

A new fractional mechanics based on fractional addition

W. Sang Chung^a, H. Hassanabadi^b, and E. Maghsoodi^c

^a*Department of Physics and Research Institute of Natural Science,
College of Natural Science, Gyeongsang National University, Jinju 660-701, Korea.
e-mail: mimip44@naver.com*

^b*Faculty of physics, Shahrood University of Technology, Shahrood, Iran.
e-mail: h.hasanabadi@shahroodut.ac.ir*

^c*Department of Physics, Faculty of Science, Lorestan University, Khoramabad, Iran.
e-mail: maghsoodi.el@lu.ac.ir*

Received 1 March 2020; accepted 7 May 2020

In this paper, we introduce a new fractional derivative to define a new fractional velocity and a new fractional acceleration with the fractional space translation symmetry, which is given by fractional addition. We also construct the fractional version for Newton mechanics with fractional space translation symmetry in one dimension. We show the conservation of fractional energy and formulate the fractional Hamiltonian formalism for the fractional mechanics with fractional space translation symmetry. We exhibit some examples for the fractional mechanics with fractional space translation symmetry.

Keywords: Fractional calculus; the new fractional derivative; the fractional Hamiltonian formalism.

PACS: 45.50.Dd; 45.05.+x; 45.20.Jj

DOI: <https://doi.org/10.31349/RevMexFis.67.68>

1. Introduction

Fractional derivative is a derivative of fractional (non-integer) order and has a long history [1]. It has attracted much attention from mathematicians, physicists, and engineers in recent decades [2-18]. In Ref. [2], the concept of the solution has been presented for a differential equation of fractional order with uncertainty. Ref. [3] concentrates on the class of fractional derivatives most important in applications, the Caputo operators, and provides a self-contained, thorough, and mathematically rigorous study of their properties and the corresponding differential equations. Also, the authors have investigated the applications of fractional calculus to first-order integral equations with power and power logarithmic kernels, and with special functions in kernels and to Euler-Poisson-Darboux's type equations and differential equations of fractional order in Refs. [4]. In Ref. [5-11], the most comprehensive developments on fractional differential and fractional integral-differential equations involving many different potentially useful operators of fractional calculus are provided. In Ref. [12], the authors propose a novel fractional-order adaptive filter structures such that the output from the conventional filtered-x least mean square algorithm is passed through a new update equation derived from a cost function based on a posteriori error and optimized using fractional derivatives. In Ref. [13-15], the steady-state fractional advection-dispersion equation on bounded domains in R^d is discussed, and the fractional differential and integral operators are defined and analyzed. A novel method as the double integral method, for obtaining the solution of the fractional differential equation in the class of second-grade fluids models is proposed [16]. Nonlocal differential and integral oper-

ators with fractional order and fractal dimension have been recently introduced in Ref. [17], and in this paper has been defined the powerful mathematical tools to model complex real-world problems that could not be modeled with classical and nonlocal differential and integral operators with a single order. In Ref. [18], the local asymptotic stability and the global asymptotic stability for the trivial equilibrium point of the fractional electrical RLC circuit have been discussed.

Fractional derivative is regarded as a powerful tool for studying nonlinear systems [19-22]. In Ref. [19], the local fractional Burgers' equation (LFBE) is investigated from the point of view of local fractional conservation laws envisaging a nonlinear local fractional transport equation with a linear non-differentiable diffusion term. Reference [20] is a review of physical models that look very promising for the future development of fractional dynamics, and the Authors suggest a short introduction to fractional calculus as a theory of integration and differentiation of noninteger order. Also, Some applications of integro-differentiations of fractional orders in physics are discussed, and the models of discrete systems with memory, lattice with long-range inter-particle interaction, dynamics of fractal media are presented [20]. In Ref. [21], the problem of robust control of uncertain fractional-order nonlinear complex systems is investigated and, after establishing a simple linear sliding surface, the sliding mode theory is used to derive a novel robust fractional control law for ensuring the existence of the sliding motion in finite time. In Ref. [22], By use of the Gibbs-Appel approach and the complementary constitutive axioms corresponding to the fractional Kelvin-Zener model of the viscoelastic body, the equations of motion were derived. Riemann and Louville [9] first constructed the fractional derivative through the integral,

$$\partial_x^{\alpha,RL} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-\xi)^{n-\alpha-1} f(\xi) d\xi, \quad (1)$$

where $n = [\alpha] + 1$ and $[x]$ is the greatest integer equal to or less than x . Riemann-Louville fractional derivative of a constant is not zero. To cure this problem, Caputo [14] modified Riemann and Louville's definition for fractional derivative, which is called Caputo fractional derivative,

$$\partial_x^{\alpha,C} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\xi)^{n-\alpha-1} \frac{d^n}{d\xi^n} f(\xi) d\xi, \quad (2)$$

where $n = [\alpha] + 1$.

Riemann-Louville fractional derivative and Caputo fractional derivative do not obey the Leibniz rule. A new fractional derivative obeying Leibniz's rule was introduced by Khalil, Horani, Yousef, and Sababheh [23]. It is called a conformable fractional derivative and depends just on the basic limit definition of the derivative. Namely, for a function $f : (0, \infty) \rightarrow \mathbb{R}$ the conformable fractional derivative of order α ($0 < \alpha \leq 1$) of f at $x > 0$ is defined by

$$\begin{aligned} (\partial_x^{\alpha,KHYS} f(x)) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x x^{1-\alpha}) - f(x)}{\Delta x} \\ &= x^{1-\alpha} \partial_x f(x), \end{aligned} \quad (3)$$

and the conformable fractional derivative at 0 is defined as $(\partial_x^{\alpha,KHYS} f)(0) = \lim_{x \rightarrow 0} (\partial_x^{\alpha} f)(x)$. Some applications of the conformable fractional derivative are given in [23-34].

Riemann-Louville fractional derivative and Caputo fractional derivative have been successfully applied to characterize the constitutive equations of viscoelastic non-Newtonian fluid, by using the fractional derivatives to replace the integer-order derivatives [35-37]. For the conformable fractional derivative, a study on the classical mechanics was given in [29], and a study on the motion in the viscoelastic medium was given in [30], where the fractional velocity was defined as

$$\begin{aligned} v(t) &:= (\partial_t^{\alpha,KHYS} x(t)) = \lim_{\Delta t \rightarrow 0} \frac{x(t + t^{1-\alpha} \Delta t) - x(t)}{\Delta t} \\ &= t^{1-\alpha} \dot{x}. \end{aligned} \quad (4)$$

The fractional velocity defined in Eq. (4) can be written as

$$v(t) = \lim_{t' \rightarrow t} \frac{x(t') - x(t)}{\frac{1}{\alpha} ((t')^\alpha - t^\alpha)}, \quad (5)$$

Eq. (5) is not invariant under the translation in time $t \rightarrow t + \tau$ while it is invariant under the fractional translation in time, $t^\alpha \rightarrow t^\alpha + \tau^\alpha$.

Instead of the fractional velocity with fractional translation in time, we can consider the fractional velocity with fractional translation in space. In this paper, we introduce a new fractional velocity defined by

$$v(t) = \lim_{t' \rightarrow t} \frac{\frac{1}{\alpha} [(x(t'))^\alpha - (x(t))^\alpha]}{t' - t}, \quad (6)$$

for $x(t') > x(t) > 0$. We refer to the fractional velocity (6) as fractional velocity with fractional space translation symmetry. The fractional velocity with fractional space translation symmetry is not invariant under the translation in position $x \rightarrow x + a$ while it is invariant under the fractional translation in position, $x^\alpha \rightarrow x^\alpha + a^\alpha$. Using the fractional velocity with fractional space translation symmetry, we define the fractional acceleration with fractional space translation symmetry and construct the fractional Newton equation with fractional space translation symmetry. This paper is organized as follows: In Sec. 2, we discuss the fractional velocity and fractional acceleration. In Sec. 3, we discuss the fractional work, fractional kinetic energy, and fractional potential energy. In Sec. 4, we discuss the fractional Hamiltonian formalism for the fractional mechanics. In Sec. 5, we discuss some examples of the fractional mechanics.

2. Fractional velocity and fractional acceleration: New definitions

Now let us invoke the average velocity when a particle lies on $x(t)$ at time t and lies on $x(t')$ at time t' where $t' > t$, which is

$$v_{ave} = \frac{\text{displacement}}{\text{change in time}} = \frac{x(t') - x(t)}{t' - t}, \quad (7)$$

where the displacement is defined as

$$\text{displacement} = x(t') - x(t), \quad (8)$$

and it is invariant under the ordinary space translation.

First let us consider the case that $x(t') > x(t) > 0$. If we define the fractional displacement as

$$\text{fractional displacement} = (x(t'))^\alpha - (x(t))^\alpha, \quad (9)$$

and define the fractional average velocity as

$$v_{ave} = \frac{\text{fractional displacement}}{\text{change in time}} = \frac{(x(t'))^\alpha - (x(t))^\alpha}{t' - t}, \quad (10)$$

we know that the fractional average velocity is invariant under the fractional space translation.

The numerator in Eq. (10) can not be regarded as the deformed subtraction because its corresponding deformed addition is not associative. For this reason, we define the fractional addition and fractional subtraction for the case of $x > y > 0$ as

$$x \oplus_\alpha y = (x^\alpha + y^\alpha)^{1/\alpha} \quad (0 < \alpha \leq 1), \quad (11)$$

$$x \ominus_\alpha y = (x^\alpha - y^\alpha)^{1/\alpha} \quad (0 < \alpha \leq 1). \quad (12)$$

The fractional addition is commutative and associative and preserves the dimension.

The fractional displacement is then written as

$$\text{fractional displacement} = [x(t') \ominus_{\alpha} x(t)]^{\alpha}, \quad (13)$$

and the fractional average velocity

$$v_{\text{ave}} = \frac{[x(t') \ominus_{\alpha} x(t)]^{\alpha}}{t' - t}. \quad (14)$$

Here we have a problem. The position $x(t)$ can take a negative value. In that case, the fractional addition or fractional subtraction becomes a complex number. To cure this problem, we should modify the definitions of fractional addition and fractional subtraction so that they may hold for any $x, y \in \mathbb{R}$.

For $x > 0, y > 0$, the fractional addition and fractional subtraction are defined as

$$\begin{aligned} x \oplus_{\alpha} y &= (x^{\alpha} + y^{\alpha})^{1/\alpha}, \\ x \ominus_{\alpha} y &= \begin{cases} (x^{\alpha} - y^{\alpha})^{1/\alpha} & (x > y) \\ -(y^{\alpha} - x^{\alpha})^{1/\alpha} & (x < y) \end{cases}. \end{aligned} \quad (15)$$

For $x > 0, y < 0$, the fractional addition and fractional subtraction are defined as

$$\begin{aligned} x \oplus_{\alpha} y &= \begin{cases} (x^{\alpha} - (-y)^{\alpha})^{1/\alpha} & (x > -y) \\ -((-y)^{\alpha} - x^{\alpha})^{1/\alpha} & (x < -y) \end{cases}, \\ x \ominus_{\alpha} y &= (x^{\alpha} + (-y)^{\alpha})^{1/\alpha}. \end{aligned} \quad (16)$$

For $x < 0, y > 0$, the fractional addition and fractional subtraction are defined as

$$\begin{aligned} x \oplus_{\alpha} y &= \begin{cases} (y^{\alpha} - (-x)^{\alpha})^{1/\alpha} & (-x < y) \\ -((-x)^{\alpha} - y^{\alpha})^{1/\alpha} & (-x > y) \end{cases}, \\ x \ominus_{\alpha} y &= -(y^{\alpha} + (-x)^{\alpha})^{1/\alpha}. \end{aligned} \quad (17)$$

For $x < 0, y < 0$, the fractional addition and fractional subtraction are defined as

$$\begin{aligned} x \oplus_{\alpha} y &= -((-x)^{\alpha} + (-y)^{\alpha})^{1/\alpha}, \\ x \ominus_{\alpha} y &= \begin{cases} ((-y)^{\alpha} - (-x)^{\alpha})^{1/\alpha} & (x > y) \\ -((-x)^{\alpha} - (-y)^{\alpha})^{1/\alpha} & (x < y) \end{cases}. \end{aligned} \quad (18)$$

Thus we can write the fractional addition and fractional subtraction as

$$\begin{aligned} x \oplus_{\alpha} y &= ||x|^{\alpha-1}x + |y|^{\alpha-1}y|^{1/\alpha-1} \\ &\quad \times (|x|^{\alpha-1}x + |y|^{\alpha-1}y), \\ x \ominus_{\alpha} y &= ||x|^{\alpha-1}x - |y|^{\alpha-1}y|^{1/\alpha-1} \\ &\quad \times (|x|^{\alpha-1}x - |y|^{\alpha-1}y). \end{aligned} \quad (19)$$

Besides, we have the relation $x \ominus_{\alpha} y = x \oplus_{\alpha} (-y)$. For the fractional addition and fractional subtraction, we have the following properties:

1. Distributivity

$$\begin{aligned} (kx) \oplus_{\alpha} (ky) &= k(x \oplus_{\alpha} y), \\ (kx) \ominus_{\alpha} (ky) &= k(x \ominus_{\alpha} y), \quad k \in \mathbb{R}. \end{aligned} \quad (20)$$

2. Expansion

$$(A \oplus_{\alpha} B)(C \oplus_{\alpha} D) = AC \oplus_{\alpha} BC \oplus_{\alpha} AD \oplus_{\alpha} BD. \quad (21)$$

From the definition of fractional average velocity, the fractional instantaneous velocity (shortly fractional velocity) is defined by

$$v(t) = \lim_{t' \rightarrow t} \frac{[x(t') \ominus_{\alpha} x(t)]^{\alpha}}{\alpha(t' - t)} \quad \text{for } x(t') > x(t), \quad (22)$$

and

$$v(t) = - \lim_{t' \rightarrow t} \frac{[x(t) \ominus_{\alpha} x(t')]^{\alpha}}{\alpha(t' - t)} \quad \text{for } x(t') < x(t). \quad (23)$$

Equation (22) and Eq. (23) are unified as follows:

$$\begin{aligned} v(t) &= D_t^{\alpha} x(t) \\ &= \lim_{t' \rightarrow t} \frac{|x(t') \ominus_{\alpha} x(t)|^{\alpha-1} [x(t') \ominus_{\alpha} x(t)]}{\alpha(t' - t)}, \end{aligned} \quad (24)$$

or

$$v(t) = |x|^{\alpha-1} \dot{x}, \quad (25)$$

which is derived in Appendix A.

The fractional average acceleration is defined as

$$\begin{aligned} a_{\text{ave}} &= \frac{\text{change in fractional velocity}}{\text{change in time}} \\ &= \frac{v(t') - v(t)}{t' - t}, \end{aligned} \quad (26)$$

hence the fractional instantaneous acceleration (shortly fractional acceleration) is given by

$$a = \frac{dv}{dt}. \quad (27)$$

The fractional version of Newton's law is then

$$F = ma = m \frac{dv}{dt} = m \frac{d}{dt} (|x|^{\alpha-1} \dot{x}). \quad (28)$$

Equation (28) is invariant under the following transformation for the fractional velocity,

$$v' = v + u \quad (u > 0), \quad (29)$$

which is the same as the ordinary case. But, Eq. (28) is not invariant under the ordinary Galilei transformation

$$x' = x - ut, \quad t' = t. \quad (30)$$

Instead, Eq. (28) is invariant under the fractional Galilei transformation, which is given by

$$x' = (x^{\alpha} + \alpha ut)^{1/\alpha} = x \oplus_{\alpha} (\alpha ut)^{1/\alpha}, \quad (31)$$

for $x, x', u > 0$. Now we will refer to the system whose coordinate x' is given by Eq. (31) as a fractional inertial frame. For the fractional inertial frame, the fractional version of Newton's law remains invariant, which is called a fractional Galilei relativity.

3. Fractional work, fractional kinetic energy, and fractional potential energy

In ordinary mechanics, work w is the product of the force and displacement. When a force F is acted on a body with a mass m , and this body moves from x to x' ($x' > x$) in a same direction as the force, the work is given by

$$w = (\text{force}) \times (\text{displacement}) = F(x' - x). \quad (32)$$

When the force varies during motion from b_i to b_f , the work is given by

$$w = \int_{b_i}^{b_f} dx F(x). \quad (33)$$

For the conserved force, we have the potential energy $F = -\partial_x V$; thus, the work reduces to

$$w = -(V(b_f) - V(b_i)). \quad (34)$$

For the fractional case, we define the fractional work W as the product of the force and fractional displacement,

$$W = (\text{force}) \times (\text{fractional displacement}) = F(x' \ominus_\alpha x)^\alpha. \quad (35)$$

If we set $x' = x + \Delta x$, we have

$$(x' \ominus_\alpha x)^\alpha \approx \alpha |x|^{\alpha-1} \Delta x. \quad (36)$$

Thus, the fractional work reads

$$W = \int_{b_i}^{b_f} d_\alpha x F(x) = \int_{b_i}^{b_f} dx (\alpha |x|^{\alpha-1}) F(x). \quad (37)$$

If we define the fractional potential energy as

$$F(x) = -D_x^\alpha V(x), \quad (38)$$

where

$$\begin{aligned} D_x^\alpha V(x) &= \lim_{x' \rightarrow x} \frac{\alpha(V(x') - V(x))}{|x' \ominus_\alpha x|^{\alpha-1} (x' \ominus_\alpha x)} \\ &= |x|^{1-\alpha} \partial_x V, \end{aligned} \quad (39)$$

we have

$$W = -(V(b_f) - V(b_i)). \quad (40)$$

Now let us find the fractional kinetic energy from the definition of the fractional work, which is given by

$$\begin{aligned} W &= \int_{b_i}^{b_f} m a d_\alpha x = \int_{b_i}^{b_f} m a (\alpha |x|^{\alpha-1}) dx \\ &= \alpha \int_{b_i}^{b_f} m \frac{dv}{dt} |x|^{\alpha-1} \frac{dx}{dt} dt = \alpha \int_{b_i}^{b_f} m \frac{dv}{dt} v dt \\ &= K(b_f) - K(b_i), \end{aligned} \quad (41)$$

where the fractional kinetic energy is defined as

$$K = \frac{1}{2} m v^2 = \frac{1}{2} m (|x|^{\alpha-1} \dot{x})^2. \quad (42)$$

Thus, we have fractional energy conservation,

$$E = K + V = \frac{1}{2} m v^2 + V. \quad (43)$$

4. Fractional Hamiltonian formalism in the fractional mechanics

The fractional classical mechanics is also constructed by the fractional Poisson bracket defined as follows:

$$\begin{aligned} \{F(x, p), G(x, p)\}_\alpha &= |x|^{1-\alpha} \\ &\times \left(\frac{\partial F}{\partial x} \frac{\partial G}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial G}{\partial x} \right), \end{aligned} \quad (44)$$

which gives

$$\{x, p\}_\alpha = |x|^{1-\alpha}. \quad (45)$$

If we introduce the fractional Hamiltonian as

$$H = \frac{p^2}{2m} + V(x), \quad (46)$$

we have the fractional Hamilton's equation of motion as

$$\begin{aligned} \dot{x} &= \{x, H\}_\alpha = |x|^{1-\alpha} \frac{p}{m}, \\ \dot{p} &= \{p, H\}_\alpha = -|x|^{1-\alpha} \partial_x V(x) = -D_x^\alpha V(x). \end{aligned} \quad (47)$$

From the first relation of Eq. (47), the fractional momentum is given by

$$p = m |x|^{\alpha-1} \dot{x} = m v. \quad (48)$$

Inserting Eq. (48) into Eq. (47), we have the same equation like Eq. (28).

Now let us find the geometrical meaning of the fractional Poisson bracket. The fractional Poisson bracket is related to the mechanics in a curved space. In one dimensional curved space, the metric is given by

$$ds^2 = g_{xx} dx^2. \quad (49)$$

In a curved space, the Poisson bracket is deformed as

$$\{x, p\} = g_{xx}^{-1/2}. \quad (50)$$

This is the same as Eq. (45) if we take

$$g_{xx}^{-1/2} = |x|^{1-\alpha}, \quad (51)$$

or

$$ds^2 = \frac{1}{|x|^{2(1-\alpha)}} dx^2. \quad (52)$$

Thus, our model is related to the mechanics in a one-dimensional curved space with a metric of the form (51).

The fractional momentum defined by Eq. (48) is not related to the ordinary translation but the fractional translation $T_\alpha(\Delta x) : x \rightarrow x \oplus_\alpha \Delta x$. Indeed we can easily check that the fractional momentum is invariant under the fractional translation. This translation obeys the product rule, $T_\alpha(\Delta x)T_\alpha(\Delta x') = T_\alpha(\Delta x \oplus_\alpha \Delta x')$. Thus, the fractional momentum is the conserved quantity for the fractional translation.

5. Some examples of the fractional mechanics

In this section, we will discuss some physical examples of fractional mechanics.

5.1. Uniform fractional velocity

Let us consider the motion of a particle with a uniform fractional velocity u . From the definition of the fractional velocity, we have

$$|x|^{\alpha-1}\dot{x} = u, \quad x(0) = x_0, \quad (53)$$

or

$$\frac{d}{dt}(|x|^{\alpha-1}x) = \alpha u. \quad (54)$$

The solution of Eq. (54) depends on the signs of u and x_0 .

Case 1. $u > 0, x_0 > 0$: In this case, we have

$$x^\alpha = \alpha ut + x_0^\alpha. \quad (55)$$

Its solution is

$$x(t) = (\alpha ut + x_0^\alpha)^{1/\alpha}. \quad (56)$$

Figure 1 shows the plot of x versus t with $u = 1, x_0 = 1$ for $\alpha = 1$ (Gray), $\alpha = 0.8$ (Brown), and $\alpha = 0.5$ (Pink).

Case 2. $u > 0, x_0 < 0$: In this case, we have

$$|x|^{\alpha-1}x = \alpha ut - |x_0|^\alpha. \quad (57)$$

Its solution is

$$x(t) = \begin{cases} -(|x_0|^\alpha - \alpha ut)^{1/\alpha} & (t < \frac{|x_0|^\alpha}{\alpha u}) \\ (\alpha ut - |x_0|^\alpha)^{1/\alpha} & (t \geq \frac{|x_0|^\alpha}{\alpha u}) \end{cases}. \quad (58)$$

Figure 2 shows the plot of x versus t with $u = 1, x_0 = -1$ for $\alpha = 1$ (Gray), $\alpha = 0.8$ (Brown), and $\alpha = 0.5$ (Pink).

Case 3. $u < 0, x_0 > 0$: In this case, we have

$$|x|^{\alpha-1}x = -\alpha|u|t + x_0^\alpha. \quad (59)$$

Its solution is

$$x(t) = \begin{cases} (x_0^\alpha - \alpha|u|t)^{1/\alpha} & (t < \frac{x_0^\alpha}{\alpha|u|}) \\ -(\alpha|u|t - x_0^\alpha)^{1/\alpha} & (t \geq \frac{x_0^\alpha}{\alpha|u|}) \end{cases}. \quad (60)$$

Figure 3 shows the plot of x versus t with $u = -1, x_0 = 1$ for $\alpha = 1$ (Gray), $\alpha = 0.8$ (Brown), and $\alpha = 0.5$ (Pink).

Case 4. $u < 0, x_0 < 0$: In this case, we have

$$|x|^{\alpha-1}x = -(\alpha|u|t + |x_0|^\alpha). \quad (61)$$

Its solution is

$$x(t) = -(\alpha|u|t + |x_0|^\alpha)^{1/\alpha}. \quad (62)$$

Figure 4 shows the plot of x versus t with $u = -1, x_0 = -1$ for $\alpha = 1$ (Gray), $\alpha = 0.8$ (Brown), and $\alpha = 0.5$ (Pink).

5.2. Uniform fractional acceleration

Let us consider the motion of a particle with a uniform fractional acceleration a . From the definition of the fractional acceleration, we have

$$\frac{dv}{dt} = a, \quad (63)$$

which gives

$$v(t) = v_0 + at. \quad (64)$$

For simplicity, let us consider the case that $v_0 = 0, a > 0$. Then we have

$$|x|^{\alpha-1}x = \frac{1}{2}\alpha at^2, \quad (65)$$

which gives

$$x(t) = \left(\frac{1}{2}\alpha at^2\right)^{1/\alpha}. \quad (66)$$

Figure 5 shows the plot of x versus t with $a = 1$ for $\alpha = 1$ (Gray), $\alpha = 0.8$ (Brown), and $\alpha = 0.5$ (Pink).

5.3. Fractional harmonic oscillator

Now let us consider the fractional harmonic oscillator problem. The fractional version of Newton's law reads

$$\frac{m}{\alpha} \left(\frac{d}{dt}\right)^2 (|x|^{\alpha-1}x) = -k(|x|^{\alpha-1}x), \quad (67)$$

with initial conditions

$$x(0) = A > 0, \quad v(0) = 0. \quad (68)$$

Solving Eq. (67), we get

$$|x|^{\alpha-1}x = A^\alpha \cos \sqrt{\frac{k\alpha}{m}}t, \quad (69)$$

which gives

$$x(t) = \begin{cases} A \left(\cos \sqrt{\frac{k\alpha}{m}}t\right)^{1/\alpha} & (x > 0) \\ -A \left(-\cos \sqrt{\frac{k\alpha}{m}}t\right)^{1/\alpha} & (x < 0) \end{cases} \quad (70)$$

or

$$x(t) = A \cos \sqrt{\frac{k\alpha}{m}}t^{1/\alpha-1} \cos \sqrt{\frac{k\alpha}{m}}t. \quad (71)$$

This gives a periodic motion with a period

$$T = 2\pi \sqrt{\frac{m}{k\alpha}}. \quad (72)$$

Figure 6 shows the plot of x versus t with $A = 1, m = 1, k = 1$ for $\alpha = 1$ (Gray), $\alpha = 0.8$ (Brown), and $\alpha = 0.5$ (Pink).

6. Conclusion

In this paper, we introduced a new fractional derivative to define a new fractional velocity with fractional space translation symmetry based on the fractional addition rule. We used the fractional addition rule to introduce the concept of fractional displacement, fractional velocity, and fractional acceleration and constructed the fractional version of Newton's law. We showed that the fractional version of Newton's law is invariant under the fractional Galilei transformation. We defined the fractional work through the fractional displacement and constructed fractional kinetic energy and fractional potential energy. We also derived the conservation of fractional mechanical energy. We discussed fractional Hamiltonian formalism for the fractional mechanics with the help of the fractional Poisson bracket. We found that the fractional mechanics with fractional space translation symmetry is related to classical mechanics in a curved space. We discussed some examples of the fractional mechanics such as uniform fractional velocity motion, uniform fractional acceleration motion, and fractional harmonic oscillator problem.

Appendix A

We derive Eq. (25) for the following four cases:

1. $x(t') > x(t) > 0$: In this case, we have

$$v(t) = \lim_{t' \rightarrow t} \frac{(x(t'))^\alpha - (x(t))^\alpha}{\alpha(t' - t)} = (x(t))^{\alpha-1} \dot{x}. \quad (73)$$

2. $x(t) > x(t') > 0$: In this case, we have

$$v(t) = - \lim_{t' \rightarrow t} \frac{(x(t))^\alpha - (x(t'))^\alpha}{\alpha(t' - t)} = (x(t))^{\alpha-1} \dot{x}. \quad (74)$$

3. $x(t) < x(t') < 0$: In this case, we have

$$v(t) = \lim_{t' \rightarrow t} \frac{(-x(t))^\alpha - (-x(t'))^\alpha}{\alpha(t' - t)} = (-x(t))^{\alpha-1} \dot{x}. \quad (75)$$

4. $x(t') < x(t) < 0$: In this case, we have

$$\begin{aligned} v(t) &= - \lim_{t' \rightarrow t} \frac{(-x(t'))^\alpha - (-x(t))^\alpha}{\alpha(t' - t)} \\ &= (-x(t))^{\alpha-1} \dot{x}. \end{aligned} \quad (76)$$

Acknowledgement

The authors thank the referee for a thorough reading of our manuscript and for constructive suggestions.

-
1. D. Valério, J. Machado, V. Kiryakova, *Fract. Calc. Appl. Anal.* **17** (2014) 552. <https://doi.org/10.2478/s13540-014-0185-1>
 2. P. Agarwal, V. Lakshmikantham, J. Nieto, *Nonlinear Anal. Theory Methods Appl.* **72** (2010) 2859. <https://doi.org/10.1016/j.na.2009.11.029>
 3. K. Diethelm, *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Springer, Berlin (2010).
 4. S. Samko, A. Kilbas and O. Marichev, *Fractional Integrals and Derivatives* (Gordon and Breach, New York, 1993), p.1.
 5. K. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations* (Wiley, New York, 1993), p.1.
 6. A. Kilbas, H. Strivatava and J. Trujillo, *Theory and Application of Fractional Differential Equations* (Wiley, New York, 1993), p.1.
 7. I. Podlubny, *Fractional Differential Equations* (Academic Press, New York, 1999), p.1.
 8. K. Oldham and J. Spanier, *The fractional Calculus* (Academic Press, New York, 1974), p.1.
 9. R. Herrmann, *Fractional Calculus: An Introduction for Physicists*, World Scientific, Singapore (2014).
 10. Y. Luchko, *Fract. Calc. Appl. Anal.* **2** (1999) 463.
 11. I. Petras, *Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulation*, Springer Science and Business Media, Berlin (2011).
 12. S. Shah, R. Samar, N. Khan and M. Raja, *Nonlinear Dyn.* **85** (2016) 1363. <https://doi.org/10.1007/s11071-016-2765-6>
 13. V. Ervin and J. Roop, *Numer. Methods Partial Differ. Equ.* **23** (2007) 256. <https://doi.org/10.1002/num.20169>
 14. M. Caputo, *Geophysical Journal International.* **13** (1967) 529. <https://doi.org/10.1111/j.1365-246X.1967.tb02303.x>
 15. M. Mittag-Leffler, *Comptes Rendus de l'Académie des Sciences* **13** (1904) 3.
 16. N. Sene, *Chaos, Solitons & Fractals* **133** (2020) 109631. <https://doi.org/10.1016/j.chaos.2020.109631>
 17. A. Atangana, A. Akgül and K. Owolabi, *Analysis of fractal fractional differential equations*, Alexandria Eng. J. (2020), <https://doi.org/10.1016/j.aej.2020.01.005>
 18. N. Sene, *Stability analysis of electrical RLC circuit described by the Caputo-Liouville generalized fractional derivative*, Alexandria Eng. J. (2020), <https://doi.org/10.1016/j.aej.2020.01.008>
 19. X. Yang, J. Machado and J. Hristov, *Nonlinear Dyn.* **84** (2016) 3. <https://doi.org/10.1007/s11071-015-2085-2>
 20. V. Tarasov, *Int. J. Mod. Phys. B* **27** (2013) 1330005. <https://doi.org/10.1142/S0217979213300053>

21. M. Aghababa, *Nonlinear Dyn.* **73** (2013) 2329. <https://doi.org/10.1007/s11071-013-0944-2>
22. N. Kovincic, D. T. Spasic, *Nonlinear Dyn.* **85** (2016) 2369.
23. R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, *J. Comput. Appl. Math.* **264** (2014) 6570. <https://doi.org/10.1016/j.cam.2014.01.002>
24. M. Abu Hammad and R. Khalil, *Amer. J. Comput. Appl. Math.* **4** (2014) 187. [10.5923/j.ajcam.20140406.01](https://doi.org/10.5923/j.ajcam.20140406.01)
25. M. Abu Hammad and R. Khalil, *J. Diff. Equ. Appl.* **13** (2014) 177. <http://dx.doi.org/10.12732/ijdea.v13i3.1753>
26. M. Abu Hammad and R. Khalil, *Internat. J. Pure Appl. Math.* **94** (2014) 215. <http://dx.doi.org/10.12732/ijpam.v94i2.8>
27. M. Abu Hammad and R. Khalil, *Internat. J. Appl. Math. Res.* **3** (2014) 214. DOI: [10.14419/ijamr.v3i3.2747](https://doi.org/10.14419/ijamr.v3i3.2747)
28. D. R. Anderson and D. J. Ulness, *J. Math. Phys.* **56** (2015) 063502. <https://doi.org/10.1063/1.4922018>
29. W. Chung, *J. Comput. Appl. Math.* **290** (2015) 150. <https://doi.org/10.1016/j.cam.2015.04.049>
30. W. Chung and H. Hassanabadi, *Int. J.Theor.Phys* **56** (2017) 851. <https://doi.org/10.1016/j.physletb.2018.07.064>
31. M. D. Ortigueira and J. A. Tenreiro Machado, *J. Comput. Phys.* **293** (2015) 4. <https://doi.org/10.1016/j.jcp.2014.07.019>
32. K. R. Prasad, B. M. B. Krushna, *J. Nonlinear Funct. Anal.* **2015** (2015), Article ID 11.
33. D. Avci, B. Eroglu and N. Ozdemir, *Thermal Science* **21** (2017) 819. <https://doi.org/10.2298/TSCI160427302A>
34. F. Silva, D. Moreira and M. Moret, *Axioms* **7** (2018) 55. DOI: [10.3390/axioms7030055](https://doi.org/10.3390/axioms7030055)
35. C. Fetecau, C. Fetecau, M. Kamran and D. Vieru, *Journal of Non-Newtonian Fluid Mechanics* **156** (2009) 189. <https://doi.org/10.1016/j.jnnfm.2008.06.005>
36. M. Khan, M. Arshad and A. Aujum, *Nuclear Engineering and Design* **243** (2012) 20. <https://doi.org/10.1016/j.nucengdes.2011.11.024>
37. C. Li, L. Zheng, X. Zhang, and G. Chen, *Computers and Fluids* **125** (2106) 25.