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Nonlinear equations describing controlled-current transients in semiconductors and insulators

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Abstract. The nonlinear equations describing the charging and discharging electrical transients under controlled-current conditions across a single-carrier semiconductor (or insulator) are analytically solved. Planar, cylindrical, and spherical geometries are studied, and the effect of traps and diffusion is investigated. The inclusion of diffusion leads to an extension of the Burgers equation, which can be solved in terms of displaced Airy functions.

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1. Introduction

The mathematical modelling of electrical transients in semiconductors and insulators involves many different problems, including: the effect of trapping and diffusion [1–3], electrode-limited transients [4], photoelectronic processes [5–7], high field conduction [8], numerical models [9,10], cylindrical and spherical geometries [11], and two-carrier conduction processes [12–14]. All of these problems are described by nonlinear partial differential equations (PDEs), with an appropriate set of initial and boundary conditions. If these boundary-value problems are examined, an interesting similarity can be found: in most of them a constant-voltage constraint is considered. Inverse processes, in which the voltage changes with time while the current is held constant, have been experimentally studied, but there are few mathematical models which describe such processes precisely.

Recently it has been shown that interesting results can be analytically obtained from the system of PDEs which describes the transient electrical conduction in two-carrier semiconducting or insulating solids, if a constant-current condition is considered [15]. In Ref. [15], however, the influence of traps and the diffusion of carriers were not considered, and only the transport between planar electrodes was studied.

In the present paper the electrical transients in a single-carrier semiconductor (or insulator) under constant-current conditions will be studied. The paper starts with the simplest problem of planar electrodes without traps and without diffusion (Sect. 2), then considers the problem with cylindrical and spherical geometries (Sects. 3 and 4), and finally studies the effect of traps and diffusion (Sects. 5 and 6).

In each of these five sections both, the charging and discharging transients, are studied.

2. Electron flow without traps and without diffusion

2.1. Charging transient

The solution of the boundary-value problem considered in this section has been presented elsewhere [15], but it is included here for the sake of completeness.

The equations describing the electron flow through a single-carrier semiconductor (or insulator) in the absence of trapping and diffusion are the total current density equation, Poisson's equation, and the equation of continuity. For planar geometry, these equations are, respectively [1]

$$J_0' = q\mu n'(x',t')E'(x',t') + \varepsilon\frac{\partial E'}{\partial t'}, \tag{1}$$

$$\frac{\partial E'}{\partial x'} = -\frac{q}{\varepsilon}\Big[n'(x',t') - n_0'\Big], \tag{2}$$

$$\frac{\partial}{\partial x'}\Big[q\mu n'(x',t')E'(x',t')\Big] = q\frac{\partial n'}{\partial t'}, \tag{3}$$

where q is the absolute value of the electronic charge, μ the electron mobility, n' the free-electron density, n_0' the free-electron density under thermal equilibrium, ε the permittivity (or dielectric constant), E' the electric field, and J_0' the total current density. Using Eqs. (2) and (3) it can be proven that J_0' is independent of x'. Once this is known, it is unnecessary to consider Eq. (3) any further, as it becomes a consequence of Eqs. (1) and (2).

If d' is the separation between the electrodes, and J_u' is a characteristic (positive) value of the total current density, we can define the following dimensionless quantities

$$x = \frac{x'}{d'}, \qquad t = \left(\frac{\mu J_u'}{\varepsilon d'}\right)^{1/2} t',$$

$$n = \left(\frac{\mu d'}{\varepsilon J_u'}\right)^{1/2} qn', \qquad J_0 = \frac{J_0'}{J_u'},$$

$$E = \left(\frac{\varepsilon\mu}{d' J_u'}\right)^{1/2} E'$$

Substituting these variables in Eqs. (1) and (2), and neglecting n_0' in comparison with n', the following dimensionless equations are obtained

$$J = n(x,t)E(x,t) + \frac{\partial E}{\partial t},$$

$$\frac{\partial E}{\partial x} = -n(x,t).$$

This pair of equations can be transformed into a single first order quasilinear PDE for the (dimensionless) electric field

$$-EE_x + E_t = J_0. \tag{4}$$

As initial and boundary conditions we shall consider the following

$$E(x,0) = 0 \quad (x > 0), \tag{5}$$

$$E(0,t) = 0 \quad (t > 0). \tag{6}$$

Equation (5) implies that the free-electron density is zero at $t = 0$, and Eq. (6) means that an ohmic contact is assumed at $x = 0$. Finally, as we are interested in controlled-current transients, we have to specify the temporal variation of the total current density. We shall consider the simplest condition: a step current excitation. Therefore, we will consider that J_0 is a negative constant for $t > 0$.

Even though the Eq. (4) is not a linear one, the boundary-value problem (4)–(6) can be exactly solved by the method of characteristics [16] due to the simplicity of the initial and boundary conditions. Applying this method, the solution is found to be

$$E(x,t) = \begin{cases} -(-2J_0x)^{1/2}, & x \le -J_0t^2/2, \\ J_0t, & x \ge -J_0t^2/2 \end{cases} \tag{7}$$

and from this expression we can obtain the free-electron density:

$$n(x,t) = \begin{cases} \left(-\dfrac{J_0}{2x}\right)^{1/2}, & x \le -J_0t^2/2, \\ 0, & x \ge -J_0t^2/2. \end{cases} \tag{8}$$

This equation shows that the characteristic curve,

$$x = -\tfrac{1}{2}J_0t^2, \tag{9}$$

is the advancing front of the electrons. Therefore, the first electrons arrive at the anode (located at $x = 1$) at the instant

$$t_0 = \left(-\frac{2}{J_0}\right)^{1/2} \tag{10}$$

and this is also the time at which the steady state is established.

If desired, the time-variation of the voltage could be obtained calculating the integral

$$V(t) = -\int_0^1 E(x,t)\,dx.$$

2.2. Discharging transient

In Subsection 2.1 we found that a steady state is reached at the instant t_0 given by Eq. (10). Now, in this part, we shall investigate what happens if the current is suddenly interrupted at an instant $t_1 > t_0$. The boundary-value problem describing this decaying transient is the following

$$-EE_x + E_t = 0, \tag{11}$$

$$E(x,t_1) = -(-2J_0x)^{1/2} \qquad (x > 0), \tag{12}$$

$$E(0,t) = 0 \qquad (t > t_1) \tag{13}$$

Again, this problem can be solved by the method of characteristics. The solution is found to be

$$E(x,t) = -J_0(t-t_1) - \Big[-2J_0x + J_0^2(t-t_1)^2\Big]^{1/2}. \tag{14}$$

To visualize the shape of this surface it is useful to observe that the constant-field trajectories in the plane $x - t$

$$E(x,t) = E_0 \qquad (E_0 = \text{constant}).$$

are just the straight lines

$$t = \left(-\frac{1}{E_0}\right)x + \left(t_1 - \frac{E_0}{2J_0}\right).$$

Therefore, the electric-field surface $E(x,t)$ stretches along these lines, which are precisely the characteristic curves corresponding to the problem (11)–(13). Figure 1 shows some of these characteristic lines.

If desired, the free-electron density, $n(x,t)$, and the transient voltage, $V(t)$, could be easily found from Eq. (14).

3. Coaxial cylindrical electrodes

3.1. Charging transient

Consider now a single-carrier semiconductor without traps and without diffusion, placed between two coaxial cylindrical electrodes of radii r_1' and r_2' ($r_1' < r_2'$), and

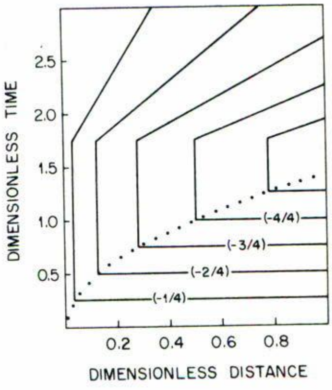

FIGURE 1. Level curves of the surface $E(x,t)$ in the absence of traps and diffusion, for $J_0 = -1$. The dotted parabola corresponds to the front of the advancing electrons.

length L'. In this case the equations describing the electron flow are the following

$$\frac{I_0'}{2\pi r' L'} = q\mu n'(r',t')E'(r',t') + \varepsilon \frac{\partial E'}{\partial t'}, \tag{15}$$

$$\frac{1}{r'}\frac{\partial}{\partial r'}(r'E') = -\frac{q}{\varepsilon} n'(r',t'), \tag{16}$$

$$\frac{1}{r'}\frac{\partial}{\partial r'}\left[q\mu r' n'(r',t')E'(r',t')\right] = q\frac{\partial}{\partial t'} n'(r',t'). \tag{17}$$

From these equations it can be proven that I_0' is independent of r'. Once this is known it is unnecessary to consider Eq. (17) any further, as it becomes a consequence of Eqs. (15) and (16).

If d' is the separation between the electrodes, and I_u' is a characteristic (positive) value of the total current, we can define the following dimensionless quantities

$$r = \frac{r'}{d'} \qquad t = \left(\frac{\mu I_u'}{\varepsilon d'^3}\right)^{1/2} t',$$

$$n = \left(\frac{\mu d'^3}{\varepsilon I_u'}\right)^{1/2} q n', \qquad I_0 = \frac{I_0'}{I_u'},$$

$$E = \left(\frac{\varepsilon \mu d'}{I_u'}\right)^{1/2} E', \qquad L = \frac{L'}{d'}.$$

In terms of these quantities, Eqs. (15) and (16) take the form

$$\frac{I_0}{2\pi r L} = nE + \frac{\partial E}{\partial t},$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE) = -n.$$

From these equations we can obtain a single nonlinear PDE for the (dimensionless) electric field:

$$E\frac{\partial}{\partial r}(rE) - \frac{\partial}{\partial t}(rE) = -\frac{I_0}{2\pi L}. \tag{18}$$

Then, if we define

$$u = rE, \tag{19}$$

Equation (18) takes the form

$$\frac{u}{r}u_r - u_t = -\frac{I_0}{2\pi L}. \tag{20}$$

If we consider the initial and boundary conditions

$$u(r,0) = 0 \qquad (r_1 \le r \le r_2), \tag{21}$$

$$u(r_1,t) = 0 \qquad (0 \le t), \tag{22}$$

and we assume that I_0 has a constant negative value, the boundary-value problem (20)-(22) can be solved by the method of characteristics, and the solution is found to be

$$u(r,t) = \begin{cases} -\left[-\frac{I_0}{2\pi L}(r^2 - r_1^2)\right]^{1/2} & t \ge f(r) \\ \frac{I_0}{2\pi L}t & t \le f(r) \end{cases}$$

where

$$f(r) = \left[-\frac{2\pi L}{I_0}(r^2 - r_1^2)\right]^{1/2}.$$

The function $u(r,t)$ implies that $E(r,t)$ and $n(r,t)$ have the form

$$E(r,t) = \begin{cases} -\left[-\frac{I_0}{2\pi L}\left(\frac{r^2 - r_1^2}{r^2}\right)\right]^{1/2}, & t \ge f(r), \\ \frac{I_0}{2\pi L}\frac{t}{r}, & t \le f(r), \end{cases} \tag{23}$$

$$n(r,t) = \begin{cases} -\frac{I_0}{2\pi L}\left[-\frac{I_0}{2\pi L}\left(r^2 - r_1^2\right)\right]^{-1/2}, & t \ge f(r), \\ 0, & t \le f(r), \end{cases} \tag{24}$$

From Eq. (24) we can see that the curve $t = f(r)$ describes the front of the advancing electrons. The first electrons arrive at the anode located at $r_2 = r_1 + 1$ at the instant

$$t_0(r_1) = f(r_1+1) = \left[-\frac{2\pi L}{I_0}(2r_1+1)\right]^{1/2}, \tag{25}$$

and at this moment a steady state is established.

The time variation of the voltage can be obtained calculating the integral

$$V(t) = -\int_{r_1}^{r_1+1} E(r,t)\,dr.$$

3.2. Discharging Transient (cylindrical electrodes)

Let us consider the discharging transient in the limit $r_1 \to 0$. From Eqs. (18) and (23) we can see that this transient is described by the equations

$$E\frac{\partial}{\partial r}(rE) - \frac{\partial}{\partial t}(rE) = 0,$$

$$E(r,t_1) = -\left(-\frac{I_0}{2\pi L}\right)^{1/2} \qquad (r>0),$$

$$E(0,t) = 0 \qquad (t \geq t_1),$$

where t_1 is any value greater than $t_0(0)$. In terms of the variable u, defined by Eq. (19), this boundary-value problem takes the form

$$\frac{u}{r}u_r - u_t = 0, \tag{26}$$

$$u(r,t_1) = -r\left(-\frac{I_0}{2\pi L}\right)^{1/2} \qquad (r>0), \tag{27}$$

$$u(0,t) = 0 \qquad (t \geq t_1). \tag{28}$$

This problem can be solved by the method of characteristics. The solution, in terms of the function $E(r,t)$, is

$$E(r,t) = E_s^2\left(\frac{t-t_1}{r}\right) + E_s\left[E_s^2\left(\frac{t-t_1}{r}\right)^2 + 1\right]^{1/2} \qquad (r>0), \tag{29}$$

where E_s is the steady state value of the electric field, that is

$$E_s = -\left(-\frac{I_0}{2\pi L}\right)^{1/2}.$$

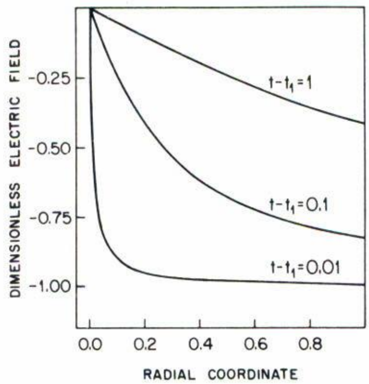

FIGURE 2. Radial dependence of the electric field with coaxial cylindrical electrodes, at different moments during the decaying transient, for $I_0 = -2\pi L$ and $r_1 = 0$.

In Fig. 2 we can observe the spatial behavior of the function $E(r,t)$ for different values of t. To visualize the surface $E(r,t)$ it is also useful to look at the constant-field trajectories

$$E(r,t) = E_0 \qquad (E = \text{constant}).$$

From Eq. (29) it follows that these trajectories are the straight lines

$$t = \left(\frac{E_0}{2E_s^2} - \frac{1}{2E_0}\right) r + t_1.$$

Therefore, the surface $E(r,t)$ stretches following these lines.

If desired, the transient voltage can be found evaluating the integral

$$V(t) = -\int_0^1 E(r,t)\, dr.$$

4. Concentric spherical electrodes

4.1. Charging transient

Next let us consider the problem with spherical symmetry. The equations describing the electron flow across a semiconductor without traps and without diffusion, placed between two concentric spherical electrodes of radii r_1' and $r_2' = r_1' + d'$ are

$$\frac{I_0'}{4\pi r'^2} = q\mu n'(r',t')E'(r',t') + \varepsilon \frac{\partial E'}{\partial t'}, \tag{30}$$

$$\frac{1}{r'^2}\frac{\partial}{\partial r'}(r'^2 E') = -\frac{q}{\varepsilon}n'(r', t'), \tag{31}$$

$$\frac{1}{r'^2}\frac{\partial}{\partial r'}\left[r'^2 q\mu n'(r', t')E'(r', t')\right] = q\frac{\partial}{\partial t'}n'(r', t') \tag{32}$$

As in the preceding section, from these equations it can be proven that I_0 does not depend on r'. Once this is known, it is not necessary to consider Eq. (32) any further, as it becomes a consequence of Eq. (30) and (31).

If I'_u is a characteristic (positive) value of the total current we can introduce the following dimensionless variables

$$r = \frac{r'}{d'} \qquad t = \left(\frac{\mu I'_u}{\varepsilon d'^3}\right)^{1/2} t'$$

$$n = \left(\frac{\mu d'^3}{\varepsilon I'_u}\right)^{1/2} qn' \qquad I_0 = \frac{I'_0}{I'_u},$$

$$E = \left(\frac{\varepsilon\mu d'}{I'_u}\right)^{1/2} E'.$$

In terms of these variables the Eqs. (30) and (31) take the form

$$\frac{I_0}{4\pi r^2} = nE + \frac{\partial E}{\partial t},$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E) = -n.$$

These equations can be combined into a single equation for the electric field

$$\frac{I_0}{4\pi r^2} = -\frac{E}{r^2}\frac{\partial}{\partial r}(r^2 E) + \frac{\partial E}{\partial t}, \tag{33}$$

which, with the change of variables

$$u = r^2 E, \tag{34}$$

becomes

$$\frac{u}{r^2}u_r - u_t = -\frac{I_0}{4\pi}. \tag{35}$$

The initial and boundary conditions for this equation will be

$$u(r, 0) = 0 \qquad (r_1 \le r \le r_2), \tag{36}$$

$$u(r_1, t) = 0 \qquad (0 \le t). \tag{37}$$

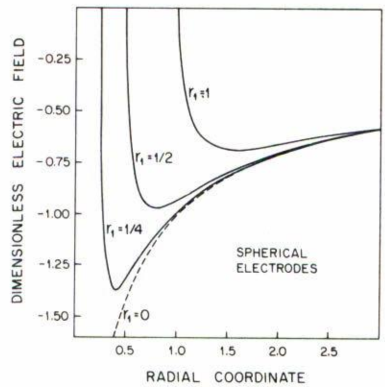

FIGURE 3. Steady state distribution of the electric field with concentric spherical electrodes, for $I_0 = -6\pi$ and different values of the inner radius r_1.

Considering that I_0 is a (negative) constant, the boundary-value problem (35)-(37) can be solved by the method of characteristics. The solution of this problem, in terms of the function $E(r,t)$, is

$$E(r,t) = \begin{cases} -\left[-\frac{I_0}{6\pi}\left(\frac{r^3 - r_1^3}{r^4}\right)\right]^{1/2} & t \geq f(r), \\ \frac{I_0}{4\pi}\frac{t}{r^2} & t \leq f(r), \end{cases} \tag{38}$$

where

$$f(r) = \left[\frac{2}{3}\left(-\frac{4\pi}{I_0}\right)(r^3 - r_1^3)\right]^{1/2}. \tag{39}$$

The spatial behavior of the function $E(r,t)$, for $t \geq f(r)$ and different values of r_1, is shown in Fig. 3.

The curve $t = f(r)$ is the advancing front of the electrons. The first electrons arrive at the anode, located at $r_2 = r_1 + 1$, at the instant

$$t_0(r_1) = f(r_1 + 1) = \left[\frac{2}{3}\left(-\frac{4\pi}{I_0}\right)(3r_1^2 + 3r_1 + 1)\right]^{1/2}, \tag{40}$$

and at this moment a steady state is established.

The voltage transient

$$V(t) = -\int_{r_1}^{r_2} E(r,t)\, dr$$

cannot be expressed in terms of elementary functions, as this equation leads to elliptic integrals. However, in the limit $r_1 \to 0$ this integral can be evaluated without problems.

4.2. Discharging transient (spherical electrodes)

As in the preceding section, we shall consider the discharging transient in the limit $r_1 \to 0$. From Eqs. (33) and (38) we can see that this transient is described by the equations

$$\frac{E}{r^2}\frac{\partial}{\partial r}(r^2 E) - \frac{\partial E}{\partial t} = 0,$$

$$E(r,t_1) = -\left(-\frac{I_0}{6\pi r}\right)^{1/2} \qquad (r > 0),$$

$$E(0,t) = 0 \qquad (t \geq t_1),$$

where t_1 is any value greater than $t_0(0)$. In terms of the variable u, defined by Eq. (34), this boundary-value problem becomes

$$\frac{u}{r^2}u_r - u_t = 0, \tag{41}$$

$$u(r,t_1) = -\left(-\frac{I_0}{6\pi}\right)^{1/2} r^{3/2} \qquad (r > 0), \tag{42}$$

$$u(0,t) = 0 \qquad (t \geq t_1). \tag{43}$$

This problem can be solved by the method of characteristics, and the solution, in terms of the function $E(r,t)$, is

$$E(r,t) = -\left(-\frac{I_0}{6\pi}\right)^{1/2}\left\{-\frac{3}{2}\frac{t-t_1}{r^2} + \left[\left(\frac{3}{2}\right)^2\left(\frac{t-t_1}{r^2}\right)^2 + \frac{1}{r}\right]^{1/2}\right\} \qquad (r > 0). \tag{44}$$

The spatial variation of this function, for different values of t, is shown in Fig. 4.

The voltage transient cannot be expressed in terms of elementary functions in this case, because the integration of the function $E(r,t)$ involves an elliptic integral.

5. Planar geometry with traps

5.1. Charging Transient

In this section we will consider a one-carrier semiconductor, with traps situated at a single energy level, and uniformly distributed in space. For planar geometry, the

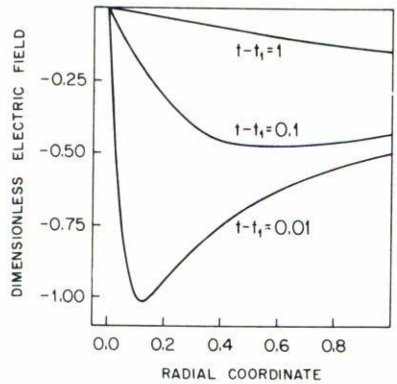

FIGURE 4. Radial dependence of the electric field with concentric spherical electrodes, at different moments during the decaying transient, in the limit with $r_1 = 0$.

equations describing the electron flow are the following

$$J_0' = q\mu n'(x',t')E'(x',t') + \varepsilon\frac{\partial E'}{\partial t'}, \tag{45}$$

$$\frac{\partial E'}{\partial x'} = -\frac{q}{\varepsilon}\left[n'(x',t') - n_0' + m'(x',t') - m_0'\right], \tag{46}$$

$$\frac{\partial m'}{\partial t'} = C\left\{n'(x',t')\left[N' - m'(x',t')\right] - \frac{N_c}{g}\exp\left[\frac{E_t - E_c}{kT}\right]m'(x',t')\right\}, \tag{47}$$

$$\frac{\partial}{\partial x'}\left[q\mu n'(x',t')E'(x',t')\right] = q\frac{\partial}{\partial t'}\left[n'(x',t') + m'(x',t')\right], \tag{48}$$

where J_0', q, ε, μ, n', n_0', and E' have the same meaning as in Section 2, and m' is the trapped electron density, m_0' the trapped electron density in thermal equilibrium, C the electron capture coefficient, N' the density of electron traps, E_t the trapping energy level, E_c the energy level at the edge of the conduction band, N_c the effective density of states in the conduction band, and g is the degeneracy factor of trap states. From the Eqs. (45), (46) and (48) it can be proven that J_0' does not depend on x'. Once this is known, it is unnecessary to consider Eq. (48) any further, as it becomes a consequence of Eqs. (45) and (46).

Following Batra and Seki [5], we will neglect the values n_0' and m_0' in Eq. (46), and introduce the approximation

$$N' - m' \approx N'$$

in Eq. (47). With these assumptions, the Eqs. (46) and (47) take the form

$$\frac{\partial E'}{\partial x'} = -\frac{q}{\varepsilon}\left[n'(x',t') + m'(x',t')\right], \tag{49}$$

$$\frac{\partial m'}{\partial t'} = CN'n'(x',t') - Cn_1'm'(x',t'), \tag{50}$$

where we have defined

$$n_1' = \frac{N_c}{g}\exp\left[\frac{E_t - E_c}{kT}\right].$$

Now, if we define the dimensionless variables x, t, n, J_0, and E exactly as in Sect. 2, and introduce the additional dimensionless quantities

$$m = \left(\frac{\mu d'}{\varepsilon J_u'}\right)^{1/2} qm', \qquad N = \left(\frac{\varepsilon d'}{\mu J_u'}\right)^{1/2} CN',$$

$$n_1 = \left(\frac{\varepsilon d'}{\mu J_u'}\right)^{1/2} Cn_1',$$

Equations (45), (49) and (50) take the form

$$J_0 = nE + E_t, \tag{51}$$

$$E_x = -n - m, \tag{52}$$

$$m_t = Nn - n_1 m. \tag{53}$$

Solving Eq. (52) for n, and substituting the resulting expression into Eqs. (51) and (53), the following equations are obtained

$$E_t - EE_x + 0 - (Em + J_0) = 0, \tag{54}$$

$$m_t + NE_x + 0 + (N + n_1)m = 0. \tag{55}$$

This system can be written in the matrix form

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x + \mathbf{B} = 0, \tag{56}$$

where **U** and **B** are the column vectors

$$\mathbf{U} = \begin{bmatrix} E \\ m \end{bmatrix}, \tag{57}$$

$$\mathbf{B} = \begin{bmatrix} -(Em + J_0) \\ (N + n_1)m \end{bmatrix}, \tag{58}$$

and **A** is the matrix

$$\mathbf{A} = \begin{bmatrix} -E & 0 \\ N & 0 \end{bmatrix}. \tag{59}$$

First order quasilinear systems of the form (56) can be classified as elliptic, parabolic, hyperbolic, strictly hyperbolic, or ultrahyperbolic systems [17], depending on the eigenvalues of the matrix **A**. In the present case, the two eigenvalues of the matrix **A** given by Eq. (59) are

$$\lambda_1 = 0, \qquad \lambda_2 = -E.$$

The fact that the two eigenvalues are real and distinct implies that the quasilinear system (54)–(55) is a strictly hyperbolic system, and therefore there exist two families of characteristic curves, given by the solutions of the equations

$$\frac{dx}{dt} = \lambda_i \qquad (i = 1, 2). \tag{60}$$

For $i = 1$ the characteristics are just the straight lines $x =$ const., but for $i = 2$ the characteristics cannot be found until the form of the function $E(x,t)$ is known.

To solve the quasilinear hyperbolic system (54)–(55) it is convenient to transform it into a single equation for the electric field. This is possible because we can obtain an expression for $m(x,t)$ in terms of the function $E(x,t)$, solving the Eq. (55) by the method of variation of parameters. In this manner we obtain

$$m(x,t) = \left[k(x) - \int_0^t N E_x e^{(N+n_1)t'} dt'\right] e^{-(N+n_1)t},$$

where $k(x)$ is an arbitrary function. If we consider the initial condition

$$m(x,0) = 0,$$

the function $k(x)$ must be identically zero, and therefore

$$m(x,t) = -N e^{-(N+n_1)t} \int_0^t E_x e^{(N+n_1)t'} dt'.$$

Substituting this expression into Eq. (54) the following nonlinear integro-differential equation is obtained

$$E_t - E E_x - J_0 - N e^{-(N+n_1)t} E \int_0^t E_x e^{(N+n_1)t'} dt' = 0. \tag{61}$$

We are interested in the particular solution of this equation consistent with the initial and boundary conditions

$$E(x,0) = 0 \qquad (x \geq 0), \tag{62}$$

$$E(0,t) = 0 \qquad (t \geq 0). \tag{63}$$

To solve this boundary-value problem notice that in the absence of traps the following equality holds

$$\int_0^t E_x e^{(N+n_1)t'} dt' = E_x \int_0^t e^{(N+n_1)t'} dt'.$$

In the presence of traps this equation is not exactly satisfied, but it remains a reasonable approximation which enables us to transform Eq. (61) into the following equation

$$E_t - \frac{n_1}{N+n_1}\left[1 + \frac{N}{n_1}e^{-(N+n_1)t}\right] EE_x = J_0. \tag{64}$$

For typical values of the parameters N and n_1, and any positive value of t, the second term in the square brackets appearing in this equation is much smaller than unity. Therefore we can neglect this term, thus obtaining

$$EE_x - AE_t = -AJ_0, \tag{65}$$

where we have defined

$$A = 1 + \frac{N}{n_1}. \tag{66}$$

Equation (65), with the initial and boundary conditions (62) and (63), can be solved by the method of characteristics, and the solution is found to be

$$E(x,t) = \begin{cases} -(-2AJ_0x)^{1/2}, & x \le -J_0t^2/2A, \\ J_0t, & x \ge -J_0t^2/2A. \end{cases} \tag{67}$$

This expression shows that a stationary state is established at the instant

$$t_0 = \left(-\frac{2A}{J_0}\right)^{1/2}. \tag{68}$$

Comparing this equation with Eq. (10) of Section 2 we can see that the presence of traps delays the establishment of a steady state, as one could have expected intuitively.

If desired, the transient voltage can be obtained from Eq. (67).

5.2. Discharging transient (with traps)

The discharging transient is described by the equations

$$E_t - EE_x + 0 + 0 - mE = 0, \tag{69}$$

$$0 + NE_x + m_t + 0 + (N+n_1)m = 0, \tag{70}$$

which is a system of the form

$$A_i E_t + B_i E_x + C_i m_t + D_i m_x + F_i = 0 \qquad (i = 1, 2).$$

Courant and Friedrichs [18] show that a system of this type can be transformed into the following system of characteristic equations

$$x_\alpha - \zeta_+ t_\alpha = 0,$$

$$x_\beta - \zeta_- t_\beta = 0,$$

$$TE_\alpha + (a\zeta_+ - S)m_\alpha + (K\zeta_+ - H)t_\alpha = 0,$$

$$TE_\beta + (a\zeta_- - S)m_\beta + (K\zeta_- - H)t_\beta = 0,$$

where ζ_+ and ζ_- are the solutions of the equation

$$a\zeta^2 - 2b\zeta + c = 0,$$

with

$$a = [AC], \quad 2b = [AD] + [BC], \quad c = [BD],$$

with the abbreviation

$$[XY] = X_1Y_2 - X_2Y_1$$

and

$$T = [AB], \quad S = [BC], \quad K = [AF], \quad H = [BF].$$

If we approximate

$$N + n_1 \approx n_1$$

in the last term of Eq. (70), which is a reasonable approximation because typically the value of N is much smaller than the value of n_1, the four characteristic equations for the system (69)–(70) become

$$\chi_\alpha = 0, \tag{71}$$

$$\chi_\beta + Et_\beta = 0, \tag{72}$$

$$NE_\alpha + Em_\alpha + n_1 mEt_\alpha = 0, \tag{73}$$

$$NE_\beta = 0. \tag{74}$$

Of these equations, the more interesting are Eqs. (72) and (74). Equation (74) shows that the electric field does not depend on β. Therefore

$$E = E(\alpha)$$

and consequently the electric field is constant along the curves α = constant. To find out the shape of the curves $\alpha(x,t)$ = const. we can employ the identity

$$\left(\frac{\partial x}{\partial t}\right)_\alpha \left(\frac{\partial t}{\partial \beta}\right)_\alpha = \left(\frac{\partial x}{\partial \beta}\right)_\alpha$$

to rewrite Eq. (72) in the form

$$\left(\frac{\partial t}{\partial x}\right)_\alpha = -\frac{1}{E(\alpha)}.$$

This equation shows that a curve $t(x, \alpha$ = const.) has a constant slope equal to $-1/E(\alpha)$. Therefore, the curves $\alpha(x,t)$ = const. are straight lines with slope $-1/E(\alpha)$. This information is sufficient to calculate the evolution of the electric field during the decaying transient, starting from any initial condition $E(x,t_1)$. In particular, if t_0 is the time given in Eq. (68), $t_1 > t_0$, and we take as initial condition the steady state electric field

$$E(x,t_1) = -(-2AJ_0x)^{1/2} \qquad (x \geq 0),$$

the electric field during the decaying transient is

$$E(x,t) = E(x-\delta, t_1) \qquad (x \geq 0, \quad t \geq t_1),$$

where the value of δ can be obtained from the equation

$$\frac{t-t_1}{\delta} = -\frac{1}{E(x-\delta,t_1)};$$

that is

$$\delta = AJ_0(t-t_1)^2 + \left[(-AJ_0)^2(t-t_1)^4 - 2AJ_0x(t-t_1)^2\right]^{1/2}.$$

From the last three equations it follows that the electric field during the discharging transient is

$$E(x,t) = -AJ_0(t-t_1) - \left[-2AJ_0x + A^2J_0^2(t-t_1)^2\right]^{1/2} \quad (x \geq 0, t \geq t_1), \quad (75)$$

and from this expression the transient voltage can be obtained.

6. Planar geometry including diffusion

6.1. Charging transient

Finally we shall consider the equations describing the electron flow in a semiconductor (or insulator), including the contribution due to the diffusion of the charge carriers. Using the same dimensionless variables introduced in Section 2, and defining a dimensionless diffusion coefficient

$$D = \left(\frac{\varepsilon}{J'_u \mu d'^3}\right)^{1/2} D',$$

these equations are

$$J_0 = n(x,t)E(x,t) + D\frac{\partial n}{\partial x} + \frac{\partial E}{\partial t},$$
$$\frac{\partial E}{\partial x} = -n(x,t).$$

From these equations an interesting nonlinear equation for the electric field can be obtained

$$E_t = DE_{xx} + EE_x + J_0. \tag{76}$$

In the following we shall obtain the particular solution of this equation consistent with the initial and boundary conditions

$$E(x,0) = 0. \qquad (x \geq 0), \tag{77}$$

$$E(0,t) = 0 \qquad (t \geq 0). \tag{78}$$

Equation (76) is an extension of the Burgers equation, and this suggests the application of the Hopf-Cole transformation

$$E = 2D\frac{u_x}{u}, \tag{79}$$

which implies that

$$u(x,t) = u(0,t)\exp\left[\frac{1}{2D}\int_0^x E(x',t)\,dx'\right]. \tag{80}$$

With the change of variable (79), Eq. (76) is transformed into the linear equation

$$u_t = Du_{xx} + \frac{J_0}{2D}xu, \tag{81}$$

and the initial and boundary conditions (77)-(78) transform into

$$u(x,0) = u_0 \quad (u_0 = \text{const.}) \quad (x \geq 0), \tag{82}$$

$$u_x(0,t) = 0 \quad (t \geq 0). \tag{83}$$

Furthermore, from Eq. (80) it follows that

$$\lim_{x\to\infty} u(x,t) = 0 \qquad (t > 0). \tag{84}$$

To obtain the solution of the boundary-value problem (81)-(84) let us begin looking for the separable solutions of Eq. (81) consistent with the boundary conditions (83)-(84). Substituting

$$u(x,t) = f(x)g(t)$$

into Eq. (81), it follows that $f(x)$ and $g(t)$ must be the solutions of the equations

$$g_t + \nu g = 0, \tag{85}$$

$$f_{xx} + (s - rx)f = 0, \tag{86}$$

where ν is a constant, and r and s are defined as

$$r = -\frac{J_0}{2D^2}, \tag{87}$$

$$s = \frac{\nu}{D}. \tag{88}$$

Equation (85) can be immediately solved, and its solution is

$$g(t) = g(0)e^{-\nu t}. \tag{89}$$

To obtain the solution of Eq. (86) it is convenient to introduce the change of variable

$$z = r^{-2/3}(rx - s), \tag{90}$$

which transforms Eq. (86) into the Airy equation

$$f_{zz} - zf = 0, \tag{91}$$

whose general solution is a linear combination of the Airy functions

$$f(z) = c_a \operatorname{Ai}(z) + c_b \operatorname{Bi}(z) \qquad (c_a \text{ and } c_b \text{ const.}). \tag{92}$$

Therefore, the solution of Eq. (86) is

$$f(x) = c_a \operatorname{Ai}\left(r^{1/3}x - \frac{\nu}{Dr^{2/3}}\right) + c_b \operatorname{Bi}\left(r^{1/3}x - \frac{\nu}{Dr^{2/3}}\right). \tag{93}$$

For this expression to be consistent with the condition (84), it is necessary that c_b be equal to zero. Therefore, $f(x)$ reduces to

$$f(x) = c_a \operatorname{Ai}\left(r^{1/3}x - \frac{\nu}{Dr^{2/3}}\right). \tag{94}$$

If we now impose the condition (83), the following equation must be satisfied

$$\operatorname{Ai}'\left(-\frac{\nu}{Dr^{2/3}}\right) = 0, \tag{95}$$

where $\operatorname{Ai}'(z)$ is the derivative of the Airy function $\operatorname{Ai}(z)$. Therefore, the parameter ν can only take discrete values ν_n, such that

$$-\frac{\nu_n}{Dr^{2/3}} = \lambda_n, \tag{96}$$

where λ_n is the n^{th} zero of the function $\operatorname{Ai}'(z)$. The solutions of Eq. (81) consistent with the boundary conditions (83)–(84) have, therefore, the form

$$u_n(x,t) = c_n \exp(\lambda_n D r^{2/3} t) \operatorname{Ai}(r^{1/3}x + \lambda_n) \quad (c_n = \text{const.}), \tag{97}$$

and these functions enable us to express the general solution of Eq. (81), satisfying the conditions (83)–(84), in the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp(\lambda_n D r^{2/3} t) \operatorname{Ai}(r^{1/3}x + \lambda_n). \tag{98}$$

The coefficients c_n are determined from the initial condition (82), which implies that

$$u_0 = \sum_{n=1}^{\infty} c_n \operatorname{Ai}(r^{1/3}x + \lambda_n). \tag{99}$$

From this equation we can obtain the values of the coefficients c_n using the orthogonal property of the displaced Airy functions [19]

$$\int_0^{\infty} \operatorname{Ai}(x + \lambda_n) \operatorname{Ai}(x + \lambda_m)\, dx = -\lambda_m \operatorname{Ai}^2(\lambda_m)\delta_{nm}. \tag{100}$$

In this way we obtain

$$c_n = -\frac{r^{1/3}u_0}{\lambda_n \operatorname{Ai}^2(\lambda_n)} \int_0^\infty \operatorname{Ai}(r^{1/3}x' + \lambda_n)\, dx'. \tag{101}$$

From Eqs. (79), (98), and (101) we can write the solution of the boundary-value problem (76)–(78) as follows

$$E(x,t) = 2Dr^{1/3} \frac{\sum_{n=1}^{\infty} a_n \exp(\lambda_n D r^{2/3} t) \operatorname{Ai}'(r^{1/3}x + \lambda_n)}{\sum_{n=1}^{\infty} a_n \exp(\lambda_n D r^{2/3} t) \operatorname{Ai}(r^{1/3}x + \lambda_n)}, \tag{102}$$

where

$$a_n = -\frac{1}{\lambda_n \operatorname{Ai}^2(\lambda_n)} \int_0^\infty \operatorname{Ai}(r^{1/3}x' + \lambda_n)\, dx'. \tag{103}$$

To appreciate the behavior of the electric field as $t \to \infty$ it is convenient to rewrite Eq. (102) in the form

$$E(x,t) = 2Dr^{1/3} \frac{a_1 \operatorname{Ai}'(r^{1/3}x + \lambda_1) + \sum_{n=2}^{\infty} a_n \exp\left[(\lambda_n - \lambda_1) D r^{2/3} t\right] \operatorname{Ai}'(r^{1/3}x + \lambda_n)}{a_1 \operatorname{Ai}(r^{1/3}x + \lambda_1) + \sum_{n=2}^{\infty} a_n \exp\left[(\lambda_n - \lambda_1) D r^{2/3} t\right] \operatorname{Ai}(r^{1/3}x + \lambda_n)}. \tag{104}$$

As $\lambda_n - \lambda_1$ is a negative number (for $n \geq 2$), all the exponentials tend to zero as $t \to 0$. Therefore, if we define

$$\lim_{t\to\infty} E(x,t) = E_0(x), \tag{105}$$

we can see that

$$E_0(x) = 2Dr^{1/3} \frac{\operatorname{Ai}'(r^{1/3}x + \lambda_1)}{\operatorname{Ai}(r^{1/3}x + \lambda_1)}. \tag{106}$$

To visualize the shape of this function we can introduce the asymptotic forms of the Airy function and its derivative [20]

$$\operatorname{Ai}'(z) = -\frac{1}{2} z^{1/4} \exp\left(-\frac{2}{3} z^{3/2}\right) (0.564190) \quad (z > 10),$$

$$\operatorname{Ai}(z) = \frac{1}{2} z^{-1/4} \exp\left(-\frac{2}{3} z^{3/2}\right) (0.564190) \quad (z > 10).$$

Substituting these expressions into Eq. (106), we obtain

$$E(x) = -2Dr^{1/3}\left(r^{1/3}x + \lambda_1\right)^{1/2} \qquad (x > 10r^{-1/3}),$$

which, with the aid of Eq. (87), can also be written as

$$E(x) = -\left[-2J_0x + \lambda_1(-4DJ_0)^{2/3}\right]^{1/2} \qquad \left(x > 10\left(-\frac{2D^2}{J_0}\right)^{1/3}\right).$$

If desired, expressions for the free-electron density and the transient voltage can be easily obtained from Eq. (102).

6.2. Discharging transient (with diffusion)

The discharging transient is described by the Burgers equation

$$E_t = DE_{xx} + EE_x. \tag{107}$$

As initial condition we can consider the stationary form of the electric field given by Eq. (106)

$$E(x, t_0) = 2D\frac{d}{dx}\ln \mathrm{Ai}(r^{1/3}x + \lambda_1) \qquad (x \geq 0), \tag{108}$$

where t_0 is any value large enough for the exponentials in Eq. (104) to be negligible (usually $t > 1$ will suffice), and as boundary condition we shall consider the same condition used in the preceding sections

$$E(0, t) = 0 \qquad (t \geq t_0). \tag{109}$$

Introducing the Hopf-Cole transformation

$$E(x, t) = 2D\frac{\partial}{\partial x}\ln u(x, t), \tag{110}$$

the boundary-value problem (107)–(109) is transformed into

$$u_t = Du_{xx}, \tag{111}$$

$$u(x, t_0) = u_0(x) \qquad (x \geq 0), \tag{112}$$

$$u_x(0, t) = 0 \qquad (t \geq t_0), \tag{113}$$

where we have defined

$$u_0(x) = \mathrm{Ai}(r^{1/3}x + \lambda_1). \tag{114}$$

The solution of the new boundary-value problem (111)-(113) is known to be [21]

$$u(x,t) = \frac{1}{2\sqrt{\pi D(t-t_0)}} \int_0^{\infty} \left\{ \exp\left[-\frac{(x-\xi)^2}{4D(t-t_0)}\right] + \exp\left[-\frac{(x+\xi)^2}{4D(t-t_0)}\right] \right\} u_0(\xi)\, d\xi, \tag{115}$$

and therefore the electric field is given by

$$E(x,t) = \frac{-\frac{1}{(t-t_0)} \int_0^{\infty} \left\{ (x-\xi)\exp\left[-\frac{(x-\xi)^2}{4D(t-t_0)}\right] + (x+\xi)\exp\left[-\frac{(x+\xi)^2}{4D(t-t_0)}\right] \right\} u_0(\xi) d\xi}{\int_0^{\infty} \left\{ \exp\left[-\frac{(x-\xi)^2}{4D(t-t_0)}\right] + \exp\left[-\frac{(x+\xi)^2}{4D(t-t_0)}\right] \right\} u_0(\xi) d\xi}. \tag{116}$$

This expression is complicated, but it reduces to

$$E(x,t) = -\frac{x}{(t-t_0)} \tag{117}$$

for times large enough to be licit to approximate

$$\exp\left[-\frac{(x \pm \xi)^2}{4D(t-t_0)}\right] u_0(\xi) \approx u_0(\xi)$$

in the integrals appearing in Eq. (116).

7. Summary

The electrical transients originated by the sudden application, or the sudden interruption, of constant currents across single-carrier semiconductors (or insulators) were mathematically analyzed. Planar, cylindrical, and spherical geometries were considered. For each of these geometries there exist a single, first-order, quasilinear PDE which describes the charging transient in the absence of traps and diffusion, and a similar equation which describes the discharging transient. Assuming an ohmic contact at the cathode, these quasilinear equations were solved by the method of characteristics.

In the presence of traps the charging transient is described by a quasilinear hyperbolic system, which can be transformed into an integro-differential equation for the electric field. An approximate solution of this equation was obtained using the method of characteristics. The discharging transient was also found using the concept of characteristics.

The diffusion of the charge carriers was also considered. The charging transient with diffusion is described by an extension of the Burgers equation, which could be solved in terms of the displaced Airy functions. The discharging transient is described by the normal Burgers equation, and therefore its solution could be obtained by means of the Hopf-Cole transformation.

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Resumen. Se resuelven analíticamente las ecuaciones no lineales que describen los transitorios eléctricos de carga y descarga bajo condiciones de corriente controlada a través de semiconductores (o aislantes) con un solo tipo de portadores de carga. Se consideran electrodos planos, cilíndricos y esféricos, y se investiga el efecto que producen la presencia de trampas y la difusión de portadores. El término difusivo conduce a una extensión de la ecuación de Burgers, la cual puede ser resuelta en términos de funciones de Airy desplazadas.