CAUSAL VERSION OF THE ABRAHAM-LORENTZ EQUATION FOR POINT PARTICLES

A.M. Cetto and L. de la Peña

Instituto de Física, UNAM
Apartado Postal 20-364. 01000 - México, D.F.

ABSTRACT

The Abraham-Lorentz equation of motion for charged point particles is beset by well-known problems which are only partly eliminated in the Lorentz-Dirac theory. Using a consistent nonrelativistic hamiltonian treatment of the radiating point particle, we obtain an expression for the self-force which can be interpreted in terms of an effective structure acquired by the particle via the radiation field. A simplifying approximation of this expression leads to a modified Abraham-Lorentz equation, exempt of the classical difficulties. One is thus led to conclude that the usual difficulties are introduced by the approximations and are not an integral part of the theory.

RESUMEN

La ecuación de Abraham-Lorentz para partículas puntuales con carga está plagada de dificultades bien conocidas, que sólo en parte pueden ser resueltas con la teoría de Lorentz-Dirac. A partir de un tratamiento hamiltoniano no relativista para la partícula puntual que radía, obtenemos una expresión para la autofuerza que puede ser interpretada en

términos de una estructura efectiva adquirida por la partícula a través
del campo de radiación. Una simplificación aproximada de esta expresión
conduce a una ecuación modificada de Abraham-Lorentz, exenta de las difi-
cultades clásicas. De ello se puede concluir que las dificultades usua-
les son producidas por las aproximaciones, y que no constituyen parte in-
tegral de la teoría.

1. THE ABRAHAM-LORENTZ EQUATION

The classical motion of a charged particle is commonly de-
scribed by the Abraham-Lorentz (AL) equation:

$$m\ddot{\vec{x}} = \vec{F} + m\tau\dddot{\vec{x}} \quad (\tau = 2e^2/3mc^3) \quad , \tag{1}$$

which takes account of the self-interaction of the particle in two ways,
namely: i) the inclusion of the electromagnetic contribution to the mass
in m, and ii) the addition of the radiation reaction force $m\tau\dddot{\vec{x}}$.

There exist numerous alternative derivations of Eq. (1). The
simplest and best known one is usually performed[1], following Lorentz[2],
by representing the retarded potentials of the radiated field as a power
series of the relaxation time t - x/c and computing the mechanical force
of the corresponding field on the particle. We recall that the first
terms of this expansion are: a) the so-called inertial reaction of the
particle, which gives rise to the mass correction $\delta m = mc\tau/a$; b) the re-
action force $m\tau\dddot{\vec{x}}$; c) higher derivative terms of the form $m(a/c)^2\ddddot{\vec{x}}$, etc.,
which are assumed small in order for the series to converge. In Eq. (1)
they have been neglected.

Here a/c is the time required by the light to traverse the ra-
dius of the particle; for an electron with classical radius, this time
would be of the order of τ.

Eq. (1) presents several difficulties, some of which can be eli-
minated by performing more careful derivations of the self-field or of
the effects produced by it. Briefly stated:
i) The first term in the Lorentz expansion diverges for point particles,
giving rise to an infinite electromagnetic mass; this is a well-known,
persistent problem of classical electrodynamics. For extended particles,
δm is finite but still has a wrong value. This problem can be eliminated

only by carrying out a non-orthodox relativistic treatment[1,3].

ii) Eq. (1) does not account for the radiation of a particle moving with constant acceleration, in contrast with Larmor's law of radiation. Moreover, the appearance of the term $m\tau\dddot{\vec{x}}$ implies the need of an additional initial condition which is due to the approximation procedure, not to the dynamics.

iii) Eq. (1) implies non-causality by allowing for preacceleration effects and run-away solutions. Either one of these problems —but not both— may be eliminated with a suitable choice of the initial or the final value of $\ddot{\vec{x}}$[4,5]. The usual procedure is to eliminate the run-away solution by a proper choice of $\ddot{\vec{x}}(t \to \infty)$; Eq. (1) leads then to

$$m\dddot{\vec{x}} = \frac{1}{\tau} \int_t^\infty dt' e^{-(t'-t)/\tau} \dot{\vec{F}}(t') \quad ,$$

which obviously implies a non-causal (advanced) relationship between force and acceleration. By using an extended-particle model, however, it is possible to recover causality under certain general restrictions[6-8]. For a survey of the various approaches to the subject, we refer the reader to the literature[9].

Summarizing, we may say that there is no satisfactory equation of motion for a radiating point charge subject to an arbitrary external field. The usual practical solution to this problem consists in introducing a new approximation which happens to remove at once all these drawbacks by a sort of compensation of errors: deriving Eq. (1) written to zero order in τ one obtains $\ddot{\vec{x}} = \vec{F}/m$, which substituted back in Eq. (1) gives a causal equation

$$m\ddot{\vec{x}} = \vec{F} + \tau\dot{\vec{F}} \quad ,$$

valid whenever $m\tau|\dddot{\vec{x}}| << |\dot{\vec{F}}|$. However, in this paper we are concerned with a matter of principle, namely, how it happens that an originally causal hamiltonian —as is the hamiltonian of classical electrodynamics— can lead to a non-causal equation of motion. Our purpose is to construct a consistent classical (nonrelativistic) description of the dynamics of the point particle without the above mentioned drawbacks.

2. EQUATION OF MOTION FOR A CHARGED POINT PARTICLE

We start with the hamiltonian treatment of a charged point particle subject to both an external field and its own radiation field. The complete hamiltonian is

$$H = (\vec{p} - \frac{e}{c} \vec{A})^2/2m + V + H_r \quad , \tag{2}$$

where $\vec{A}$ is the vector potential of the total electromagnetic field, H_r is the hamiltonian of the radiation field, namely

$$H_r = \frac{1}{8\pi} \int d^3x (\vec{E}^2 + \vec{B}^2) \quad , \tag{3}$$

and V is the total scalar potential.

Without loss of generality, we may assume that there are no charges and currents present other than those due to the charged particle. In this case the field is simply a radiation field, with V=0 and $\nabla \cdot \vec{A} = 0$ in the Coulomb gauge. The term V in Eq. (2) refers then to the external potential only, and $\vec{A}$ is the vector potential associated to the (transverse) radiation field, which can be conveniently represented in terms of plane travelling waves. Assuming the field to be contained in a cavity of volume L^3 with perfectly conducting walls (we shall eventually let $L \rightarrow \infty$), we write as usual[10]

$$\vec{A} = \sqrt{\frac{4\pi c^2}{L^3}} \sum_{n\sigma} \hat{\epsilon}_{n\sigma} (q_{n\sigma} \cos \vec{k}_n \cdot \vec{x} - \frac{p_{n\sigma}}{\omega_n} \sin \vec{k}_n \cdot \vec{x}) \quad , \tag{4}$$

where

$$k_n = \omega_n/c \quad , \qquad \vec{k}_n = 2\pi(n_1\hat{i} + n_2\hat{j} + n_3\hat{k})/L \quad ,$$

$$\sum_\sigma \epsilon_{n\sigma i} \epsilon_{n\sigma j} = \delta_{ij} - k_i k_j/k^2 \quad , \qquad \hat{\epsilon}_{n\sigma} \cdot \hat{\epsilon}_{n\sigma'} = \delta_{\sigma\sigma'} \quad , \tag{5}$$

$$\vec{k}_n \cdot \hat{\epsilon}_{n\sigma} = 0 \quad , \qquad \sigma, \sigma' = 1, 2 \quad ;$$

and in the free-space limit

$$\frac{1}{L^3} \sum_n \xrightarrow{L \to \infty} \frac{1}{(2\pi c)^3} \int_o^\infty d\omega\, \omega^2 \int_\Omega d\Omega_k \quad .$$ (6)

The field hamiltonian is then, by Eqs. (3) and (4),

$$H_r = \frac{1}{2} \sum_{n\sigma} (p_{n\sigma}^2 + \omega_n^2 q_{n\sigma}^2) \quad ,$$ (7)

i.e., it is the hamiltonian of an infinite number of independent field modes of frequency ω_n and polarization σ. The complete hamiltonian is therefore, by Eqs. (2) and (7),

$$H = (\vec{p} - \frac{e}{c} \vec{A})^2/2m + V + \frac{1}{2} \sum_{n\sigma} (p_{n\sigma}^2 + \omega_n^2 q_{n\sigma}^2) \quad .$$ (8)

By deriving Hamilton's equations in terms of the pairs of canonical variables $(\vec{x}, \vec{p})$ and $(q_{n\sigma}, p_{n\sigma})$ we obtain the set of coupled dynamical equations:

$$m\dot{\vec{x}} = \vec{p} - \frac{e}{c} \vec{A} \quad ,$$

$$\dot{\vec{p}} = \vec{F} - e' \sum_{n\sigma} (\dot{\vec{x}} \cdot \hat{\epsilon}_{n\sigma}) \vec{k}_n (q_{n\sigma} \sin \vec{k}_n \cdot \vec{x} + \frac{p_{n\sigma}}{\omega_n} \cos \vec{k}_n \cdot \vec{x})$$ (9)

and

$$\dot{q}_{n\sigma} = p_{n\sigma} + e' \dot{\vec{x}} \cdot \hat{\epsilon}_{n\sigma} \frac{\sin \vec{k}_n \cdot \vec{x}}{\omega_n} \quad ,$$

$$\dot{p}_{n\sigma} = - \omega_n^2 q_{n\sigma} + e' \dot{\vec{x}} \cdot \hat{\epsilon}_{n\sigma} \cos \vec{k}_n \cdot \vec{x} \quad ,$$ (10)

where $\vec{F} = -\nabla V$ and $e' = e\sqrt{4\pi/L^3}$.

In terms of the new (field) variable

$$Z_{n\sigma} = p_{n\sigma} + i\omega_n q_{n\sigma}$$ (11)

542

and its complex conjugate, Eqs. (10) can be rewritten in the form

$$\dot{Z}_{n\sigma} = i\omega_n Z_{n\sigma} + e' \, \vec{\dot{x}} \cdot \hat{\epsilon}_{n\sigma} e^{i\vec{k}_n \cdot \vec{x}} \quad , \tag{12}$$

and its complex conjugate. The first integral of this equation is

$$Z_{n\sigma} = e^{i\omega_n t} \left[Z_{n\sigma}(0) + e' \int_o^t dt' \, \vec{\dot{x}}' \cdot \hat{\epsilon}_{n\sigma} e^{i(\vec{k}_n \cdot \vec{x}' - \omega_n t')} \right] \; , \tag{13}$$

where $\vec{x}' = \vec{x}(t')$, and $t = 0$ is chosen as the starting time of the particle-field interaction.

On the other hand, by combining Eqs. (9) and using (13) we have

$$m\vec{\ddot{x}} = \vec{F} - \frac{e'}{2} \sum_{n\sigma} (\vec{\epsilon}_{n\sigma} + \vec{\dot{x}} \times \frac{\vec{k}_n \times \vec{\epsilon}_{n\sigma}}{\omega_n}) \left[Z_{n\sigma}(0) e^{-i\vec{k}_n\vec{x}} + \right.$$
$$\left. + e' \int_o^t dt' \, \vec{\dot{x}}' \cdot \vec{\epsilon}_{n\sigma} e^{-i\vec{k}_n (\vec{x} - \vec{x}')} + c.c. \right] \quad , \tag{14}$$

where $\vec{k}_n \vec{x} = \vec{k}_n \cdot \vec{x} - \omega_n t$. This equation shows that the Lorentz force consists of two parts: a free-field force —which we may either ignore or absorb into the external force $\vec{F}$, since its effects are irrelevant for our purposes— and an additional contribution arising from the particle itself. Denoting the latter by $\vec{F}_a$,

$$\vec{F}_a = - \frac{e'^2}{2} \sum_{n\sigma} (\vec{\epsilon}_{n\sigma} + \vec{\dot{x}} \times \frac{\vec{k}_n \times \vec{\epsilon}_{n\sigma}}{\omega_n}) \int_o^t dt' \vec{\dot{x}}' \cdot \vec{\epsilon}_{n\sigma} e^{-i\vec{k}_n (\vec{x}-\vec{x}')} + c.c.$$
$$\tag{15}$$

we have

$$m\vec{\ddot{x}} = \vec{F} + \vec{F}_a \quad ,$$

which is an exact equation of motion for the self-interacting charged point particle acted on by the external force $\vec{F}$.

3. CALCULATION OF THE SELF-FORCE

Eq. (16) is exact, but it cannot be solved exactly. It is therefore reasonable to analyze the self-force $\vec{F}_a$ and try to rewrite it in a more convenient, though approximate form. As a first step we sum over the polarization index σ and take the limit $L \to \infty$; we then have

$$\vec{F}_a = -\frac{e^2}{4\pi^2 c^3} \int_0^\infty d\omega\, \omega^2 \int_\Omega d\Omega_k \int_0^t dt'\, \vec{u}\, e^{-i\vec{k}(\vec{x} - \vec{x}')} + c.c. \quad , \qquad (17)$$

where

$$\vec{u} = (1 - \frac{\vec{k} \cdot \dot{\vec{x}}}{c})\dot{\vec{x}}' + (\frac{\dot{\vec{x}} \cdot \dot{\vec{x}}'}{c} - \hat{k} \cdot \dot{\vec{x}}')\hat{k} \quad ; \quad \hat{k} = \vec{k}/k \quad . \qquad (18)$$

After performing the angular integration we are left with*

$$\vec{F}_a = \frac{4e^2}{\pi c^3} \int_0^\infty d\omega\, \omega^2 \int_0^t dt'\, \dot{\vec{x}}'\, k \cos \omega(t - t') \quad , \qquad (19)$$

where

$$k = \frac{\cos \omega s}{\omega^2 s^2} - \frac{\sin \omega s}{\omega^3 s^3} \quad ; \quad s = \frac{|\vec{x} - \vec{x}'|}{c} \quad . \qquad (20)$$

It is evident that $\vec{F}_a$ cannot be calculated exactly, since both $\vec{x}$ and k depend on the trajectory of the particle, i.e., on the solution of Eq. (16). This is —though somewhat rephrased— the old problem of classical electrodynamics, which gives rise to the Lorentz approximation and other similar treatments of the self-action of the charged particle. In fact, the usual AL equation can be derived by neglecting the retardation in $s = |\vec{x} - \vec{x}'|/c$, i.e., taking $s = 0$ and hence $k = -1/3$ in Eq. (19).

---

* In writing Eq. (19) we have neglected a small extra force that depends on the changes in the instantaneous direction of the velocity that occur in the time interval $0 < t' < t$. Therefore, Eq. (19) is exact only for one-dimensional problems. It is always possible to add this extra term if required, but since we are here concerned with a matter of principle, we have preferred to omit it for the sake of simplicity.

The result thus obtained (after an integration by parts):

$$\vec{F}_a^{\,o} = \frac{2e^2}{3c^3} \left(-\frac{2}{\pi} \ddot{\vec{x}} \int_0^\infty d\omega + \dddot{\vec{x}} \right) \quad , \tag{21}$$

contains both the (infinite) electromagnetic contribution to the mass:

$$\delta m^o = \frac{2m\tau}{\pi} \int_0^\infty d\omega$$

and the (acausal) Lorentz radiation reaction force:

$$\vec{F}_r^{\,o} = m\tau \dddot{\vec{x}} \quad . \tag{23}$$

The system of Eqs. (16) and (19) is, however, causal, with $\vec{F}_a$ finite; we see then that the usual problems connected with the AL equation can be traced back to the approximation made above, i.e., to the assumption that the spatial structure of the field can be neglected, which is a reasonable assumption for the low-frequency modes, but not for every ω.

We therefore propose an approximation which still allows us to handle Eq. (19), but avoids the problems created by the (too rough) dipole approximation. Considering that k is a complicated function of ω and t, which starts at $k(0) = -1/3$ and oscillates with an amplitude decreasing as $(\omega s)^{-2}$ for large values of ωs, we propose to use instead a simpler function given by

$$k_\lambda(\omega) = -\frac{1}{3} \frac{1}{1 + \lambda^2\omega^2} \quad , \tag{24}$$

where λ is a positive parameter measured in units of time, and which may be estimated by introducing some appropriate additional requirement (see Sect. 5). Since k_λ does not depend on s, Eq. (19) may be integrated by parts:

$$\vec{F}_a = \frac{4e^2}{\pi c^3} \int_0^\infty d\omega \; \omega k_\lambda \left[\dot{\vec{x}}_o \sin \omega t + \int_0^t dt' \; \dddot{\vec{x}}' \sin \omega(t - t') \right] \quad . \tag{25}$$

Explicit calculation gives

$$\vec{F}_a = -\frac{m\tau}{\lambda^2} \left[\ddot{\vec{x}}_o e^{-t/\lambda} + \int_o^t dt' \; \dddot{\vec{x}}' \cdot e^{-(t-t')/\lambda} \right] \quad , \tag{26}$$

which can be once more integrated by parts to give

$$\vec{F}_a = \frac{m\tau}{\lambda} \left[(\ddot{\vec{x}}_o - \frac{\dddot{\vec{x}}_o}{\lambda}) e^{-t/\lambda} - \dddot{\vec{x}} + \int_o^t dt' \; \ddddot{\vec{x}}' \; e^{-(t-t')/\lambda} \right] , \tag{27}$$

where $\ddot{\vec{x}}_o$, $\dddot{\vec{x}}_o$ denote the initial values of $\ddot{\vec{x}}(t)$, $\dddot{\vec{x}}(t)$.

4. MODIFIED ABRAHAM-LORENTZ EQUATION

By Eq. (26), the equation of motion (16) takes the form of a modified AL equation:

$$m\ddot{\vec{x}} = \vec{F} - \frac{m\tau}{\lambda^2} \left[\ddot{\vec{x}}_o e^{-t/\lambda} + \int_o^t dt' \; \dddot{\vec{x}}' \cdot e^{-(t-t')/\lambda} \right] \quad , \tag{28}$$

which can be written alternatively as a third-order equation by combining Eq. (27) and its time derivative, to eliminate the integral term; one thus obtains

$$m(1 + \tau/\lambda)\ddot{\vec{x}} = \vec{F} + \lambda\dot{\vec{F}} - \lambda m\dddot{\vec{x}} \quad . \tag{29}$$

One must recall, however, that no additional integration constant is re-quired, since the physical solutions of Eq. (29) must also satisfy Eq. (28). As Eq. (29) shows, the electromagnetic contribution to the mass is now finite:

$$\delta m = (\tau/\lambda)m \quad , \tag{30}$$

and the radiation reaction force is

$$\vec{F}_r = \lambda(\dot{\vec{F}} - m\dddot{\vec{x}}) \tag{31}$$

instead of $m\tau\dddot{\vec{x}}$. Notice that this expression for the radiation reaction

depends not only implicitly, but also explicitly on the external force.

Eq. (28) shows that the introduction of the self-force does not violate causality, since only retarded effects are present. To see this more clearly, one can perform one more integration by parts to obtain from Eq. (28):

$$m\ddot{\vec{x}} = \vec{F} - \frac{m\tau}{\lambda^2} \left[\dot{\vec{x}} - \frac{1}{\lambda} \int_0^t dt' \ \dot{\vec{x}}' \ e^{-(t-t')/\lambda} \right] \quad .$$

This result shows that the acceleration depends on the force and on the velocity evaluated at times $t' \leqslant t$ only.

In order to further analyze the implications of this modified self-force, let us write the first integral of Eq. (28):

$$(m + \delta m)\dot{\vec{x}} = (1 + \frac{\tau}{\lambda} e^{-\Omega t})m\dot{\vec{x}}_0 + \int_0^t dt' \ [\ 1 + \frac{\tau}{\lambda} e^{-\Omega(t-t')} \ ]\vec{F}(t') \quad , \qquad (32)$$

where $\Omega = (1 + \tau/\lambda)/\lambda$. In the radiationless limit ($\tau \to 0$) as well as for large λ, Eq. (32) goes to the newtonian equation

$$m\dot{\vec{x}} = m\dot{\vec{x}}_0 + \int_0^t dt'\vec{F}(t') \quad ,$$

while in the "structureless-field" limit (dipole approximation) we recover the usual AL results, as a Taylor series development shows. With both τ and λ different from zero, the r.h.s. of Eq. (32) contains (besides the newtonian terms) a transient contribution and an additional force, both of which reflect the existence of memory.

5. DISCUSSION AND CONCLUSIONS

The finite result for δm and the presence of memory may both be interpreted as non-local effects having their origin in the space-dependence of the interaction of the particle with its own radiation field. This in its turn amounts to ascribing to the particle an effective (induced) structure, λ being a measure of the corresponding radius. To see this more clearly, let us compare Eq. (25) with the expression

for the self-force of an extended particle with a fixed, spherically symmetrical charge distribution characterized by the form factor $f(\omega/c)$. Following Bohm and Weinstein[11], one obtains in the dipole (long-wavelength) approximation:

$$\vec{F'}_a = -\frac{4e^2}{3\pi c^3} \int_0^\infty d\omega\, \omega\, |f(\tfrac{\omega}{c})|^2 [\, \vec{x}_0 \sin \omega t + \int_0^t dt'\, \vec{x'} \sin \omega(t - t')\,] \quad . \quad (33)$$

A direct comparison would lead us to conclude that our point particle has acquired an effective structure such that

$$|f(\tfrac{\omega}{c})|^2 = -3k_\lambda(\omega) \qquad (34)$$

as a result of its interaction with the (space-dependent) self-field.

We should be, however, more cautious with this interpretation. The self-force expressed in Eq. (25) is due to the action of the field emitted by an originally point particle, upon the particle itself. The form factor should therefore appear only once —not twice, as in Eq. (34)— to take account of the effective structure <u>acquired</u> by the particle via the field. To further motivate our argument, we recall that the form factor of a uniform charge distribution of radius a is[11]

$$f_a(\tfrac{\omega}{c}) = -3 \left[\frac{\cos \omega a/c}{(\omega a/c)^2} - \frac{\sin \omega a/c}{(\omega a/c)^3} \right] \quad , \qquad (35)$$

which coincides in its functional form with the exact expression for k given in Eq. (20). Hence, our expression for $\vec{F}_a$, Eq. (19), can be read as the force acting on a particle of uniform charge distribution of (variable) radius $|\vec{x} - \vec{x'}| = cs$ due to the action of the radiation field taken in its long-wavelength limit. In other words, the particle has acquired an effective structure of radius cs as a result of the action of the self-field.

The transition from Eq. (20) to (24) implies then taking a time-averaged, rigid (Yukawa[8]) charge distribution, the size of which is of the order of λc. For the electron we would expect this size to be not smaller than the classical radius and not larger than Compton's wavelength,

whence

$$\hbar/mc^2 > \lambda > e^2/mc^3 \sim \tau \quad ,$$

i.e., λ may be of the order of τ or larger, and can therefore not be neg-
lected in Eqs. (28) - (32). It should be remarked that at present-day
energies, quantum electrodynamics seems to be compatible with an electron
radius smaller than 10^{-16}cm, which is several orders of magnitude smaller
than the classical radius, e^2/mc^2 [12]. However, the effective radius
here discussed is obviously velocity-dependent and hence, no real contra-
diction arises within the nonrelativistic theory.

The specific choice of the function $k_\lambda(\omega)$ is actually arbitrary;
we have here selected a model which is mathematically simple and has the
convenience of adjusting quite well to the exact function $k(\omega s)$, with only
one free parameter. A different choice would not have altered the quali-
tative results —such as the causal behaviour—, though the details of
the dynamics would differ, especially at very short times.

Finally it should be pointed out that for practical purposes,
Eq. (28) or (29) serves to describe classical radiation damping as well
as does the AL equation after renormalization and elimination of the run-
away solutions. From a conceptual point of view, however, our derivation
has the advantage of explaining the origin of the classical difficulties
and showing how these can be avoided with a careful treatment of the
self-force that does not artificially remove causality.

REFERENCES

1. J.D. Jackson, Classical Electrodynamics. John Wiley, N.Y. (1967)
   Sect. 17.3.
2. H.A. Lorentz, The Theory of Electrons, B.G. Teubner, Leipzig (1909)
   Note 18.
3. P.A.M. Dirac, Proc. Roy. Soc. A167 (1938) 148.
4. D. Iwanenko and A. Sokolow, Klassische Feldtheorie. Akademie Verlag,
   Berlin (1953).
5. F. Rohrlich, Ann. Phys. (N.Y.) 13 (1961) 93.
6. D.J. Kaup, Phys. Rev. 152 (1966) 1130; H. Levine, E.J. Moniz and D.H.
   Sharp, Am. J. Phys. 45 (1977) 75 and references therein.
7. H.M. França, G.C. Marques and A.J. da Silva, Nuovo Cim. 48A (1978) 65.
8. L. de la Peña, J.L. Jiménez and R. Montemayor, Nuovo Cim. 69B (1982)

71.
9. See, e.g., J.D. Jackson (loc. cit.) and R. Rohrlich, Classical Charged
   Particles, Addison-Wesley, Reading, Mass. (1965). Two reviews on the
   treatment of point particles are: G.N Plass, Rev. Mod. Phys. 33 (1961)
   37; C. Teitelboim, D. Villarroel and Ch. G. van Weert, Riv. Nuovo Cim.
   3 (1980) 1. A more ample reference on extended particles may be found
   in Ref. 6 and in T. Erber, Fortschr. Phys. 9 (1961) 343.
10. We follow the notation of W. Louisell, Radiation and Noise in Quantum
    Electronics. McGraw-Hill, N.Y. (1964).
11. D. Bohm and M. Weinstein, Phys. Rev. 74 (1948) 1789.
12. V. Hepp, in Present Status and Aims of Quantum Electrodynamics, G.
    Gräff, E. Klempt and G. Werth, editors. Lecture Notes in Physics 143,
    Springer Verlag, Berlin (1981).