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On spatial dimensions in physical laws

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Abstract. We describe some of the fundamental physical laws that we encounter in textbooks in arbitrary spatial dimensions in order to study the dimensionality dependences in these physical laws. We also review the recent studies on the possibility that our world has noninteger spatial dimensions slightly deviated from 3.

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1. Introduction

Physical laws are generally described in three spatial dimensions and the number of dimensions does not appear in these laws explicitly. How would the fundamental laws be modified if the world had other dimensions? The idea of generalizing physical law(s) to other or arbitrary dimensions is by no means new. For instance, Ehrenfest (and independently Whitrow) [1] solved the Keplerian problem in arbitrary dimensions. Kaluza and Klein proposed 5 dimensional space-time to unify electromagnetic force and gravity [2]. In recent years, various models, which are based on the original suggestion of Kaluza and Klein, have been developed with the aim of unifying all the known interactions in a higher dimensional space-time. According to these models, the four dimensions of our space-time emerge as the result of compactification of multi-dimensional space-time at a certain moment during the early evolution of the universe. Arbitrary dimensionality (including fractal dimensions [3]) has been widely used as well in the theory of phase transition, renormalization theory and various lattice models [4]. Nowadays on the frontiers of physics, it has become rather common to study the phenomena and underlying physics in arbitrary dimensions. Nevertheless the dimensionality dependences in the fundamental laws of physics which we encounter in textbooks are not necessarily familiar. Multi-dimensional description of these physical laws would help us to understand their nature more profoundly and, when combined with the so-called anthropic principle [5], can give an answer to why our universe possesses three dimensions and not other dimensions.

The main purpose of the paper is to collect and review the fundamental physics laws from the dimensional point of view. We organize the paper as follows. We first describe the fundamental physics laws and equations such as the Planck radiation law, Stefan-Boltzmann law, Wien law, Maxwell equations, Lorentz force, Coulomb law, Schrödinger equation and Newton law of universal gravitation, in any, d, spatial dimensions. Although some of these results are known in literature, we do not indicate all the original references, since most of them can be derived rather straightforwardly (as an exercise) by generalizing $d = 3$ formulas to any spatial dimensions. In Sect. 3 we review the recent studies on the upper bound on the value of ε which characterizes the possible noninteger dimensions $3 + \varepsilon$ of our space. The concluding section contains a summary and discussion from the standpoint of physics education. Throughout the paper we adopt standard notation for the physical quantities.

2. Typical laws of physics in arbitrary spatial dimensions

2.1. The Planck radiation law, Stefan-Boltzmann law and Wien law

In three spatial dimensions, the energy density of black-body radiation $u(\omega, T)$ per unit angular frequency ω at absolute temperature T is described by

$$u(\omega, T) = \frac{g\hbar\omega^3}{2\pi^2 c^3}\frac{1}{\exp(\hbar\omega/kT) - 1}, \tag{1}$$

where g is the number of photon polarizations ($g = 2$).

In d spatial dimensions, one can derive the corresponding equation by following closely the derivation of Eq. (1) [6],

$$u(\omega, T) = g(\omega)\varepsilon(\omega, T), \tag{2}$$

where $g(\omega)$ is the density of states of photon in unit volume,

$$g(\omega) = \frac{g\omega^{d-1}\Omega_d}{(2\pi c)^d}, \tag{3}$$

$\varepsilon(\omega, T)$ is the average photon energy,

$$\varepsilon(\omega, T) = \frac{\hbar\omega}{\exp(\hbar\omega/kT) - 1}, \tag{4}$$

and Ω_d is the solid angle of d dimensional sphere,

$$\Omega_d = \frac{d\pi^{d/2}}{\Gamma(1 + d/2)}. \tag{5}$$

Using Eqs. (3) and (4), we obtain Planck radiation law in d dimensions as

$$u(\omega, T) = \frac{g\hbar\omega^d\Omega_d}{(2\pi c)^d}\frac{1}{\exp(\hbar\omega/kT) - 1}, \tag{6}$$

which reproduces Eq. (1), when $d = 3$. Integrating Eq. (6) over ω, we get the overall energy density $U(T)$

$$\begin{aligned} U(T) &= \int u(\omega, T)\, d\omega \\ &= g\Omega_d d! \frac{\zeta(d+1)}{(ch)^d}(kT)^{d+1} \propto T^{d+1}, \end{aligned} \tag{7}$$

where

$$\zeta(d) = \sum_{n=1}^{\infty} \frac{1}{n^d}. \tag{8}$$

Equation (7) is the Stefan-Boltzmann law in d spatial dimensions [7]. In the case of $d = 3$, we reproduce the familiar (three dimensional) Stefan-Boltzmann law by substituting $\zeta(4) = \pi^4/90$ into Eq. (7)

$$U(T) = \frac{4g\pi^5}{15(ch)^3}(kT)^4. \tag{9}$$

Using Eq. (6), we find the wavelength $\lambda_{\max}$ at which the energy density per wavelength $u(\lambda, T)$ ($\equiv u(\omega, T)\, d\omega/d\lambda$) takes its maximum value

$$\lambda_{\max} kT = \frac{ch}{b}. \tag{10}$$

This is the Wien law in d dimensions. b is a solution of transcendent equation

$$\exp(-b) + \frac{b}{d+2} - 1 = 0. \tag{11}$$

When $d = 3$ we obtain $b = 4.965$ as a solution of Eq. (11), hence Eq. (10) reproduces the three dimensional Wien law.

2.2. Maxwell equations and Lorentz force

The Maxwell equations expressed in the tensorial form [8] can be immediately generalized to d dimensions as follows

$$\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = \mu_0^{(d)} j^{\mu}, \tag{12.1}$$

$$\frac{\partial \varepsilon_{\mu_1 \ldots \mu_{d-2}\nu\rho\sigma} F^{\nu\rho}}{\partial x^{\sigma}} = 0, \tag{12.2}$$

where $\mu_0^{(d)}$ is the magnetic permeability of vacuum. Electric and magnetic fields in d dimensions are expressed in terms of the antisymmetric tensor by the following relations

$$E_i = F_{0i} \qquad (i = 1, 2, \ldots, d), \tag{13.1}$$

$$B_{i_1 i_2 \ldots i_{d-2}} = \tfrac{1}{2}\varepsilon_{i_1 i_2 \ldots i_{d-2} jk} F^{jk}. \tag{13.2}$$

Then the d dimensional Lorentz force is derived by taking into account that the force acting on a point charge at rest is

$$F_i = eE_i, \tag{14}$$

and by Lorentz transforming the system to the one moving with velocity v as

$$F_i = e\{E_i + (v \times B)_i\}, \tag{15}$$

where

$$(v \times B)_i = \frac{1}{(d-2)!}\varepsilon_{ikj_1 j_2 \ldots j_{d-2}} v^k B^{j_1 j_2 \ldots j_{d-2}}. \tag{16}$$

2.3. Coulomb law

In this section we present the formulas both in the MKSA and in the Gauss unit system. The results expressed in Gauss units are put in parentheses.

The electric potential created by a point charge e in d spatial dimensions is obtained by solving the d dimensional Poisson equation,

$$\Delta\phi = -\frac{e}{\varepsilon_0^{(d)}}\delta^{(d)}(r) \tag{17}$$

$$(\Delta\phi = -4\pi e r_0^{d-3}\delta^{(d)}(r)), \tag{17.1}$$

where $\varepsilon_0^{(d)}$ is the dielectric constant of vacuum having a dimension $[\mathrm{J^{-1}C^2m^{2-d}}]$. Its value is determined by the definition of charge in d dimensions. r_0, appearing in Eq. (17.1), has the dimension of length and is related to the definition of charge in d dimensions. $\delta^{(d)}$ is the d dimensional δ-function. Let us express the Laplace operator in the polar coordinate system

$$\Delta\phi = \left[\frac{1}{r^{d-1}}\frac{\partial}{\partial r}r^{d-1}\frac{\partial}{\partial r} - \frac{\hat{L}^2}{r^2}\right]\phi$$

$$= \left[\frac{\partial^2}{\partial r^2} + \frac{(d-1)}{r}\frac{\partial}{\partial r} - \frac{\hat{L}^2}{r^2}\right]\phi, \tag{18}$$

where $\hat{L}$ is a generalized angular momentum operator which contains only angular variables. In the case when $d \neq 2$, a spherically symmetric solution of the Poisson equation (17) with boundary condition $\phi(\infty) = 0$ for $d > 2$, $\phi(0) = 0$ for $d < 2$ reads as

$$\phi = \frac{e}{\varepsilon_0^{(d)}(d-2)\Omega_d}\frac{1}{r^{d-2}} \tag{19}$$

$$\left(\phi = \frac{4\pi e}{(d-2)\Omega_d}\left(\frac{r_0}{r}\right)^{d-3}\frac{1}{r}\right). \tag{19.1}$$

Putting $d = 3$ one can reproduce the usual Coulomb law by using $\varepsilon_0^{(3)} = \varepsilon_0$ and $\Omega_3 = 4\pi$. The solution of Eq. (17) for $d = 2$ is

$$\phi = -\frac{e}{\varepsilon_0^{(2)}\Omega_2}\ln\left(\frac{r}{r_s}\right) \tag{20}$$

$$\left(\phi = -\frac{4\pi e}{r_0\Omega_2}\ln\left(\frac{r}{r_s}\right)\right), \tag{20.1}$$

where $\Omega_2 = 2\pi$, and r_s is a reference point of the potential which satisfies $\phi(r_s) = 0$.

2.4. The Schrödinger equation

The Schrödinger equation for a d dimensional hydrogen atom is written as [9]

$$H\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + e\phi\right)\psi$$

$$= \left(-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial r^2} + \frac{(d-1)}{r}\frac{\partial}{\partial r} - \frac{\ell\{\ell+1+(d-3)\}}{r^2}\right] + e\phi\right)\psi$$

$$= E\psi, \tag{21}$$

where neither relativistic nor spin-orbit corrections are taken into consideration. In

Eq. (21), the effects of dimensionality appear in the second and third terms of the generalized angular momentum. Note that the third term in Eq. (21) contains the coefficient $\ell(\ell + 1 + d - 3)$ instead of $\ell(\ell + 1)$ in the 3 dimensional case. Assuming that the Coulomb potential takes the form $\phi = -e/(4\pi\varepsilon_0 r)$ we obtain the energy eigenvalue [9] of Eq. (21) as

$$E_n = -\frac{E_0}{[n + (d-3)/2]^2}, \tag{22}$$

for $n \geq \ell + 1$. E_0 is the ground state energy of the hydrogen atom,

$$E_0 = \tfrac{1}{2}\alpha^2 mc^2, \tag{23}$$

with α being a fine structure constant,

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} \quad \text{in MKSA units}$$

$$\left(\alpha = \frac{e^2}{\hbar c} \quad \text{in Gauss units}\right). \tag{24}$$

It would be interesting to try to derive Eq. (22) from the Bohr-Sommerfeld quantum conditions. Gurevich and Mostepanenko [10] and Tangherlini [11] have shown, by solving the Schrödinger equation for the generalized hydrogen atom in d dimensions (*i.e.* by using Eq. (19) as the Coulomb potential), that there are no stable bound orbits for $d > 3$. Thus we see that the dimensionality of the world is a reason for the existence of the stable atoms, chemistry and therefore of life.

2.5. Newton law of universal gravitation

One can derive the gravitation law in d dimensions from the d dimensional Poisson equation in a similar manner as in Sec. 2.3. Here we derive the Poisson equation using the Einstein field equation in d dimensions which reads as (in the unit of $c = 1$) [8]

$$R_{\mu\nu} = 8\pi G^{(d)} \left[T_{\mu\nu} - \frac{g_{\mu\nu}T}{d-1}\right]. \tag{25}$$

$G^{(d)}$ is a constant with the dimension $[\mathrm{Jkg^{-2}m^{d-2}}]$, the value of which is related to the definition of a mass in d dimensions. As is well known, the Newton gravitation law is obtained from the Einstein field, Eq. (25), by assuming that the gravitational field is very weak and static. This corresponds to putting

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{26}$$

and to ignoring quadratic terms of $h_{\mu\nu}$ as well as $\partial h_{\mu\nu}/\partial t = 0$.

Substitution of

$$T_{00} = \rho \quad \text{and} \quad T_{\mu\nu} = 0 \quad (\text{except for} \quad \mu = \nu = 0), \tag{27}$$

into Eq. (25) leads to the following Poisson equation

$$-\frac{1}{2}\Delta h_{00} = \Delta\phi = 8\pi G^{(d)}\frac{(d-2)}{(d-1)}\rho. \tag{28}$$

Solving this equation we obtain the Newton gravitational potential in d dimensions

$$\phi = -\frac{8\pi G^{(d)}m}{(d-1)\Omega_d}\frac{1}{r^{d-2}} \qquad (d > 2). \tag{29}$$

With $d = 3$, Eq. (29) is reduced to the ordinary Newton gravitation potential

$$\phi = -\frac{Gm}{r}. \tag{30}$$

3. $3 + \varepsilon$ dimensions

As mentioned in the Introduction, generalization of a theory to arbitrary dimensionality is widely used in various fields of physics, such as elementary particle physics, theory of phase transitions and renormalization theory. Also analyses of physical phenomena in terms of the noninteger (fractal) dimensions are being carried out frequently using lattice models. From such point of view, it is not a priori obvious whether the space-time dimensions of our world are exactly four or are deviated, even if very slightly, from four. From the data of perihelion shift, Jarlskog and Ynduráin [12] have estimated the bound on ε, characterizing the deviation of the space-time dimensions from four, as $\varepsilon < 1.5 \times 10^{-8}$. Müller and Schäfer [13] have also obtained the bound on ε both from the analyses of the similar planetary motion and the Lamb shift in hydrogen atoms as 10^{-8} and 3.6×10^{-11}, respectively.

4. Summary

We have described some of the familiar physical laws (Planck radiation law, the Stefan-Boltzmann law, Maxwell equations, the Lorentz force, Coulomb law, the Schrödinger equation and Newton law of universal gravitation) in d spatial dimensions. From the equations presented in the text which contain the number of spatial dimensions, d, explicitly one can recognize how the dimensionality of the world is reflected in these equations and laws. Therefore consideration on the dimensionality in the physical laws allows us to understand these laws more profoundly. From the standpoint of physics education one can formulate various suitable problems on higher dimensional physics (for example, effects of the dimensionality on the planetary motion, atomic spectral series, stability of atoms and so on) which may stimulate the students' curiosity and imagination. Multi-dimensional description of the physical laws, when supplemented with the so-called anthropic principle [5], can provide a possible resolution to the fundamental question "why do we observe the world as possessing three dimensions?".

References

1. P. Ehrenfest, *Proc. Amst. Acad.* **20** (1917) 200; *Ann. Phys.* **61** (1920) 440; G.J. Whitrow, *Br. J. Phil. Sci.* **6** (1955) 13.
2. T. Kaluza, *Sitz. Preuss. Akad. Wiss.*, Berlin (1921) p. 966; O. Klein, *Z. Phys.* **37** (1926) 895.
3. A. Zeilinger and K. Svozil, *Phys. Rev. Lett.* **24** (1985) 2553.
4. K.G. Wilson and J. Kogut, *Phys. Reports* **12** (1974) 75; M.E. Fisher, *Rev. Mod. Phys.* **46** (1974) 597.
5. J.D. Barrow and F.J. Tipler, *Anthropic Cosmological Principle*, Clarendon Press, Oxford (1986).
6. See, for example, K. Huang, *Statistical Mechancis*, John Wiley & Sons Inc., New York (1963).
7. M. Chaichain, R. Hagedorn and M. Hayashi, *Nucl. Phys.* **B92** (1975) 445.
8. See, for example, L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, Addison-Wesley, Reading, Massachusetts (1962).
9. M.M. Nieto, *Am. J. Phys.* **47** (1979) 1067.
10. L. Gurevich and V. Mostepanenko, *Phys. Lett.* **35A** (1971) 201.
11. F.R. Tangherlini, *Nuovo Cim.* **27** (1963) 636.
12. C. Jarlskog and F.J. Ynduráin, *Europhys. Lett.* **1** (1986) 51.
13. B. Müller and A. Schäfer, *Phys. Rev. Lett.* **56** (1986) 1215.

Resumen. Describimos algunas de las leyes fundamentales de la física que encontramos en los libros de texto, en dimensiones espaciales arbitrarias, para estudiar la dependencia en la dimensionalidad de las mismas. También revisamos los estudios recientes sobre la posibilidad de que nuestro mundo tenga dimensiones espaciales no enteras un poco diferentes de 3.