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AN EXTENSION OF LANDAU DIAGRAMS FOR CLASSICAL COLLISIONS

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ABSTRACT

The classical collision of two particles has been studied in the case of arbitrary kinematical initial conditions for them. Because the angle $(\alpha_1+\alpha_2)$ between the final velocities of the particles is not a Galilean invariant the process was studied directly in the laboratory system.

An expression is given for $(\alpha_1+\alpha_2)$ as a function of η, which is the angle that carries the dynamical information.

The special case of equal masses, also has been analyzed and a pictorical representation of the process, related with the Landau collision diagrams is given.

RESUMEN

La colisión clásica de dos partículas ha sido estudiada para el caso de condiciones iniciales cinemáticas arbitrarias. Debido a que el án

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gulo $(\alpha_1+\alpha_2)$ entre las velocidades finales de las partículas no es invariante galileano, el proceso fue estudiado directamente en el sistema del laboratorio.

Se da una expresión para $(\alpha_1+\alpha_2)$ como función de η, que es el ángulo que contiene la información dinámica.

El caso particular de masas iguales, también ha sido analizado y se da una representación gráfica del proceso, relacionada con los diagramas de colisión de Landau.

INTRODUCTION

A general analysis of the classical elastic collision of two particles has been carried out starting from Landau diagrams(1) for the description of kinetic energy and momentum conservation. In this paper both particles have been considered to be initially moving in the laboratory system. By means of a Galilean transformation this case can be reduced to the very well known collision description with the target particle at rest in the initial state.

Nevertheless, if the general case with arbitrary kinematical initial state in the laboratory system in studied knowing η, the dynamical parameter of the collision process and the total angular momentum vector, it is possible to extract conclusions concerning the angle between trajectories in the final state as a function of η parameter.

The angles of emergence α_1 and α_2 of each particle are given relative to the direction of the total momentum lineal vector of the system, which is privileged for calculation purposes in the laboratory system.

The general expression for $(\alpha_1+\alpha_2)$ has been obtained and the particular case of two equal mass particles is later discussed, to make an estimate of the departure of $(\alpha_1+\alpha_2)$ from $\pi/2$.

This angle between trajectories in the final state $(\alpha_1+\alpha_2)$ is a function of another angle η which is related to the scattering angle as expected.

For a better visualization of the kinematical state new diagrams have been introduced for which reason this analysis offers a pictorical representation of pedagogical value.

THE LANDAU DESCRIPTION

If the force of interaction is along the line of center, or the interaction potential U(r) depends only on the distance between the two particles, the angular momentum vector of the two particles system is concerved during the encounter, and it is orthogonal to the plane of collision.

In the following the scattering angle χ will be taken as a parameter[2] thus allowing to solve completely the classical collision.

In the laboratory system the final momenta[7] (primed vectors) of each particle are given by

$$\vec{P}_1' = \mu|\vec{V}|\hat{n}_0 + \frac{\mu}{m_2}\ \vec{P}_T \tag{1}$$

and

$$\vec{P}_2' = -\mu|\vec{V}|\hat{n}_0 + \frac{\mu}{m_1}\ \vec{P}_T\ , \tag{2}$$

where μ is the reduced mass, $\vec{P}_T$ the total momentum of the two body system, $\hat{n}_0$ is the direction in which the relative velocity has turned[4], and m_1, m_2 are respectively the masses of particles 1 and 2.

Let us represent the collision process geometrically through the Landau diagram where $\vec{P}_1$, $\vec{P}_2$ are respectively the initial momenta of each particle in the laboratory system (Fig. 1).

As $\vec{P}_T$ is a constant of motion , it defines a direction in the laboratory system which we will use to define the angles of emergence α_1 and α_2 of the final momenta $\vec{P}_1'$ and $\vec{P}_2'$, respectively.

Angle η is defined between the direction of the total momentum and that of the final relative velocity, that is

$$\cos\eta = \hat{n}_0 \cdot \hat{P}_T. \tag{3}$$

The problem is then reduced to finding the relation between $(\alpha_1+\alpha_2)$, η and χ .

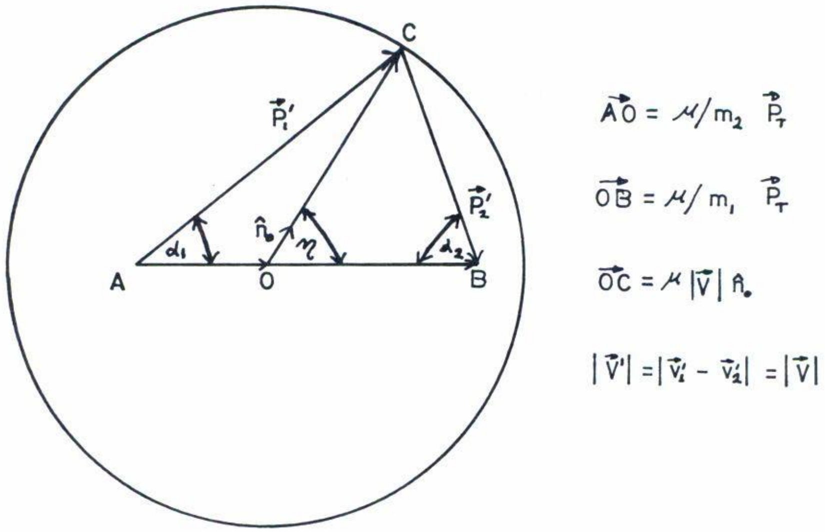

Fig. 1. Graphical representation of conservation of linear momentum in an elastic collision due to Landau.

TRIGONOMETRIC RELATION FOR THE ANGLE BETWEEN TRAJECTORIES IN THE FINAL STATE $(\alpha_1+\alpha_2)$

For example, it will be supposed that $|\vec{v}_1| > |\vec{v}_2|$; this imposes no restriction to the problem.

In order to simplify notation let us define a new variable $\vec{P}_R \equiv \mu\vec{V}$.

It can be seen from Fig. 1 that

$$\tan\alpha_1 = \frac{|P_R|\,\sin\eta}{\mu/m_2\,|P_T| + |P_R|\cos\eta}$$

and

$$\tan\alpha_2 = \frac{|P_R|\sin\eta}{\mu/m_1\,|P_T| - |P_R|\cos\eta} \qquad (4.a)$$

To find the relation between $(\alpha_1+\alpha_2)$ and η, the well known relation of the addition of the tangents can be employed, this relation give

$$\tan\,(\alpha_1+\alpha_2) = \frac{|P_R|\ |P_T|\sin\,\eta}{\mu^2/m_1m_2\,|P_T|^2 + |P_T|\,|P_R|\left(\frac{m_1-m_2}{m_1+m_2}\right)\cos\eta - |P_R|^2}\,. \qquad (4.b)$$

It is interesting to check this expression with some known result. A very well known one is that in which two equal mass particles collide, one of them being at rest in the laboratory system, in which case $(\alpha_1+\alpha_2)$ is $\pi/2$.

If $m_1 = m_2$ and $\vec{v}_2 = 0$, it can be seen that Eq. (4.b) yields the espected result (in this case $\left|\frac{P_T}{2}\right| = |P_R|$) .

RELATION BETWEEN η AND THE SCATTERING ANGLE χ

Before considering the functional expression for the relation between η and χ it is convenient to review their definitions.

η is the angle between the final relative velocity vector and the total momentum lineal, that is an angle measured in the laboratory system.

χ is defined as the scattering angle measured in the C.M. system. Nevertheless, as this angle measures the amount that the relative velocity vector has turned it can also be determined in the laboratory system[6], that is , according to their definitions:

$$\cos\,\eta = \hat{V}'\cdot\hat{P}_T \quad , \qquad \cos\,\chi = \hat{V}'\cdot\hat{V} \quad . \qquad (5)$$

In what follows, a graphical representation of these angles measured in the laboratory system will be given. For the special case of having $\vec{V}_2 = 0$, it is possible to compare the magnitudes of these angles, in which case $\hat{V} = \hat{v}_1$ and $\hat{P}_T = \hat{v}_1$, which implies $\eta=\chi$, then when target is at rest the Eq. (4.a) give the very well known relation:

$$2\alpha_2 + \eta = \pi \quad ,$$

$$\tan\,\alpha_1 = \frac{\sin\,\eta}{\cos\,\eta + m_1/m_2} \quad .$$

The relation sought between η and χ adopts a simple expression as it is seen in Fig. 2 :

$$\chi + \eta = \pi - \beta_i \qquad \text{if} \qquad \chi < \pi - \beta_i$$

and (6)

$$\chi - \eta = \pi - \beta_i \qquad \text{if} \qquad \chi > \pi - \beta_i \ ,$$

where β_i has the following expression:

$$\cos \beta_i = \frac{-\vec{P}_R \cdot \vec{P}_T}{|P_R|\,|P_T|} \ . \tag{7}$$

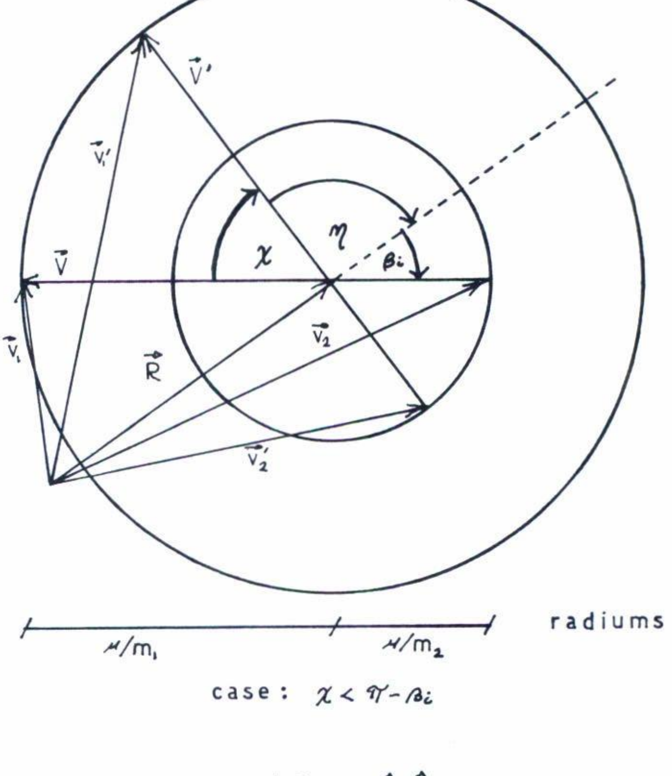

Fig. 2. Relation between χ and η angles, for the general case. As we see the angle β_i is completely given by the initial conditions. The relations $\vec{v}_1' = \mu/m_1 |\vec{V}| \hat{n}_0 + \vec{R}$ and $\vec{v}_2' = -\mu/m_2 |\vec{V}| \hat{n}_0 + \vec{R}$ are always satisfied.

It is possible to see that two simple configurations can be obtained when:

A) $\vec{v}_2 = 0$, which implies $\cos\beta_i = -1$, that is, $\beta_i = \pi$, and then $\chi = \eta$.

B) When $m_1 = m_2$ and $|\vec{v}_1| = |\vec{v}_2|$, which implies $\cos\beta_i = 0$, that is, $\beta_i = \pi/2$ and then

$$\chi + \eta = \pi/2 \qquad \text{if} \qquad \chi < \pi/2$$

and

$$\chi - \eta = \pi/2 \qquad \text{if} \qquad \chi > \pi/2 \ .$$

Special Case, $m_1 = m_2$.

Taking $m_1 = m_2$ in Eq. (4.b) a very simple expression is obtained for the angle between trajectories in the final state $(\alpha_1+\alpha_2)$:

$$\begin{aligned} \tan(\alpha_1+\alpha_2) &= \frac{|P_R| \ |P_T| \sin\eta}{\left|\frac{P_T}{2}\right|^2 - |P_R|^2} \\ &= \frac{|\vec{v}_1 - \vec{v}_2||\vec{v}_1 + \vec{v}_2|}{|\vec{v}_1 + \vec{v}_2|^2 - |\vec{v}_1 - \vec{v}_2|^2} \, 2\sin\eta \ , \end{aligned} \tag{8}$$

in which a great difference is evident with respect to the case commonly studied where the target particle is supossed to be at rest in the initial state, being $(\alpha_1+\alpha_2)$ equal to $\pi/2$ the difference here found is that the angle $(\alpha_1+\alpha_2)$ appears as a function of the dynamics of the system.

When $\vec{v}_2 = 0$, Eq. (8) trivially leads to $(\alpha_1+\alpha_2) = \pi/2$.

It is now desidered to evaluate to what extent the angle $(\alpha_1+\alpha_2)$ departs from the value $\pi/2$. It is qualitatively possible to see that this discrepancy turns appreciable when $|\vec{v}_1|$ approaches $|\vec{v}_2|$, being strongly dependent upon the initial orientation of these vectors.

Nevertheless, when $|\vec{v}_1| >> |\vec{v}_2|$, the difference between $|\vec{v}_1 - \vec{v}_2|$ and $|\vec{v}_1 + \vec{v}_2|$ turns negligible and $(\alpha_1+\alpha_2)$ can be taken as $\pi/2$. The approximate amount of the correction to the value $\pi/2$ will be found in the following particular case.

Supposing the order of the quotient of the moduli of the initial velocities to be: $|\vec{v}_2|/|\vec{v}_1| = \varepsilon$; then it is possible to estimate the angle $(\alpha_1+\alpha_2)$ when $\varepsilon << 1$.

Within an error of the same order ε, the quantities $|\vec{v}_1 - \vec{v}_2|$ and $|\vec{v}_1 + \vec{v}_2|$ can be taken simply as the difference and sum of the moduli of the initial velocities, and then

$$|\vec{v}_1 - \vec{v}_2| \cong |\vec{v}_1| \ (1 - \varepsilon)$$

and (9)

$$|\vec{v}_1 + \vec{v}_2| \cong |\vec{v}_1| \ (1 + \varepsilon) \quad ,$$

so, the expression $\tan(\alpha_1+\alpha_2)$ is given by the following expression as a function of the order ε^{-1} neglecting terms of order ε^2:

$$|\tan(\alpha_1+\alpha_2)| \cong \frac{1}{2\varepsilon} |\sin \eta| \ . \quad (10)$$

Within an error of the same order ε, χ can be taken as η and suppossing $\sin \eta$ to assume the largest value, it is possible to estimate $|\tan(\alpha_1+\alpha_2)|$.

When quotient of the moduli of the initial velocities are 0.01 or 0.1, the discrepancy of the angle $(\alpha_1+\alpha_2)$ relative to the value $\pi/2$ is approximately 2% and 13% respectively.

DISCUSSION

The general study carried out concerning the classical collision with arbitrary kinematical initial conditions for both particles is useful for classroom discussion of the kinematical aspects of the collision process.

Although the equation found for the angle between trajectories in the final state has a rather complicated expression, it is easy to check with the commonly studied case $\vec{v}_2 = 0$. Nevertheless, the equation assumes a simple form for $m_1 = m_2$, for which reason the analysis of this particular case was carried out.

The benefits obtained in this complete analysis case of arbitrary kinematical initial conditions for two particles are just pedagogic, and in the same maner the graphical representation takes a simple form of didactic value.

ACKNOWLEDGEMENTS

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NOTES

1. L.D. Landau & E.M. Lifshitz. Course of Theoretical Physics, Vol. 1 Mechanics, Addison Wesley Pub. Co. (1960) Ch. 4.
2. It can be seen that the relation between the scattering angle, χ, and the interaction potential is given by Eq. 18 2 (Landau & Lifshitz):

$$\chi = |\pi - 2\phi_0|; \qquad \phi_0 = \int_{r_{min}}^{\infty} \frac{M/r^2 \, dr}{\sqrt{2m(E - U(r) - M^2/r^2}} \quad .$$

3. In an elastic collision, the final relative velocity vector is given by $\vec{V}' = |\vec{V}|\hat{n}_0$.
4. Or the direction in which particle 1 has turned in the center of mass system.
5. It can be seen that it is perfectly valid to redefine the target particle and the projectile not invalidating the equation.
6. Bearing in mind that this angle, instead, is a Galilean invariant.
7. From now on, all primed vectors will belong to final states in the Laboratory System.