles in a consistent way.

As for the background field, it is taken to be isotropic, homogeneous and characterized by a temperature T. The distribution of the fluctuations is assumed to be Gaussian. All of these assumptions are quite well justified when the system under consideration is far from other matter; but they are clearly inadequate when other particles are near. For instance, the SED treatment of the double slit will evidently be based on the anisotropy of the field fluctuations felt by a particle going through one slit whenever the other slit is open. To put it in the anthropomorphic language habitual in the discussion of this problem, in SED it is neither the particle nor the slit it is going through that "knows" whether the other slit is open, it is the random background field that "transmits this information" and influences the particle's motion. But we are not yet in a position to work with random fields having a more complicated distribution.

The inadequacies of the Abraham-Lorentz term, on the other hand, are by now well understood. The term, in Lorentz' derivation, is computed by considering an extended structure for the particle (with a charge density $\chi(\underline{r})$ which is a continuous function of the position vector), and at the end going to zero particle radius. In this last operation an infinite electromagnetic contribution to the particle's mass appears, run-away solutions become possible where the acceleration increases exponentially in the absence of all external forces, and apparently acausal behaviour may be manifested. But taking this limit may be avoided if the particle is accepted as possessing extension and hence internal structure; the question of how this structure and its stability is to be explained remains open —unless perhaps Casimir's suggestion (see above) turns out to work, after all. Within classical electrodynamics this is discussed by Kaup (1966), Moniz and Sharp (1977), França, Marques and da Silva (1978), de la Peña, Jiménez and Montemayor (1982), and others quoted in these papers. For SED the equation of motion now takes the form

$$m\mu\ddot{\underline{r}}(t) = \underline{f}(\underline{r},t) + e\underline{E}(\underline{r},t) - m\eta \int_{-\infty}^{t} g(t-t')\ddot{\underline{r}}(t')dt' \quad , \qquad (5.11)$$

where

$$g(t) = \int_0^\infty k\,dk\,|\tilde{\chi}(k)|^2 \sin kct \quad , \tag{5.12}$$

and normally falls off very rapidly with t. It is derived from the form factor $\tilde{\chi}(k)$ of the charge density, the details of which turn out to be unimportant provided its radius is larger than $\tau c = \frac{2}{3} r_o$, where r_o is the classical electron radius. The factor η is given by

$$\eta = 16\pi^2 \mu \tau^2 \quad , \tag{5.13}$$

and

$$\mu = 1 + 16\pi^2 \tau c \int_0^\infty g(t')dt' \tag{5.14}$$

provides the (finite) electromagnetic mass correction.

Unfortunately (5.11), with its extended memory term, is even less easy to work with than (4.13). So far suitable techniques have been developed only for the case when the external force $\underline{f}$ depends on nothing but t. Thus a consistent description of the free particle has been given, and for a square-well potential results have been obtained that are in entire agreement with quantum mechanics; this is, indeed, the first strongly non-linear problem for which satisfactory answers have been found in SED, suggesting that this approach is along the right lines.

The form of (5.11) presupposes that the particle has spherical symmetry and is completely rigid; the approach described here can in principle be extended to remove these rather unphysical restrictions, by writing the charge distribution as

$$\chi = \chi(\underline{r}, \dot{\underline{r}}, \ddot{\underline{r}}, \ldots) \quad , \tag{5.15}$$

so as to make it depend on the instantaneous conditions of the motion. No explicit dependence on t should appear, unless the particle's motion is already fixed for all time. Of course, the resulting equations are even less tractable.

Multi-component systems

The result obtained by Blanco and Santos, though disconcerting

at first sight, can be shown to be due to the special assumptions made in setting up (5.1), and in particular due to the dipole approximation. If we write M,Q for the total mass and charge of the n-particle system, $\underline{F}$ for the external force and $\underline{R},\underline{\rho}$ for the position vectors of the centre of mass and the centre of charge, the sum of all the equations (5.1) is

$$M\ddot{\underline{R}} = \underline{F} + \frac{2}{3c^3} Q^2 \dddot{\underline{\rho}} + Q\underline{E}(t) \quad , \tag{5.16}$$

and subtracting this from (5.1) we have

$$m_i \ddot{\underline{r}}_i - \frac{e_i}{Q} M\ddot{\underline{R}} = \underline{f}_i - \frac{e_i}{Q} \underline{F} \quad , \qquad (i = 2...n) \quad , \tag{5.17}$$

where one equation is redundant. Thus only the motion of the centre of mass has a stochastic component, while the individual variables

$$\underline{r}_i' = \underline{r}_i - \frac{Me_i}{m_i Q} \underline{R}$$

are purely classical in nature. Clearly the dipole approximation, in making the radiation reaction identical for all particles, has removed an essential part of the problem; but a more satisfactory formulation remains to be worked out. Note, by the way, that the need for both $\underline{R}$ and $\underline{\rho}$ here is an indication that we are dealing with an open system, which may acquire a global angular momentum from the random background.

The hydrogen atom

The failure of SED in predicting no stable bound state for the hydrogen atom is in a sense inverse to that of the Schrödinger equation, which —lacking any radiative interaction— predicts an infinite number of stable states. This comparison suggests that the nature of stability in SED (and also in quantum mechanics!) requires reexamination. All the methods employed so far are in some sense equivalent to solving the Fokker-Planck equation under the condition that the time derivative of the probability density vanishes. This yields the equilibrium distribution provided the system is ergodic; indeed, even stronger ergodic sup-

positions are tacitly made, in that the system is judged to reach this equilibrium in a very short time, of the order of 10^{-23}s for the electron. (The pre-equilibrium evolution of a system in SED is not expected to agree in any sense with quantum theory, and may even in the future provide the opportunity for experimental validation of the theory). But for an isolated hydrogen atom a simple handwaving argument serves to show that this cannot hold. If the electron in a ground-state orbit, presumably near - 13.6 eV, is subject to the background field of SED, there is a small but non-zero probability that it will suffer a fluctuation of positive energy that allows it to escape. This is true for any potential for which

$$\lim_{r\to\infty} V(\underline{r}) < \infty \quad . \tag{5.18}$$

And since astrophysical arguments show that for any lifetime much longer than about 20 years the autoionization of neutral cold monoatomic hydrogen would be unobservable, we may conclude that the SED prediction, though apparently grossly mistaken, could well be the physically correct one. The normally observed stability would then be due to the nearby presence of other atoms. Note that for $T > 0$ quantum mechanics already predicts spontaneous ionization, as was first shown by Brillouin (1930); see also Fermi (1924), Farley and Wing (1981).

The usual hydrogen ground state is then a long-lived metastable state which the present methods of SED do not allow us to recognize. A possible way out is suggested by the fact that, as we saw above, the stochastic process involved is strongly non-Markovian. Now if, from a Markovian process in several dimensions, we project out a lower-dimensional one, this is not in general Markovian any longer. The inverse is not necessarily true, of course; but a hint that it might help arises when attempting to derive a Fokker-Planck equation from (5.11): this is best done, not in the usual phase space, but in an extended one with $\underline{r},\dot{\underline{r}},\ddot{\underline{r}}$ as coordinates. This suggest that a higher-dimensional space, perhaps $(\underline{r},\dot{\underline{r}},\ddot{\underline{r}})$, perhaps an even larger one, is appropriate even when better approximations than (5.11) are used. Classical mechanics in such a space will have new integrals of motion, and a suitable choice of their values

could provide just the restriction which avoids the escape to infinity. The metastable ground state would then become stable in the higher-dimensional phase space so restricted, since now the electron motion recovers ergodicity; the usual methods would now apply to give appropriate solutions. This is entirely analogous to what is currently done in statistical mechanics, where the Hamiltonian flows, non-ergodic over the usual phase space, become ergodic when restricted to the hypersurface of constant energy. An additional advantage of such a procedure is that the process is probably more nearly Markovian in the $(\underline{r},\dot{\underline{r}},\ddot{\underline{r}})$ space.

We might add here that such considerations could become even more relevant if we take into account that the random background field is often, perhaps usually, not in full equilibrium, even if it is close to it; what effect this will have on quantum-like behaviour as described by SED remains an open question.

Neutral particles

From the point of view of SED, neutral particles fall into two categories: those with rest mass greater than zero, and those with zero (or almost zero) rest mass. The first present little problem; they are considered to be composite, and their components have charges; they have, in general, a non-zero magnetic moment; they have extension and are therefore polarizable. These are various ways of stating that they interact with the electromagnetic zero-point field much as charged particles do, and their quantum-like behaviour is therefore explained. Particles with no rest mass, on the other hand, cannot at present be described by SED, which almost everywhere makes the assumption that particle speeds are well below the extreme relativistic limit.

Quite a number of speculative suggestions relevant in this connection have been made; but only one, it seems to the present author, merits serious attention. It is due to Santos (1975b, 1979). For every field known at present, elementary particles exist that both interact with it and possess electric charges. Santos argues that thrcugh the mediation of such particles all fields acquire a background of random fluctuations. If, furthermore, these fluctuations reach (or at least approach)

thermodynamical equilibrium, their statistical characteristics, and in particular, their energy amplitudes, will be shared. Hence we can understand why Planck's constant, which describes this amplitude (see Eqs. (4.1) and (4.2)), is a universal constant. Santos' universal stochastic theory would explain why particles show the typical quantum fluctuations even when they do not interact with the electromagnetic field.

Santos' theory is also relevant to another matter which has not received much discussion. Many of the arguments used above to establish the reality of the random component in the electromagnetic field could, and indeed have been, applied to the gravitational field, and conceptions such as random fluctuations in the metric tensor or in space-time have been proposed (see, e.g., March 1934, 1937; Yukawa 1966; Blokhintsev 1975; Frederick 1976; Namsrai 1980a, 1980b, 1981; Vigier 1982). If we accept Santos' view, any discussion of the relative merits of such theories and SED would be wide of the mark: they do not exclude each other. The great advantage of SED is then simply that a non-relativistic version is possible, on the basis of a well understood classical theory, and with considerable experimental evidence available for the behaviour of the random fluctuations.

6. CONCLUSIONS

In order to assess the value of what has so far been achieved, it must be stressed that the aims of SED —in this unlike the formal stochastic theories— are not simply to reproduce the results of standard quantum mechanics, and even less to reproduce them exactly. SED arose out of the need to provide a physically plausible theory that should complete the picture offered by the ensemble interpretation of quantum mechanics; its aim must therefore be that of fitting experiment at least as well as quantum mechanics does, and to offer a basis for going beyond the present limits of this theory.

SED has certainly achieved a clarity and simplicity of conceptual structure quite out of reach of quantum mechanics (in its Copenhagen version, at least). It has established that systems showing quantum-like

behaviour must be treated as open systems, with a stochastic interaction with the rest of the universe. It has also shown that this interaction may be treated as an electromagnetic zero-point field with well established statistical properties. It is, furthermore, a theory completely devoid of the conceptual paradoxes we associate with quantum theory. Finally, it contains (as does quantum mechanics) only one undetermined quantity, namely Planck's constant; with, however, an entirely different physical meaning.

SED has also given adequate and detailed accounts of a large number of phenomena, ranging from the van der Waals forces through the Planck distribution for cavity radiation to the level structure of the quantum harmonic oscillator. It has also given partial accounts of several other problems. The series of difficulties discussed above that it still faces show, however, that it is in no sense yet a finished theory.

Indeed, what is perhaps most striking is the variety of approximations that must be made before any specific problem can be tackled. Occasionally, these approximations may even be excessive, as in the case of the coupled harmonic oscillators. Moreover, even when the approximations seem appropriate, the mathematical apparatus is complicated, sometimes clumsy, rarely transparent. Thus SED offers a striking contrast to quantum mechanics: a clear and physically plausible conceptual structure combined with an unduly approximate and somewhat unclear mathematical formalism on the one hand, and obscure and contradictory conceptual background together with a highly developed, indeed elegant, formalism on the other.

This is not to suggest, of course, that some sort of compromise could be effected to take advantage of what each side has to offer. It is intended, rather, to underline the direction that, in the author's view, future efforts could usefully take. For it is evident from the many suggestions for further work commented upon in the preceding section that ample possibilities for development and for new ideas in SED remain open. But, whatever improvements these may bring with them in the mathematical methods used, there is every reason to believe that all this will be achieved without any major shift in the basic conceptions of the theory.

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