

Cooper pairing and superconductivity as non-perturbative phenomena

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Abstract. The parallels between the Cooper electron-pair problem and the two-dimensional potential well problem, which is anomalous, are highlighted. Their common signature is binding for arbitrarily weak coupling accompanied by an essential-singularity, non-perturbative behavior of the binding energy as function of coupling strength. This singularity severely limits superconducting transition-temperature magnitudes in BCS theory. However, analysis of the Cooper problem in both 1D and 3D shows that the singularity persists. Thus, contrary to the conventional wisdom, the singularity cannot be related to the two-dimensionality implied by the BCS model interaction used.

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1. Introduction

The two-dimensional (2D), quantum-mechanical, many-body electron gas problem has again become the focus of intense research, both theoretical as well as experimental, since the recent discovery of high-temperature superconductivity [1] in certain ceramic copper-oxide compounds, with greatly enhanced current densities being observed in these materials within planes composed of copper and oxygen atoms.

Prior to this, the subject was already of prime importance due to: 1) observations of a 2D electron gas on liquid helium surfaces, or on films deposited on solid dielectric materials [2,3]; and 2) detection of localized electron states at interfaces between GaAs and GaAlAs, or between a metal oxide and a semiconductor [4] displaying integer as well as fractional quantum Hall effects. Furthermore, the occurrence of electron localization in a random set of ionic potential wells, for arbitrarily small disorder, in 1D or 2D but not 3D, has recently been suggested [5,6] to be mathe-

matically equivalent to the textbook problem of finding (or not) a bound state in an arbitrarily shallow potential well in 1D, 2D or 3D, respectively.

The "standard theory" of superconductivity goes back to Bardeen, Cooper and Schrieffer (BCS) [7], and may yet prove to be applicable, with extensions and modifications, to the novel high-transition temperature substances as successfully as in the conventional low-temperature ($\lesssim 23$ K) elements and alloys, the first of which (Hg) was discovered in 1911. At the heart of this theory lies the Cooper electron-pair problem which exhibits a strong resemblance to that of one particle in a simple potential well in 2D. Both problems support a bound state for arbitrarily weak attraction and display a non-analytic (non-perturbative) functional dependence on the coupling strength. These connections and their significance are discussed and clarified in this paper, in an elementary way. The inapplicability of perturbation theory to the problem has often been cited [8,9] as the main stumbling block to develop a viable theory of superconductivity, and why consequently it took almost a half-century (from 1911 to 1957) for such a theory to emerge, though of course quantum mechanics proved essential.

2. Uniqueness of two-dimensional potential wells

It is well known from elementary wave mechanics that for a 3D attractive potential well to support at least one bound state a critical well depth and/or range is needed. This is in marked contrast to 1D where a bound state always exists, no matter how shallow and/or short-ranged the attractive well. This follows in each case from the solution of the Schrödinger equation for its negative energy eigenvalues $E \equiv -|E|$, which relies on selecting regular (*i.e.*, finite-probability-density) wave-function solutions both inside and outside the well, and then matching wave functions and their first derivatives at the well boundary.

For the ground state energy $E \equiv -|E|$, of a particle of mass m in a (1D or 3D) well of radius a from the origin and depth $-V_0$, with $V_0 > 0$, the matching equations reduce to

$$\sqrt{\frac{2m(V_0 - |E|)}{\hbar^2}} \tan \sqrt{\frac{2m(V_0 - |E|)}{\hbar^2}} a = \sqrt{\frac{2m|E|}{\hbar^2}} \quad (1D) \quad (1)$$

and

$$-\sqrt{\frac{2m(V_0 - |E|)}{\hbar^2}} \cot \sqrt{\frac{2m(V_0 - |E|)}{\hbar^2}} a = \sqrt{\frac{2m|E|}{\hbar^2}} \quad (3D) \quad (2)$$

These transcendental equations for $|E|$ are derived, and generally solved graphically, in many standard textbooks [10]. Eq. (1) has a solution $|E| > 0$ for arbitrarily small V_0 (and consequently arbitrarily small $|E| < V_0$). Expanding the tangent for small

x [$\tan x \simeq x + O(x^3)$], and then squaring, permits us to find that

$$E \xrightarrow{V_0 \rightarrow 0} -\frac{2ma^2V_0^2}{\hbar^2} + O(V_0^3) \quad (1D) \quad (3)$$

is a perturbative result. In other words, the bound state can in principle be found in terms of successively smaller systematic corrections which are powers of the perturbation well depth V_0 . On the other hand, as a simple sketch of both sides of Eq. (2) shows, a solution $|E| \geq 0$ exists only provided that

$$V_0a^2 \geq \frac{\hbar^2\pi^2}{8m} \quad (3D). \quad (4)$$

The 3D result is also perturbative since from Eq. (2) it follows that, if $V_0a^2 \equiv \hbar^2\pi^2/8m + \eta$ with $\eta \geq 0$, then $E \xrightarrow{\eta \rightarrow 0} -m\eta^2/2\hbar\pi^2 + O(\eta^3)$.

The two-dimensional case is *qualitatively* different. It is much less familiar in the textbook literature, probably because of the cumbersome need for two kinds of Bessel equations —one for inside, the other for outside the (cylindrically-symmetric) well of radius a and depth $-V_0$. For the ground state, the respective radial Schrödinger equations (upper sign = “inside”; lower sign = “outside”) are

$$\rho^2 R''(\rho) + \rho R'(\rho) \pm \rho^2 R(\rho) = 0, \quad (5)$$

where the dimensionless real distance variables

$$\begin{aligned} \rho &\equiv \sqrt{2m(V_0 - |E|)/\hbar^2}r && \text{(inside)} \\ &\equiv \sqrt{2m(V_0 + |E|)/\hbar^2}r && \text{(outside)} \end{aligned} \quad (6)$$

have been introduced. Eq. (5) is just the zeroth-order Bessel equation, with solutions which are a) regular at $r = 0$: given by the Bessel functions $J_0(\rho)$ (inside well, upper sign) and b) regular at $r = \infty$: given by the “hyperbolic” (or “modified”) Bessel function $K_0(x)$ (outside well, lower sign). For excited states, an extra term with a rotational quantum number $n = 1, 2, \dots$ appears in Eq. (5) and gives rise to the higher-order Bessel functions [11] $J_n(x)$ and $K_n(x)$. This general case is discussed by Boas [12]. Using the identities [11] $J'_0(x) = -J_1(x)$ and $K'_0(x) = -K_1(x)$, the continuity of $R'(r)/R(r)$ at $r = a$ immediately gives:

$$\begin{aligned} &\sqrt{2m(V_0 - |E|)/\hbar^2} J_1 \left(\sqrt{2m(V_0 - |E|)/\hbar^2}a \right) / J_0 \left(\sqrt{2m(V_0 - |E|)/\hbar^2}a \right) \\ &= \sqrt{2m|E|/\hbar^2} K_1 \left(\sqrt{2m|E|/\hbar^2}a \right) / K_0 \left(\sqrt{2m|E|/\hbar^2}a \right). \end{aligned} \quad (2D) \quad (7)$$

The limit $|E| < V_0 \rightarrow 0$ can now easily be performed by recalling the asymptotic results [11]:

$$J_n(x) \xrightarrow{x \rightarrow 0} \frac{x^n}{2^n n!}, \quad K_1(x) \xrightarrow{x \rightarrow 0} x^{-1}, \tag{8}$$

$$K_0(x) \xrightarrow{x \rightarrow 0} -\ln x, \tag{9}$$

and solving for $|E|$ again by exponentiation. One quickly arrives at

$$E \xrightarrow{V_0 \rightarrow 0} -\frac{\hbar^2}{2ma^2} e^{-2\hbar^2/mV_0a^2} \tag{10}$$

which because of the minus sign means the existence of a bound state, and for arbitrarily shallow well depth as in 1D. What is new and different in 2D, as opposed to 1D where (3) holds, is the essential singularity (10) in V_0 which precludes a perturbative expansion in powers of V_0 to obtain the eigenvalue, since the n th Maclaurin expansion coefficient E_n of Eq. (10) expressed as

$$E = E_0 + E_1V_0 + E_2V_0^2 + \dots, \tag{11}$$

would be

$$E_n = \frac{\hbar^2}{2ma^2} \frac{1}{n!} \left[\frac{d^n}{dV_0^n} e^{-2\hbar^2/mV_0^2} \right]_{V_0=0}, \tag{12}$$

which clearly vanishes for all $n = 0, 1, 2, \dots$

The outcome (10) in fact holds for *any central well* $V(r)$, not necessarily of rectangular cross section, if V_0a^2 is replaced by $\frac{1}{2} |\int_0^\infty dr rV(r)|$. This is proved very elegantly by Landau and Lifshitz [13] (though their result omits a divisor of 2). Finally, a variational trial function consisting of a shifted and “stretched” exponential establishes [14] the fact that binding occurs in 2D for any negative definite well $V(r, \phi)$ whatsoever, central or not, no matter how shallow, with the characteristic exponential term $e^{-|\text{const.}|/V_0a^2}$ persisting, if V_0a^2 is replaced by $\frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^\infty dr r|V(r, \phi)|$.

It is interesting to recall that this particular behavior also occurs in a many-body context in solid-state physics: the 2D perfect (*i.e.*, noninteracting) Fermi gas at low temperatures. Its chemical potential μ as a function of temperature cannot be expressed as the so-called Sommerfeld expansion in powers of T , very familiar in the 3D case, namely

$$\mu(T) \underset{T \rightarrow 0}{=} E_F + \mu_2 T^2 + \mu_4 T^4 + \dots, \tag{13}$$

where E_F is the Fermi energy and μ_2, μ_4, \dots are certain constants. The reason is that, because $g(\epsilon) = \text{const.}$, the exact expression [15] for $\mu(T)$ is found to be

$$\begin{aligned} \mu(T) &= E_F - kT \ln(1 + e^{-\mu/kT}) \\ &\underset{T \rightarrow 0}{=} E_F - kT \left[e^{-\mu/kT} - \frac{1}{2} e^{-2\mu/kT} + \dots \right], \end{aligned} \tag{14}$$

which therefore, by Eq. (12), cannot be written in the form (13) either, except for the trivial case $T = 0$.

3. Cooper pairs and superconductivity

The most striking aspect of the Cooper [16] pair problem is that it can form a bound state for an arbitrarily weak (but attractive) effective interaction. This happens in either 1, or 2 or 3D because of the Pauli principle imposed on the N electron system, which manifests itself in terms of a constant-energy (so-called Fermi) surface, on top of but very near which the net effective electron-electron interaction is different from zero.

Two conduction electrons of opposite spins orbit around each other with a center-of-mass momentum $\hbar\mathbf{K} \equiv \hbar(\mathbf{k}_1 + \mathbf{k}_2)$. They are submersed in the "Fermi sea" of $N - 2$ passive (or background) electrons, and all N electrons obey the Pauli exclusion principle. The partners of the chosen pair repel each other through the usual Coulomb electrostatic force, but also attract in a very particular way thought to be characteristic of all metals. Namely, a conduction electron can absorb or emit "phonons" created by the vibrations of the underlying ionic lattice, produced in turn by the distorting motion of the two electrons themselves. If the net combined effect of the Coulomb repulsion plus phononic attraction is repulsive we have a "normal" metal, of which, for example, there are at least 85 of the 106 or so elements of the periodic table. On the other hand, if the phononic attraction overwhelms the Coulomb repulsion we have a "superconductor", of which there are at least 28 of those 85 elemental metals at atmospheric pressure (plus a few more under higher pressures or in the form of thin films).

Let this net *effective* interaction $V(r)$, where $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$ is the relative coordinate of the two electrons, be represented by the schematic model interaction of BCS. This model captures the essential physical features of the combined effect of the electron-electron coulombic repulsion and the electron-phonon collision process whereby an electron pair is scattered from a pair state $(\mathbf{k}^\uparrow, -\mathbf{k}^\downarrow)$ to another pair state $(\mathbf{k}'^\uparrow, -\mathbf{k}'^\downarrow)$. It is very simply written as

$$\frac{1}{\Omega} \int d^D \mathbf{r} e^{-i\mathbf{k}' \cdot \mathbf{r}} V(r) e^{i\mathbf{k} \cdot \mathbf{r}} \equiv V_{\mathbf{k}', \mathbf{k}} = \begin{cases} -V & \text{if } E_F < \epsilon_{\mathbf{k}}, \epsilon_{\mathbf{k}'} < E_F + \hbar\omega_D \\ 0 & \text{otherwise,} \end{cases} \quad (15)$$

where D is the dimensionality and Ω the D -dimensional "volume" of the sample, $V > 0$ is a positive constant coupling parameter for a net, effective attractive interaction, $\epsilon_{\mathbf{k}}$ the single-electron energy, and $\hbar\omega_D$ is the maximum (Debye) energy that an ionic crystal phonon can possess to transfer to a scattering electron. The resulting Schrödinger equation in momentum space for the two electrons in a Fermi sea is then straightforward [17,18] to reduce to the relation

$$1 = V \sum_{\mathbf{k}}' \frac{1}{2\epsilon_{\mathbf{k}} - E}, \quad (16)$$

where the “prime” over summation sign reminds us to restrict the sum to those \mathbf{k} values associated with a single-electron energy $\epsilon_{\mathbf{k}}$ within the limits specified by Eq. (15). This is the fundamental equation for the possible energy eigenvalues E of the Cooper electron pair. The sum can be converted to an integral over the single-particle energies $\epsilon_{\mathbf{k}}$ if we introduce the degeneracy (or “density of states”) $g(\epsilon_{\mathbf{k}})$, so that Eq. (16) becomes

$$1 = V \int_{E_F}^{E_F + \hbar\omega_D} d\epsilon \frac{g(\epsilon)}{2\epsilon - E} \simeq Vg(E_F) \int_{E_F}^{E_F + \hbar\omega_D} \frac{d\epsilon}{2\epsilon - E}. \tag{17}$$

The last step is ensured if $g(\epsilon)$ is reasonably smooth over the integration interval of width $\hbar\omega_D$, and if this interval is sufficiently small compared with E_F . The former condition holds since $g(\epsilon) \propto \epsilon^{1/2}$, or const, or $\epsilon^{-1/2}$ (for $\nu = 3$, or 2, or 1 dimensions, respectively) (see Appendix) as shown in elementary solid-state physics texts such as Burns [19]. The latter condition, namely $\hbar\omega_D \ll E_F$, holds because the characteristic Fermi temperature $T_F \equiv E_F/k_B = \hbar^2(3\pi^2n)^{2/3}/2mk_B$, (where k_B is Boltzmann’s constant and n the concentration of conduction electrons) is typically 10^4 to 10^5 K for most conductors, whereas the characteristic Debye temperature $T_D \equiv \hbar\omega_D/k_B$ (proportional to an averaged crystal lattice sound speed) is typically only hundreds of degrees K. Consequently, the BCS interaction (8) operates only very near the Fermi surface, within an energy shell of thickness 10^{-2} to 10^{-3} times the value of the Fermi energy E_F itself. It is claimed in the textbook literature [17] that to the empirical extent that $\hbar\omega_D \ll E_F$, Cooper pairing will essentially be a two-dimensional phenomenon. Qualitatively, this is clear since “things look planar very near a spherical surface”. Quantitatively, the last step of Eq. (17) is justified either if $\hbar\omega_D/E_F \ll 1$ or if $g(\epsilon)$ is constant as occurs in two dimensions. The integral left in Eq. (17) is now elementary, and equal to

$$\frac{1}{2} \ln \left[\frac{(2E_F - E) + 2\hbar\omega_D}{(2E_F - E)} \right]. \tag{18}$$

Inserting this into Eq. (17) and solving by exponentiation for the pair-energy eigenvalue E , we obtain

$$E = 2E_F - \frac{2\hbar\omega_D}{e^{2/g(E_F)V} - 1}. \tag{19}$$

As expected, for $V = 0$, $E = 2E_F$, the energy of two electrons at the Fermi surface. For $V > 0$, $E < 2E_F$ so that the pair is (quasi) bound, and has a binding energy Δ_0 given by

$$2E_F - E \equiv \Delta_0 \equiv \frac{2\hbar\omega_D}{e^{2/g(E_F)V} - 1} \xrightarrow{V \rightarrow 0} 2\hbar\omega_D e^{-2/g(E_F)V}, \tag{20}$$

with the latter expression holding for weak coupling. As in Eq. (10), the binding energy Δ_0 is not an analytic function of V .

If the first integral in Eq. (17) is carried out without approximation by taking $g(\epsilon) \propto \epsilon^{1/2}$ (in 3D) and $\propto \epsilon^{-1/2}$ (in 1D), somewhat harder integrals result than the ones which lead to Eq. (18). However, these can also be performed analytically (see Appendix) and lead to longer expressions than before, but $E \rightarrow 0^-$ and $\hbar\omega_D \ll E_F$, the 3D and 1D cases *result in precisely the same asymptotic limit given by the last expression in Eq. (20)*, regardless of dimensionality. This means that the essential singularity is independent of dimensionality.

Historically, it was not an easy matter to embed the Cooper pair idea into a truly many-electron theory. This BCS did in 1957 by a *tour-de-force* construction, for the same schematic interaction model (15), of an antisymmetrized N -electron variational wave function depicting a “charged macromolecule” of N superconducting electrons, of which those near the Fermi surface form Cooper pairs highly correlated among themselves. This electronic macromolecule extends over the entire sample volume and constitutes the “supercurrent” that flows (when an external electric field is applied) resistanceless throughout the material, indefinitely in time. Indeed, the BCS theory is one of the most elegant, seminal and successful applications of quantum many-body physics known to date. It leads [20] to a temperature-dependent “energy gap” order parameter $\Delta(T)$, which for $T = 0$ is found to be

$$\Delta(0) = \frac{\hbar\omega_D}{\sinh[1/g(E_F)V]} \xrightarrow{V \rightarrow 0} 2\hbar\omega_D e^{-1/g(E_F)V}. \quad (21)$$

Only the very last expression is reliable since the BCS theory is restricted to weak coupling, in which limit the zero-temperature gap parameter is seen to be similar to the weak-coupling Cooper pair binding energy (20) valid in 1, 2 or 3D.

The “normal metal”-to-“superconductor” transition temperature T_c is then determined in the BCS theory by the vanishing of the function $\Delta(T)$, namely $\Delta(T_c) = 0$, and is found to be related to $\Delta(0)$ through the very simple result for the empirically observable energy gap:

$$E_g \equiv 2\Delta(0) = 2\pi e^{-\gamma} kT_c \simeq 3.53kT_c, \quad (22)$$

where $\gamma \simeq 0.5772$ is the Euler constant. This amount of energy “excites” the highly-ordered electronic macromolecule out of its ground state, consequently destroying the superconducting phase altogether. Note that Eqs. (21) and (22) predict that T_c can be arbitrarily small, a fact empirically illustrated in a striking way by the metallic element rhodium (Rh), which apparently has the lowest transition temperature (3.25×10^{-4} K) measured to date. In a lucid tutorial review, Little [21] shows how the weak-coupling BCS relation (22) is remarkably consistent with recent energy-gap measurements even in the new, highest- T_c copper-oxide materials. Unfortunately, BCS-based formalisms [22] using the phonon mechanism presently appear unable to predict values of T_c greater than about 40 K in solids; this limitation has come to be known as the “phonon barrier”. So, it is still not clear whether these formalisms may eventually provide clues for material scientists on how to actually engineer the long-sought room-temperature superconducting substance that some believe will

rival the invention of the transistor. On the other hand, as Schrieffer [23] himself argues, the inapplicability of the "pairing theory" in solids over a temperature range between 10's to 100's of degrees Kelvin would seem highly unlikely in view of clear successes of the theory to explain phenomena over 13 orders of magnitude in critical temperature —from 10^{-3} K where liquid- ^3He becomes superfluid, to 10^{10} K where nuclei show clear-cut evidence of nucleonic pairing. The relevance in high- T_c superconductivity of the BCS pairing theory is thus one of the most hotly pursued questions in present-day condensed-matter physics.

4. Conclusion

For the one-particle potential well problem in either one or two dimensions a bound state exists no matter how shallow the well—in contrast with the three-dimensional case requiring a critical well depth to bind a level. But whereas the ground-state eigenvalue energy can be expressed as a perturbative power series in the well depth strength in one dimension, such is not the case in two dimensions where an essential (non-power-series) singularity emerges. This particular type of singularity (which carries over into the many-electron, superconductivity treatment of BCS) also appears in the Cooper two-electron problem in either 1, 2, or 3D and is thus not related to the apparent two-dimensionality of the BCS interaction model.

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Appendix: Cooper pairing in one-, two- and three- dimensions

We briefly sketch how the basic weak-coupling expression (20) for the Cooper pair binding energy follows explicitly regardless of the value of D (space dimensionality), whether 1, 2 or 3, provided only that $2\nu \equiv \hbar\omega_D/E_F \ll 1$.

For any D , the summation over k in Eq. (16) leads to an integral over

$$\frac{L^D}{(2\pi)^D} d^D k \equiv g(\mathcal{E}) d\mathcal{E}, \quad (\text{A.1})$$

where L is the size length of the system. Since $\mathcal{E} = \hbar^2 k^2/2m$, this immediately leads to

$$\begin{aligned} g(\mathcal{E}) &= C_1 \mathcal{E}^{-1/2} & (D = 1) \\ &= C_2 & (D = 2) \\ &= C_3 \mathcal{E}^{1/2} & (D = 3), \end{aligned} \quad (\text{A.2})$$

where the constants $C_1 = (m/2)^{1/2}L/\pi\hbar$, $C_2 = mL^2/2\pi\hbar$ and $C_3 = 2^{-1/2}(m/\hbar^2)^{3/2} \times L^3/\pi^2$. For $D = 2$, the first integral in (17) was already carried out in the text. For $D = 1$, the first integral in (17) leads to

$$\int_{E_F}^{E_F+\hbar\omega_D} \frac{d\mathcal{E}}{\mathcal{E}^{1/2}(2\mathcal{E} - E_0)} = \frac{1}{2\sqrt{\epsilon}E_F} \ln \left[\left(\frac{\sqrt{1+2\nu} - \sqrt{\epsilon}}{\sqrt{1+2\nu} + \sqrt{\epsilon}} \right) \left(\frac{1 + \sqrt{\epsilon}}{1 - \sqrt{\epsilon}} \right) \right], \quad (A.3)$$

where $\epsilon \equiv E_0/2E_F$. Putting $\epsilon \equiv 1 - \delta$ with $\delta \equiv \Delta/2E_F$, the rhs of (A.3) can be expanded for $\nu \ll 1$ and $\delta \ll 1$ and, after a little algebra, leads to

$$\delta = 2\nu e^{-2/g(E_F)V}, \quad (A.4)$$

where $g(E_F) = C_1 E_F^{-1/2}$, i.e., of the same form as the weak-coupling limit of Eq. (20). Similarly, for $D = 3$ we have

$$\int_{E_F}^{E_F+\hbar\omega_D} d\mathcal{E} \frac{\mathcal{E}^{1/2}}{2\mathcal{E} - E_0} = \sqrt{E_F} \left\{ 2 \left(\sqrt{1+2\nu} - 1 \right) + \frac{\sqrt{\epsilon}}{2} \ln \left[\left(\frac{\epsilon + 1 + 2\nu - 2\sqrt{\epsilon(1+2\nu)}}{\epsilon - 1 - 2\nu} \right) \left(\frac{\epsilon - 1}{\epsilon + 1 - 2\sqrt{\epsilon}} \right) \right] \right\}. \quad (A.5)$$

In the limit of ν and $\delta \ll 1$, this again leads to the form (A.4), but with $g(E_F) = C_3 E_F^{1/2}$. We remark that both integrations (A.3) and (A.5) were done using tables, and checked [24] via the computer-algebra package MATHEMATICA. Note, however, that in either 1D or 3D the $e^{-1/\lambda}$ essential singularity emerges before the “two-dimensional” property of the BCS interaction $\nu \ll 1$ is imposed. Thus, the $e^{-1/\lambda}$ singularity is not a property of dimensionality, in spite of a tempting analogy with the 2D quantum binding problem.

Finally, we wish to recover the well-known properties of the first bound state of a particle in a 1-, 2- and 3- [Eq. (3) and below Eq. (4)] dimensional attractive well, namely the features [13,10] that: a) in 1- and 2- dimensions a bound level is always present, no matter how shallow the well, whereas in 3-dimensions a critical depth (4) is required; and b) the binding energy is analytic in the well-depth, as the well-depth vanishes, in 1- and 3- [Eq. (3) and below Eq. (4)] but not in 2-dimensions. To get this from the Cooper pair problem, the “blocking effect” of the Fermi sea has to be phased out appropriately, so that we deal with two particles (or one particle, with a specific effective mass) interacting in the vacuum. The non-analyticity referred to in (b) above in the 2D case immediately follows from the form of the first integral in Eq. (17) since in this instance E_F does not appear in the rhs and can thus be taken to be zero (the vacuum limit). For $D = 1$, let $E_0 \equiv -|E_0|$ and take $E_F = 0$ in the lhs of (A.3). The first integral in Eq. (17) then becomes

$$1 = VC_1 \int_0^{\hbar\omega_D} \frac{d\mathcal{E}}{\mathcal{E}^{1/2}(2\mathcal{E} + |E_0|)}, \quad (A.6)$$

which evaluates to

$$\sqrt{\frac{2}{|E_0|}} \tan^{-1} \sqrt{\frac{2\hbar\omega_D}{|E_0|}} \xrightarrow{|E_0| \rightarrow 0} \frac{\pi}{\sqrt{2|E_0|}}, \tag{A.7}$$

so that (A.6) yields

$$1 \simeq \frac{C_1 V \pi}{\sqrt{2|E_0|}}. \tag{A.8}$$

This can be satisfied for V arbitrarily small. By contrast, for $D = 3$ the lhs of (A.5) with $E_F = 0$ gives

$$\int_0^{\hbar\omega_D} d\mathcal{E} \frac{\mathcal{E}^{1/2}}{2\mathcal{E} + |E_0|} = \sqrt{\hbar\omega_D} - \sqrt{\frac{|E_0|}{2}} \tan^{-1} \sqrt{\frac{2\hbar\omega_D}{|E_0|}} \xrightarrow{|E_0| \rightarrow 0} \sqrt{\hbar\omega_D}. \tag{A.9}$$

This together with the first equality in Eq. (17) means that a non-vanishing threshold depth of magnitude

$$V = \frac{1}{C_3 \sqrt{\hbar\omega_D}} \tag{A.10}$$

is needed to support a bound level, as expected.

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Resumen. En este trabajo se reseñan los paralelismos entre el problema de Cooper para dos electrones y el problema de ligadura cuántica en dos dimensiones. Ambos problemas manifiestan una singularidad anómala esencial, es decir, un comportamiento *no perturbativo*, en el parámetro de acoplamiento. Según la teoría de BCS, una singularidad similar limita severamente la magnitud de temperaturas de transición posibles en la superconductividad. Al analizar el problema de Cooper en una y tres dimensiones se concluye que la singularidad anómala *no* está vinculada con la bidimensionalidad implícita en el modelo dinámico electrón-fonón de BCS, contrario a lo que se pudiera sospechar.