# Little group generators for Dirac neutrino one-particle states

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Assuming neutrinos to be of the Dirac type, the little group generators for the one-particle states, created off the vacuum by the field operator, are obtained, both in terms of the one-particle states themselves and in terms of creation/annihilation operators. It is shown that these generators act also as rotation operators in the Hilbert space of the states, providing three types of transformations: a helicity flip, the standard charge conjugation, and a combination of the two, up to phases. The transformations' properties are provided in detail and their physical implications discussed. It is also shown that one of the transformations continues to hold for chiral fields without mixing them. It is argued that these results provide support for the Majorana nature of massive neutrinos.

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## 1. Introduction

It is now established that neutrinos oscillate in flavor and are therefore massive [1–7], and one fundamental aspect still unresolved is the determination of their nature, whether they are Dirac or Majorana particles [8–12]. Perturbative calculations do not help in this regard because of the vanishing small ratio $m/E$ that all differences between the two types are proportional to [10, 13–15]. Experimentally, the observation of neutrinoless double beta decay processes would confirm their Majorana nature, and there are already various types of experiments, both planned and underway, set up with that purpose [16–21]. However, the non-observation of such processes do not necessarily imply that neutrinos are of the Dirac type [22]. On the theoretical side, Majorana neutrinos are preferred because they are central in the various types of the see-saw mechanism [23–29], and also in leptogenesis models [30–32].

Associated with the nature of neutrinos is the question of lepton-number conservation. Let us consider mass eigenstates for both neutrino types. These are one-particle states of definite energy and momentum created off the vacuum by the relevant field operator. In terms of creation operators, $a^{\dagger}(\mathbf{p})$ ($b^{\dagger}(\mathbf{p})$) for Dirac neutrinos (anti-neutrinos) and $\hat{a}^{\dagger}(\mathbf{p})$ for Majorana neutrinos, the alternatives are, respectively,

$$a^{\dagger}_{-}(\mathbf{p}) \neq b^{\dagger}_{+}(\mathbf{p}) \,, \tag{1}$$

$$\hat{a}^{\dagger}_{+}(\mathbf{p}) \neq \hat{a}^{\dagger}_{-}(\mathbf{p}) \,, \tag{2}$$

where the subscripts $\pm$ respectively denotes positive and negative helicity. Thus, if lepton number conservation holds, different helicity neutrinos and anti-neutrinos are different particles (Dirac case), and Eq. (1) is the right choice (there is also $a^{\dagger}_{-}(\mathbf{p}) \neq b^{\dagger}_{+}(\mathbf{p})$, but these modes have not been observed). On the other hand, if lepton number is violated neutrinos and anti-neutrinos are just two helicity states of the same particle (Majorana case), and Eq. (2) is the right one. Neutrinos and anti-neutrinos are related by the anti-unitary and discrete $\mathcal{CPT}$ transformation (charge conjugation, parity and time reversal) [11]. If $\nu(\boldsymbol{p}, h)$ represents a Dirac neutrino with momentum $\boldsymbol{p}$ an helicity $h$ then, setting the overall phase to one, for simplicity, we have

$$\nu(\boldsymbol{p}, h) \xrightarrow{\mathcal{CPT}} \overline{\nu}(\boldsymbol{p}, -h) \,, \tag{3}$$

where $\overline{\nu}(\boldsymbol{p}, -h)$ is the anti-neutrino with the same momentum and opposite helicity. For the Majorana case the $\mathcal{CPT}$ transformation just reverses the helicity

$$\nu(\boldsymbol{p}, h) \xrightarrow{\mathcal{CPT}} \nu(\boldsymbol{p}, -h) \,. \tag{4}$$

In this work we take Eq. (1) as the premise and consider neutrino and anti-neutrino one-particles states, labeled by their momentum and helicity. That is, we restrict the discussion to the Hilbert space of the free one-particle states of a given momentum, created off the vacuum by the relevant field operator, which in the case of Dirac neutrinos it is four dimensional. Let us, for now, generically denote them by $|\mathbf{p}, h\rangle$ (they will be properly defined in the next section). They are degenerate in the four-momentum eigenvalues and, in particular, if $\mathcal{H}$ is the Hamiltonian operator we have

$$\mathcal{H}\,|\mathbf{p}, h\rangle = E_{\boldsymbol{p}}\,|\mathbf{p}, h\rangle \,, \tag{5}$$

with $E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$. Now let us consider a unitary transformation $U$ such that

$$U\,|\mathbf{p}, h\rangle = \alpha\,|\mathbf{p}, h'\rangle \,, \tag{6}$$

where $\alpha$ is a phase, $|\alpha|^2 = 1$. That is, $U$ transforms the one-particle states among themselves. Then it is easy to see that $(\mathcal{H}U - U\mathcal{H})\,|\mathbf{p}, h\rangle = 0$. Thus,

$$[U, \mathcal{H}] = 0, \tag{7}$$

and we conclude by Eqs. (5) and (6) that $U$ is a unitary transformation leaving the four-momentum invariant, which is precisely the definition of a little group transformation [33–36]. Since there are four one-particle states, two helicity values for each of the neutrino and the anti-neutrino, there are three different types of unitary transformations we can consider, these are: a transformation that flips the helicity without mixing particles and anti-particles, the standard charge conjugation and a combination of these two. Also, because the one-particle states are fermionic and massive, the little group is SU(2), the rotation group for $SL(2,C)$ in the $(1/2,0)\oplus(0,1/2)$ representation [34, 36].

It is the purpose of this paper to define the $U$ transformations, both in terms of the one-particle states themselves and in terms of creation/annihilation operators, and exhibit their properties, which are of physical interest. Among other properties, we show that the transformations are Hermitian besides being unitary, and that they do indeed satisfy the SU(2) Lie algebra. Physically, the three transformations correspond to helicity flip, charge conjugation and a combination of the two, up to phases. This last transformation will be also shown to hold for chirally projected fields.

The organization is as follows: We first present the states, operators, and spinors and establish the conventions in section II, we then proceed to present and discuss the three unitary transformations in Sec. 3, first in terms of the one-particle states, assuming a finite volume quantization, and then in terms of creation/annihilation operators, which is more fundamental. In Sec. 4 we show that the Dirac field operator, both for the unconstrained case and for the left-chiral one, is consistently transformed under one of the little group transformations. Finally, we further discuss the physical implications of the results and provide concluding remarks.

## 2. Free field conventions

Let us assume that a free massive neutrino is of the Dirac type, so that Eq. (1) holds. It is thus described by a massive Dirac field operator, here given in the helicity basis [37]

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E}}\sum_{\lambda=\pm}\Big(u_\lambda(\mathbf{p})\,a_\lambda(\mathbf{p})\,e^{-ip.x} + v_\lambda(\mathbf{p})\,b_\lambda^\dagger(\mathbf{p})\,e^{ip.x}\Big), \tag{8}$$

where the operators $a_\pm^\dagger(\mathbf{p})$, $b_\pm^\dagger(\mathbf{p})$, respectively, create particles and anti-particles of the given helicity, labeled by the subscript, off the vacuum. The equal-time anti-commutation relations are the canonical ones [38]

$$\begin{aligned}
\left\{\Psi_\alpha(\mathbf{x}), \Psi_\beta^\dagger(\mathbf{y})\right\} &= \delta^3(\mathbf{x}-\mathbf{y})\,\delta_{\alpha\beta},\\
\left\{\Psi_\alpha(\mathbf{x}), \Psi_\beta(\mathbf{y})\right\} &= \left\{\Psi_\alpha^\dagger(\mathbf{x}), \Psi_\beta^\dagger(\mathbf{y})\right\} = 0,\\
\left\{a_\lambda(\mathbf{p}), a_{\lambda'}^\dagger(\mathbf{q})\right\} &= (2\pi)^3\,\delta^3(\mathbf{p}-\mathbf{q})\,\delta_{\lambda\lambda'}.\\
\left\{b_\lambda(\mathbf{p}), b_{\lambda'}^\dagger(\mathbf{q})\right\} &= (2\pi)^3\,\delta^3(\mathbf{p}-\mathbf{q})\,\delta_{\lambda\lambda'}.
\end{aligned} \tag{9}$$

The one-particle sates

$$\begin{aligned}
|\mathbf{p},-\rangle &= a_-^\dagger(\mathbf{p})\,|0\rangle,\\
|\mathbf{p},+\rangle &= a_+^\dagger(\mathbf{p})\,|0\rangle,\\
|\overline{\mathbf{p}},-\rangle &= b_-^\dagger(\mathbf{p})\,|0\rangle,\\
|\overline{\mathbf{p}},+\rangle &= b_+^\dagger(\mathbf{p})\,|0\rangle,
\end{aligned} \tag{10}$$

correspondingly represent left- and right-handed[i] neutrinos, and left- and right-handed anti-neutrinos, with the anti-particle states distinguished by an over bar, and $|0\rangle$ denoting the free vacuum state.

The bispinors in the field expansion in Eq. (8) are expressed in terms of the two-component Weyl spinors

$$\begin{aligned}
\xi_+(\mathbf{p}) &= \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\left(\frac{\theta}{2}\right)\\ e^{i\frac{\varphi}{2}}\sin\left(\frac{\theta}{2}\right)\end{pmatrix},\\
\xi_-(\mathbf{p}) &= \begin{pmatrix} -e^{-i\frac{\varphi}{2}}\sin\left(\frac{\theta}{2}\right)\\ e^{i\frac{\varphi}{2}}\cos\left(\frac{\theta}{2}\right)\end{pmatrix},
\end{aligned} \tag{11}$$

which satisfy $\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\,\xi_\lambda(\mathbf{p}) = \lambda\xi_\lambda(\mathbf{p})$, $\lambda = \pm$, with three-momentum $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}| = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. Explicitly

$$\begin{aligned}
u_\lambda(\mathbf{p}) &= \begin{pmatrix}\sqrt{E-\lambda|\mathbf{p}|}\,\xi_\lambda(\mathbf{p})\\ \sqrt{E+\lambda|\mathbf{p}|}\,\xi_\lambda(\mathbf{p})\end{pmatrix},\\
v_\lambda(\mathbf{p}) &= \begin{pmatrix}-\lambda\sqrt{E+\lambda|\mathbf{p}|}\,\xi_{-\lambda}(\mathbf{p})\\ \lambda\sqrt{E-\lambda|\mathbf{p}|}\,\xi_{-\lambda}(\mathbf{p})\end{pmatrix},
\end{aligned} \tag{12}$$

and they are normalized according to the relations

$$\begin{aligned}
\overline{u}_\lambda(\mathbf{p})\,u_{\lambda'}(\mathbf{p}) &= 2m\delta_{\lambda,\lambda'}\\
\overline{v}_\lambda(\mathbf{p})\,v_{\lambda'}(\mathbf{p}) &= -2m\delta_{\lambda,\lambda'}\\
\overline{u}_\lambda(\mathbf{p})\,v_{\lambda'}(\mathbf{p}) &= \overline{v}_\lambda(\mathbf{p})\,u_{\lambda'}(\mathbf{p}) = 0.
\end{aligned} \tag{13}$$

Here, the over bar represents the Dirac adjoint $\overline{u} \equiv u^\dagger\gamma^0$. The bispinors satisfy the momentum-space Dirac equations $\left(\not{p}-m\right)u_\lambda(\mathbf{p}) = 0$ and $\left(\not{p}+m\right)v_\lambda(\mathbf{p}) = 0$, with $\not{p} \equiv \gamma^\mu p_\mu$. We use the Weyl representation of the gamma matrices, as given in Ref. [38].

The Hamiltonian, momentum, and lepton-number operators are respectively given by

$$\mathcal{H}=\int \frac{d^3p}{(2\pi)^3} \sum_{\lambda=\pm} E_{\mathbf{p}} \left( a_\lambda^\dagger(\mathbf{p}) a_\lambda(\mathbf{p}) + b_\lambda^\dagger(\mathbf{p}) b_\lambda(\mathbf{p}) \right), \quad (14)$$

$$\boldsymbol{P}=\int \frac{d^3p}{(2\pi)^3} \sum_{\lambda=\pm} \boldsymbol{p} \left( a_\lambda^\dagger(\mathbf{p}) a_\lambda(\mathbf{p}) + b_\lambda^\dagger(\mathbf{p}) b_\lambda(\mathbf{p}) \right), \quad (15)$$

$$\mathrm{L}=\int \frac{d^3p}{(2\pi)^3} \sum_{\lambda=\pm} \left( a_\lambda^\dagger(\mathbf{p}) a_\lambda(\mathbf{p}) - b_\lambda^\dagger(\mathbf{p}) b_\lambda(\mathbf{p}) \right). \quad (16)$$

## 3. Little group transformations

### 3.1. One-particle states

To simplify calculations and obtain a rapid overview of the transformations let us for now resort to a discrete volume quantization [11], so that the one-particle states in Eq. (10) can be taken orthonormal. It is then straightforward to consider operators of the form $\sum \alpha_{hh'} |\mathbf{p}, h\rangle \langle \mathbf{p}, h'|$, with $\alpha_{hh'}$ adequately chosen phases so that the operators are unitary and satisfy the SU(2) algebra. These constitute the $U$ operators in Eqs. (5) and (6). Then we have

$$\begin{aligned} U_1 &= i|\mathbf{p},-\rangle\langle\mathbf{p},+| - i|\mathbf{p},+\rangle\langle\mathbf{p},-| \\ &\quad - i|\overline{\mathbf{p}},-\rangle\langle\overline{\mathbf{p}},+| + i|\overline{\mathbf{p}},+\rangle\langle\overline{\mathbf{p}},-|, \end{aligned} \quad (17)$$

$$\begin{aligned} U_2 &= |\mathbf{p},-\rangle\langle\overline{\mathbf{p}},-| + |\overline{\mathbf{p}},-\rangle\langle\mathbf{p},-| \\ &\quad + |\mathbf{p},+\rangle\langle\overline{\mathbf{p}},+| + |\overline{\mathbf{p}},+\rangle\langle\mathbf{p},+|, \end{aligned} \quad (18)$$

$$\begin{aligned} U_3 &= |\mathbf{p},+\rangle\langle\overline{\mathbf{p}},-| - |\overline{\mathbf{p}},+\rangle\langle\mathbf{p},-| \\ &\quad - |\mathbf{p},-\rangle\langle\overline{\mathbf{p}},+| + |\overline{\mathbf{p}},-\rangle\langle\mathbf{p},+|, \end{aligned} \quad (19)$$

which respectively produce

$$\begin{aligned} U_1|\mathbf{p},-\rangle &= -i|\mathbf{p},+\rangle, \\ U_1|\mathbf{p},+\rangle &= i|\mathbf{p},-\rangle, \\ U_1|\overline{\mathbf{p}},-\rangle &= i|\overline{\mathbf{p}},+\rangle, \\ U_1|\overline{\mathbf{p}},+\rangle &= -i|\overline{\mathbf{p}},-\rangle, \end{aligned} \quad (20)$$

$$\begin{aligned} U_2|\mathbf{p},-\rangle &= |\overline{\mathbf{p}},-\rangle, \\ U_2|\mathbf{p},+\rangle &= |\overline{\mathbf{p}},+\rangle, \\ U_2|\overline{\mathbf{p}},-\rangle &= |\mathbf{p},-\rangle, \\ U_2|\overline{\mathbf{p}},+\rangle &= |\mathbf{p},+\rangle, \end{aligned} \quad (21)$$

$$\begin{aligned} U_3|\mathbf{p},-\rangle &= -|\overline{\mathbf{p}},+\rangle, \\ U_3|\mathbf{p},+\rangle &= |\overline{\mathbf{p}},-\rangle, \\ U_3|\overline{\mathbf{p}},-\rangle &= |\mathbf{p},+\rangle, \\ U_3|\overline{\mathbf{p}},+\rangle &= -|\mathbf{p},-\rangle. \end{aligned} \quad (22)$$

To further establish their properties, it is easier to work with a matrix representation, obtained from the matrix elements

$$\begin{pmatrix} \langle -,\mathbf{p}|U_i|\mathbf{p},-\rangle & \langle +,\mathbf{p}|U_i|\mathbf{p},-\rangle & \langle -,\overline{\mathbf{p}}|U_i|\mathbf{p},-\rangle & \langle +,\overline{\mathbf{p}}|U_i|\mathbf{p},-\rangle \\ \langle -,\mathbf{p}|U_i|\mathbf{p},+\rangle & \langle +,\mathbf{p}|U_i|\mathbf{p},+\rangle & \langle -,\overline{\mathbf{p}}|U_i|\mathbf{p},+\rangle & \langle +,\overline{\mathbf{p}}|U_i|\mathbf{p},+\rangle \\ \langle -,\mathbf{p}|U_i|\overline{\mathbf{p}},-\rangle & \langle +,\mathbf{p}|U_i|\overline{\mathbf{p}},-\rangle & \langle -,\overline{\mathbf{p}}|U_i|\overline{\mathbf{p}},-\rangle & \langle +,\overline{\mathbf{p}}|U_i|\overline{\mathbf{p}},-\rangle \\ \langle -,\mathbf{p}|U_i|\overline{\mathbf{p}},+\rangle & \langle +,\mathbf{p}|U_i|\overline{\mathbf{p}},+\rangle & \langle -,\overline{\mathbf{p}}|U_i|\overline{\mathbf{p}},+\rangle & \langle +,\overline{\mathbf{p}}|U_i|\overline{\mathbf{p}},+\rangle \end{pmatrix}. \quad (23)$$

Thus,

$$U_1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \quad (24)$$

$$U_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (25)$$

$$U_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad (26)$$

Where, with a slight abuse of notation, we label the matrix representation with the same symbol as the corresponding operator. With these matrices, the following properties are readily verified

$$\left.\begin{aligned} U_i^{-1} &= U_i^\dagger, \\ U_i &= U_i^\dagger, \\ \det U_i &= 1, \\ \operatorname{tr} U_i &= 0, \\ [\mathcal{H}, U_j] &= 0, \\ [U_i, U_j] &= 2i\varepsilon_{ijk} U_k. \end{aligned}\right\} \quad i = 1,2,3 \quad (27)$$

Being both unitary and Hermitian, the transformations are also observables. The second last property follows from comparing Eqs. (17) to (19) with Eq. (6) and using Eqs. (5) and (7). This property verifies that the transformations leave the four-momentum invariant and are conserved. The last property establishes that the transformations fulfill the SU(2) algebra, so they can be identified with the little group generators. In this regard they are analogous to the Pauli matrices that play the dual role of being SU(2) generators and $2\pi$ rotation operators for spin $1/2$ particles.

The transformation's physical content is read directly from Eqs. (20) - (22): $U_1$ flips the helicity without mixing particles and anti-particles, $U_2$ is charge conjugation with the conventional phases [38], and $U_3$ is a combination of the previous two, up to a phase. Thus, $U_3$ relates particles and anti-particles with opposite helicities and, in particular, connects a LH neutrino one-particle state with a RH anti-neutrino one.

We also emphasize that, being a little group rotation, the transformation does not flip the three-momentum, as it would necessarily be the case for a $\mathcal{CP}$ (charge conjugation and parity) transformation [11, 40]. In fact, using the standard transformation properties of the one-particle states under $\mathcal{CPT}$, we see from Eq. (22) that, up to phases, $U_3$ produces the same outcome as a $\mathcal{CPT}$ transformation. This, of course, does not mean that these two operators are equivalent, since the later is a discrete and anti-unitary spacetime transformation, while the former is a rotation in spin-space.

We can readily verify the commutation properties between the little group rotations with $\mathcal{CP}$ and $\mathcal{CPT}$. Setting phase factors to one for simplicity, *e.g.*, $\mathcal{CP}\left|\mathbf{p},-\right\rangle = \left|-\overline{\mathbf{p}},+\right\rangle$ and $\mathcal{CPT}\left|\mathbf{p},-\right\rangle = \left|\overline{\mathbf{p}},+\right\rangle$, and remembering we are restricting the discussion to the Hilbert state of the free one-particle states, we obtain using Eq. (22)

$$\begin{aligned} U_3\mathcal{CP}\left|\mathbf{p},-\right\rangle &= -\left|-\mathbf{p},-\right\rangle, \\ \mathcal{CP}U_3\left|\mathbf{p},-\right\rangle &= -\left|-\mathbf{p},-\right\rangle. \end{aligned} \tag{28}$$

Analogously,

$$\begin{aligned} U_3\mathcal{CPT}\left|\mathbf{p},-\right\rangle &= -\left|\mathbf{p},-\right\rangle, \\ \mathcal{CPT}U_3\left|\mathbf{p},-\right\rangle &= -\left|\mathbf{p},-\right\rangle. \end{aligned} \tag{29}$$

Hence,

$$[U_3,\mathcal{CP}] = [U_3,\mathcal{CPT}] = 0. \tag{30}$$

Similar results hold for $U_1$ and $U_2$, so we conclude

$$[U_i,\mathcal{CP}] = [U_i,\mathcal{CPT}] = 0, \;\; i = 1,2,3. \tag{31}$$

### 3.2. Creation and annihilation operators

The little group rotations can also be given in terms of creation and annihilation operators. They are

$$\begin{aligned} U_1 = \exp\left(-i\frac{\pi}{2}\right)\exp\Bigg\{i\frac{\pi}{2}\int\frac{d^3p}{(2\pi)^3}\Bigg(a_+^\dagger(\mathbf{p})\,a_-(\mathbf{p}) \\ + a_-^\dagger(\mathbf{p})\,a_+(\mathbf{p}) + b_+^\dagger(\mathbf{p})\,b_-(\mathbf{p}) + b_-^\dagger(\mathbf{p})\,b_+(\mathbf{p})\Bigg)\Bigg\} \\ \times\exp\Bigg\{i\frac{\pi}{2}\int\frac{d^3p}{(2\pi)^3}\Bigg(a_-^\dagger(\mathbf{p})\,a_-(\mathbf{p}) - a_+^\dagger(\mathbf{p})\,a_+(\mathbf{p}) \\ + b_+^\dagger(\mathbf{p})\,b_+(\mathbf{p}) - b_-^\dagger(\mathbf{p})\,b_-(\mathbf{p})\Bigg)\Bigg\}, \end{aligned} \tag{32}$$

$$\begin{aligned} U_2 = \exp\Bigg\{i\frac{\pi}{2}\int\frac{d^3p}{(2\pi)^3}\sum_{\lambda=\pm}\left[b_\lambda^\dagger(\mathbf{p}) - a_\lambda^\dagger(\mathbf{p})\right] \\ \times\left[a_\lambda(\mathbf{p}) - b_\lambda(\mathbf{p})\right]\Bigg\}, \end{aligned} \tag{33}$$

$$\begin{aligned} U_3 = \exp\Bigg\{i\frac{\pi}{2}\int\frac{d^3p}{(2\pi)^3}\Bigg(\left[a_-^\dagger(\mathbf{p}) + b_+^\dagger(\mathbf{p})\right] \\ \times\left[a_-(\mathbf{p}) + b_+(\mathbf{p})\right] + \left[b_-^\dagger(\mathbf{p}) - a_+^\dagger(\mathbf{p})\right] \\ \times\left[a_+(\mathbf{p}) - b_-(\mathbf{p})\right]\Bigg)\Bigg\}, \end{aligned} \tag{34}$$

where we have again slightly abused the notation and keep the same labels for the transformations. Acting on the creation operators they give

$$\begin{aligned} U_1 a_-^\dagger(\mathbf{p})\,U_1^\dagger &= -i a_+^\dagger(\mathbf{p}), \\ U_1 a_+^\dagger(\mathbf{p})\,U_1^\dagger &= i a_-^\dagger(\mathbf{p}), \\ U_1 b_-^\dagger(\mathbf{p})\,U_1^\dagger &= i b_+^\dagger(\mathbf{p}), \\ U_1 b_+^\dagger(\mathbf{p})\,U_1^\dagger &= -i b_-^\dagger(\mathbf{p}), \end{aligned} \tag{35}$$

$$\begin{aligned} U_2 a_-^\dagger(\mathbf{p})\,U_2^\dagger &= b_-^\dagger(\mathbf{p}), \\ U_2 a_+^\dagger(\mathbf{p})\,U_2^\dagger &= b_+^\dagger(\mathbf{p}), \\ U_2 b_-^\dagger(\mathbf{p})\,U_2^\dagger &= a_-^\dagger(\mathbf{p}), \\ U_2 b_+^\dagger(\mathbf{p})\,U_2^\dagger &= a_+^\dagger(\mathbf{p}), \end{aligned} \tag{36}$$

$$\begin{aligned} U_3 a_-^\dagger(\mathbf{p})\,U_3^\dagger &= -b_+^\dagger(\mathbf{p}), \\ U_3 a_+^\dagger(\mathbf{p})\,U_3^\dagger &= b_-^\dagger(\mathbf{p}), \\ U_3 b_-^\dagger(\mathbf{p})\,U_3^\dagger &= a_+^\dagger(\mathbf{p}), \\ U_3 b_+^\dagger(\mathbf{p})\,U_3^\dagger &= -a_-^\dagger(\mathbf{p}). \end{aligned} \tag{37}$$

From Eqs. (32) - (34) it is clear that $U_i\left|0\right\rangle = \left|0\right\rangle$, $i = 1,2,3$, with $\left|0\right\rangle$ the vacuum state, then the one-particle state transformations in Eqs. (20) - (22) follow from Eqs. (35) - (37). Explicit construction of $U_3$ and its action is provided in the appendix. It is also straightforward to verify that, as required, the transformations commute with the Hamiltonian and the momentum operators in Eqs. (14) and (15)

$$[\mathcal{H},U_i] = [\boldsymbol{P},U_i] = 0 \;\; i = 1,2,3. \tag{38}$$

$U_2$ and $U_3$ anti-commute with the lepton-number operator in Eq. (16)

$$\{L,U_i\} = 0 \;\; i = 2,3, \tag{39}$$

while $U_1$ commutes with it. It can also be shown that the anti-commutation relations remain invariant under the transformations (the case for $U_3$ is shown in the appendix).

$$\begin{aligned} U_i\left\{\Psi_\alpha(\mathbf{x}),\Psi_\beta^\dagger(\mathbf{y})\right\}U_i^\dagger = \delta^3(\mathbf{x}-\mathbf{y})\,\delta_{\alpha\beta}, \\ i = 1,2,3. \end{aligned} \tag{40}$$

## 4. Field operator transformation

In this section we show that the Dirac field operator is consistently transformed under $U_3$, both for the unconstrained field and the chirally projected one.

### 4.1. Unconstrained field

The Dirac field in Eq. (8) is consistently transformed under $U_3$ by appropriately transforming the bispinors. For that purpose, let us consider the rotation matrix that implements a counterclockwise rotation by an angle $2\varphi$ around the positive $p_z$ axis

$$\mathcal{R}(\boldsymbol{p}) = \begin{pmatrix} R_C & 0 \\ 0 & R_C \end{pmatrix} = \exp\left\{-i\,2\varphi\frac{\Sigma^3}{2}\right\}, \tag{41}$$

with $\Sigma^3$ being the third component of

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix},$$

and $R_C$ the SU(2) matrix

$$R_C = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix} = \exp\left\{-i\,2\varphi\frac{\sigma^3}{2}\right\} \tag{42}$$

yielding

$$R_C\xi^*_\pm(\mathbf{p}) = \xi_\pm(\mathbf{p}) \tag{43}$$

on the two-component spinors. Analogously, the matrix $\mathcal{R}(\boldsymbol{p})$ in Eq. (41) produces, for the bispinors

$$\begin{aligned} \mathcal{R}(\boldsymbol{p})u^*_\lambda(\mathbf{p}) &= u_\lambda(\mathbf{p}), \\ \mathcal{R}(\boldsymbol{p})v^*_\lambda(\mathbf{p}) &= v_\lambda(\mathbf{p}). \end{aligned} \tag{44}$$

We also make use of the chiral matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{45}$$

and the relations

$$\begin{aligned} -\gamma^5 u_\lambda(\mathbf{p}) &= \lambda\, v_{-\lambda}(\mathbf{p}), \\ -\gamma^5 v_\lambda(\mathbf{p}) &= -\lambda\, u_{-\lambda}(\mathbf{p}). \end{aligned} \tag{46}$$

Combining Eqs. (44) and (46) we get

$$\begin{aligned} -\gamma^5\mathcal{R}(\boldsymbol{p})u^*_\lambda(\mathbf{p}) &= \lambda\, v_{-\lambda}(\mathbf{p}), \\ -\gamma^5\mathcal{R}(\boldsymbol{p})v^*_\lambda(\mathbf{p}) &= -\lambda\, u_{-\lambda}(\mathbf{p}). \end{aligned} \tag{47}$$

The field transformation is then obtained from Eqs. (8), (37) and (47) as

$$U_3\Psi(x)U_3^\dagger = -\gamma^5\mathcal{R}(\boldsymbol{p})\Psi^*(x), \tag{48}$$

and it constitutes a consistent transformation of the field operator, since its right-hand side induces a transformation of the Dirac equation in momentum-space: from

$$-\gamma^5\mathcal{R}(\boldsymbol{p})\left(\not{p}^* - m\right)\left(-\gamma^5\mathcal{R}(\boldsymbol{p})\right)^\dagger = -\left(\not{p} + m\right)$$

and Eq. (47) we get

$$(\not{p}-m)\,u_\lambda(\mathbf{p})=0 \quad \xrightarrow{-\gamma^5\mathcal{R}(\boldsymbol{p})\mathcal{K}} \left(\not{p}+m\right)v_{-\lambda}(\mathbf{p})=0, \tag{49}$$

where $\mathcal{K}$ represents the operation of conjugating to the right. We also have that a second application of the $U_3$ transformation corresponds to no transformation at all, as can be seen from Eq. (37) and the fact that

$$\begin{aligned} \left(-\gamma^5\mathcal{R}(\boldsymbol{p})\right)\mathcal{K}\left(-\gamma^5\mathcal{R}(\boldsymbol{p})\right)\mathcal{K} = \left(-\gamma^5\mathcal{R}(\boldsymbol{p})\right) \\ \times\left(-\gamma^5\mathcal{R}(\boldsymbol{p})\right)^* = \mathbf{1}. \end{aligned}$$

The case for $U_2$, being the charge conjugation operator, is textbook matter, and the case for $U_1$ proceeds in a similar fashion.

### 4.2. Chirally projected field

Let us apply the chiral projection operators $L = 1/2\left(\mathbf{1} - \gamma^5\right)$ and $R = 1/2\left(\mathbf{1} + \gamma^5\right)$ to the field expansion in Eq. (8), to obtain

$$\begin{aligned} \Psi_L(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E}}\Big\{ &\left(a_+(\mathbf{p})\,e^{-ip.x} + b^\dagger_-(\mathbf{p})\,e^{ip.x}\right) \\ &\times \xi_+(\mathbf{p})\sqrt{E-|\mathbf{p}|} + \left(a_-(\mathbf{p})\,e^{-ip.x} - b^\dagger_+(\mathbf{p})\,e^{ip.x}\right) \\ &\times \xi_-(\mathbf{p})\sqrt{E+|\mathbf{p}|}\Big\}, \end{aligned} \tag{50}$$

$$\begin{aligned} \Psi_R(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E}}\Big\{ &\left(a_+(\mathbf{p})\,e^{-ip.x} - b^\dagger_-(\mathbf{p})\,e^{ip.x}\right) \\ &\times \xi_+(\mathbf{p})\sqrt{E+|\mathbf{p}|} + \left(a_-(\mathbf{p})\,e^{-ip.x} + b^\dagger_+(\mathbf{p})\,e^{ip.x}\right) \\ &\times \xi_-(\mathbf{p})\sqrt{E-|\mathbf{p}|}\Big\}. \end{aligned} \tag{51}$$

We thus see that each chiral field and its Hermitian conjugate produce both helicity states, and we can continue to use the one-particle states in Eq. (10), with one set for each chirality.

In the high energy limit, with $E \gg m$, states with weight $\sqrt{E-|\mathbf{p}|} \approx m/\sqrt{2E}$ are suppressed, while the ones with $\sqrt{E+|\mathbf{p}|} \approx \sqrt{2E}$ are favored, so in general, and if helicity is not measured, a left-chiral neutrino of energy $E$ produced by the weak interaction will be in a superposition of both helicities. It can then be described by the density matrix [8]

$$\begin{aligned} \rho_\nu(E) = &\left(\frac{E+|\mathbf{p}|}{2E}\right)|\mathbf{p},-\rangle\langle\mathbf{p},-| \\ &+\left(\frac{E-|\mathbf{p}|}{2E}\right)|\mathbf{p},+\rangle\langle\mathbf{p},+|. \end{aligned} \tag{52}$$

The anti-neutrino density matrix is accordingly given by

$$\rho_{\bar{\nu}}(E) = \left(\frac{E+|\mathbf{p}|}{2E}\right) |\bar{\mathbf{p}},+\rangle \langle \bar{\mathbf{p}},+| + \left(\frac{E-|\mathbf{p}|}{2E}\right) |\bar{\mathbf{p}},-\rangle \langle \bar{\mathbf{p}},-| , \tag{53}$$

and from Eq. (22) we have

$$U_3 \rho_\nu(E) U_3^\dagger = \rho_{\bar{\nu}}(E), \tag{54}$$

and in particular we can again conclude that a LH neutrino and a RH anti-neutrino, created by a left-chiral field, are connected by the $U_3$ transformation.

For the field operator in Eq. (50) we get, using Eqs. (37) and (43)

$$U_3 \Psi_L(x) U_3^\dagger = R_C \Psi_L^*(x), \tag{55}$$

so the left-chiral field transforms appropriately under $U_3$. On the other hand, $U_2$ mixes the chiral fields, as it must for charge conjugation

$$U_2 \Psi_L(x) U_2^\dagger = -i\sigma_2 \Psi_R^*(x). \tag{56}$$

As for $U_1$ it is not possible to obtain a consistent transformation of the fields and at the same time maintain the SU(2) algebra, so this transformation is lost for chiral fields.

## 5. Concluding remarks

We have obtained the little group generators, which act also as symmetry operators, for massive Dirac neutrino one-particle states, provided their properties in detail and discuss their physical interpretations. The most interesting result comes from $U_3$ because it connects a LH neutrino state with a RH anti-neutrino one, by a rotation in spin space that violates lepton number conservation. The other two transformations involve states that have not been observed, namely a RH-neutrino and a LH anti-neutrino, but which are in principle not precluded by any fundamental consideration. Regarding $U_2$, which as stated is just the standard charge conjugation operator, what we have obtained is consistent with the fact that this transformation is actually an internal transformation [41], not related to spacetime at all.

Let us consider $U_2$ and $U_3$ for free charged fermions. In this case charge conjugation is also a symmetry of the free theory since it commutes with the Hamiltonian, but of course this does not imply that a charged fermion can spontaneously change to its anti-fermion, since such a process is precluded by the charge conservation selection rule. The same applies for $U_3$. No such selection rule exists for strictly neutral fermions so, to the extent that the transformations here presented are superseded by charge conservation, these apply only to strictly neutral, elementary fermions, of which the neutrino is the only particle known to exist so far, in the free theory. The elementary part is guaranteed by the fundamental aspect of the field and the use of one-particle states created off the vacuum by the field operator.

As for lepton number, even though total lepton number has never been observed to be violated, it is a classical global symmetry, and there is a priori no reason, either from unitarity, renormalizability, or otherwise, that prevents it to be broken by quantum effects. Flavor lepton number, on the other hand, is already known to be violated by the flavor basis in neutrino oscillations.

## Appendix

### A. Realization of the $U_3$ transformation

In this appendix we explicitly derive Eq. (37). Let us consider the unitary transformations

$$\hat{\omega}_1 = \exp(i\alpha A), \qquad \hat{\omega}_2 = \exp(i\beta B) \tag{A.1}$$

with $\alpha$ and $\beta$ real numbers, and $A$ and $B$ the Hermitian operators

$$A = \int \frac{d^3p}{(2\pi)^3} \left\{ a_-^\dagger(\mathbf{p})\, a_-(\mathbf{p}) - a_+^\dagger(\mathbf{p})\, a_+(\mathbf{p}) + b_+^\dagger(\mathbf{p})\, b_+(\mathbf{p}) - b_-^\dagger(\mathbf{p})\, b_-(\mathbf{p}) \right\}, \tag{A.2}$$

$$B = \int \frac{d^3p}{(2\pi)^3} \left\{ a_-^\dagger(\mathbf{p})\, b_+(\mathbf{p}) + b_+^\dagger(\mathbf{p})\, a_-(\mathbf{p}) + b_-^\dagger(\mathbf{p})\, a_+(\mathbf{p}) + a_+^\dagger(\mathbf{p})\, b_-(\mathbf{p}) \right\}. \tag{A.3}$$

Then, using the operator identity

$$\left[\hat{A}\hat{B}, \hat{C}\right] = \hat{A}\left\{\hat{B}, \hat{C}\right\} - \left\{\hat{A}, \hat{C}\right\}\hat{B},$$

we have

$$[A, a_-(\mathbf{p})] = -a_-(\mathbf{p}),$$
$$[A, [A, a_-(\mathbf{p})]] = a_-(\mathbf{p}),$$

and so on recursively. Thus, by the Baker–Campbell–Hausdorff relation we get $\hat{\omega}_1 a_-(\mathbf{p})\, \hat{\omega}_1^\dagger = a_-(\mathbf{p})\, e^{-i\alpha}$. Performing analogous calculations for the rest of the operators, and choosing $\alpha = \pi/2$, yields

$$\begin{aligned} \hat{\omega}_1 a_-(\mathbf{p})\, \hat{\omega}_1^\dagger &= -\, i a_-(\mathbf{p}), \\ \hat{\omega}_1 a_+(\mathbf{p})\, \hat{\omega}_1^\dagger &= i a_+(\mathbf{p}), \\ \hat{\omega}_1 b_-(\mathbf{p})\, \hat{\omega}_1^\dagger &= i b_-(\mathbf{p}), \\ \hat{\omega}_1 b_+(\mathbf{p})\, \hat{\omega}_1^\dagger &= -\, i b_+(\mathbf{p}). \end{aligned} \tag{A.4}$$

As for the $B$ operator we get

$$[B, a_-(\mathbf{p})] = -b_+(\mathbf{p}),$$
$$[B, [B, a_-(\mathbf{p})]] = a_-(\mathbf{p}),$$

and so on recursively, yielding

$$\hat{\omega}_2 a_-(\mathbf{p})\,\hat{\omega}_2^\dagger = a_-(\mathbf{p})\cos\beta - ib_+(\mathbf{p})\sin\beta.$$

Performing analogous calculations for the rest of the operators, and choosing $\beta = \pi/2$, result in

$$\begin{aligned}\hat{\omega}_2 a_-(\mathbf{p})\,\hat{\omega}_2^\dagger &= -\,ib_+(\mathbf{p}),\\ \hat{\omega}_2 a_+(\mathbf{p})\,\hat{\omega}_2^\dagger &= -\,ib_-(\mathbf{p}),\\ \hat{\omega}_2 b_-(\mathbf{p})\,\hat{\omega}_2^\dagger &= -\,ia_+(\mathbf{p}),\\ \hat{\omega}_2 b_+(\mathbf{p})\,\hat{\omega}_2^\dagger &= -\,ia_-(\mathbf{p}).\end{aligned} \tag{A.5}$$

Now, with the help of the identity

$$\left[\hat{A}\hat{B},\hat{C}\hat{D}\right] = \hat{A}\left\{\hat{B},\hat{C}\right\}\hat{D} - \left\{\hat{A},\hat{C}\right\}\hat{B}\hat{D} + \hat{C}\hat{A}\left\{\hat{B},\hat{D}\right\} - \hat{C}\left\{\hat{A},\hat{D}\right\}\hat{B},$$

it is straightforward to check that the operators $A$ and $B$ commute and so, with $\alpha = \beta = \pi/2$, we have $\hat{\omega}_1\hat{\omega}_2 = \exp\{i(\pi/2)(A+B)\} = U_3$, with $U_3$ given in Eq. (34), and where the last equality is directly verified after factorizing. Thus, combining Eqs. (A.4) and (A.5) and taking the Hermitian conjugate, Eq. (37) follows directly.

To check the invariance of the equal-time anti-commutation relations let us define the unitary matrix

$$\Gamma = i\gamma^2 \exp\left\{i\frac{\pi}{2}\left[\hat{\boldsymbol{n}}\cdot\boldsymbol{\Sigma}^*\right]\right\}, \tag{A.6}$$

with $\hat{\boldsymbol{n}} = (-\sin\varphi, \cos\varphi, 0)$. Using Eqs. (41) and (42) it is straightforward to check that $\Gamma\gamma^0 = -\gamma^5\mathcal{R}(\boldsymbol{p})$. Then the rhs of Eq. (48) is rewritten as

$$U_3\Psi(x)U_3^\dagger = \Gamma\bar{\Psi}^T(x). \tag{A.7}$$

Thus,

$$\begin{aligned}U_3\{\Psi_\alpha(\mathbf{x}),\Psi_\beta^\dagger(\mathbf{y})\}U_3^\dagger &= \left\{\left(\Gamma\bar{\Psi}^T(x)\right)_\alpha, \left(\Psi^T(x)\gamma^0\Gamma^\dagger\right)_\beta\right\}\\ &= \Gamma_{\alpha\mu}\gamma^0_{\nu\mu}\left\{\Psi_\nu(\mathbf{x}),\Psi_\sigma^\dagger(\mathbf{y})\right\}\gamma^0_{\sigma\tau}\Gamma^\dagger_{\tau\beta}\\ &= \delta^3(\mathbf{x}-\mathbf{y})\left(\Gamma\gamma^0\gamma^0\Gamma^\dagger\right)_{\alpha\beta} = \delta^3(\mathbf{x}-\mathbf{y})\,\delta_{\alpha\beta}.\end{aligned} \tag{A.8}$$

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$i$. In this work, the terms left-handed (LH) and right-handed (RH) always refer to helicity $\mp$, respectively.


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