Two-dimensional boson oscillator under a magnetic field in the presence of a minimal length in the non-commutative space

Z. Selama* and A. Boumali**
Laboratoire de Physique Appliquée et Théorique, LPAT, Université Larbi-Tébessi, Tébessa, Algeria.
**e-mail: zina.slama@univ-tebessa.dz; zinaslama@gmail.com*
***e-mail: abdelmalek.boumali@univ-tebessa.dz; boumali.abdelmalek@gmail.com*

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In this paper, we consider a two-dimensional Klein-Gordon oscillator in the presence of a magnetic field in non-commutative phase space in the framework of relativistic quantum mechanics with minimal length. The eigensolutions are found and the system is mapped to the well-known Schrödinger equation in a Pöschl-Teller potential.

Keywords: Klein-Gordon oscillator; minimal length; magnetic field; non commutative phase-space.

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1. Introduction

The studies of the relativistic generalization of the harmonic oscillator has drawn much attention in recent years. The well-known relativistic model of the harmonic oscillator was revived by Moshinsky and Szczepaniak [1], who gave it the name of Dirac oscillator (DO) because, in the non-relativistic limit, it becomes a harmonic oscillator with a very strong spin-orbit coupling term. The Dirac relativistic oscillator is an important potential both the theoretical and application implications. It was for the first time studied by Ito *et al.* [2]. They considered a Dirac equation in which the momentum $\vec{p}$ is replaced by $\vec{p} - im\beta\omega\vec{r}$, with $\vec{r}$ being the position vector, m the mass of particle, and ω the frequency of the oscillator. Physically, it can be shown that the DO interaction represents a physical system, which can be interpreted as the interaction of the anomalous magnetic moment with a linear electric field [3, 4]. The electromagnetic potential associated with the DO has been found by Benitez *et al.* [5]. The Dirac oscillator has attracted a lot of interest both because it provides one of the examples of the Dirac's equation exact solvability and because of its numerous physical applications (see [6] and references therein). Recently, Franco-Villafane *et al.* [7] exposed the proposal of the first experimental microwave realization of the one-dimensional DO. The experiment relies on a relation of the DO to a corresponding tight-binding system. The experimental results obtained, concerning the spectrum of the one-dimensional DO with and without the mass term, are in good agreement with those obtained in the theory. In addition, Yang and Piekarewicz [8] illustrate the power and flexibility of the Dirac oscillator and they suggest extensions to the study of systems without spherical symmetry as required in constrained calculations of nuclear excitations. Self-consistent calculations of binding energies and ground-state densities for a selected set of doubly-magic magic are performed using the Dirac-oscillator basis and are then compared against results obtained with the often-used Rung-Kutta method. The results obtained using the Dirac oscillator basis reproduced with high accuracy those derived using the Rung-Kutta method.

The unification between the general theory of relativity and the quantum mechanics is one of the most important problems in theoretical physics. This unification predicts the existence of a minimal measurable length on the order of the Planck length. All approaches of quantum gravity show the idea that near the Planck scale, the standard Heisenberg uncertainty principle should be reformulated. The minimal length uncertainty relation has appeared in the context of the string theory, where it is a consequence of the fact that the string cannot probe distances smaller than the string scale $\hbar\sqrt{\beta}$, where β is a small positive parameter called the deformation parameter. This minimal length can be introduced as an additional uncertainty in position measurement, so that the usual canonical commutation relation between position and momentum operators becomes

$$[\hat{x}, \hat{p}] = i\hbar\left(1 + \beta p^2\right). \tag{1}$$

This commutation relation leads to the standard Heisenberg uncertainty relation

$$\triangle\hat{x}\triangle\hat{p} \geq i\hbar\left(1 + \beta\left(\triangle p\right)^2\right), \tag{2}$$

which clearly implies the existence of a non-zero minimal length $\triangle x_{\text{min}} = \hbar\sqrt{\beta} \sim l_p$ where l_p is the Planck length. This modification of the uncertainty relation is usually termed the generalized uncertainty principle (GUP) or the minimal length uncertainty principle [9–12]. Note here that Saavedra and Utreras [13] were the first to propose a generalization of the canonical commutation relations of quantum mechanics which should be important at high energies. So a new (high energy) uncertainty principle was obtained.

In this direction, some remarks can be made about Eqs. (1) and (2): (i) according to the woks of [14–18], one important observation was that various observable effects of

the minimal length uncertainty relation are non-perturbative in the deformation parameter β, (*i.e.*, contain all orders in β) even though β appears only to linear order in Eqs. (1) and (2) [16–18] . (ii) In Ref. [15], the authors study the effect of the minimal length on the thermal properties of a Dirac oscillator when the position and momentum obey Eq. (1), and studied the possible constraint that can be placed on β. By using the properties of the Epstein zeta function, this study leads to a minimal length in the interval of $10^{-16} < \Delta x < 10^{-14}$ m with the following physically acceptable condition $\beta > \beta_0 = (1/m^2c^2)$. (iii) Kempf [10, 11, 19] has shown that Eq. (1) follows naturally when the canonical commutation relation between position and momentum operators is modified according to the Eq. (1). So, we may select the position $\hat{x}$ and momentum $\hat{p}$ operators in terms of the undeformed observables x and p, *i.e.* $[x, p] = i\hbar$, as

$$\hat{x} = \left(1 + \beta p^2\right) x, \text{and}\, \hat{p} = p. \tag{3}$$

One should note that the construction of models in these spaces would not be an easy task as the operators $\hat{x}$ and $\hat{p}$ in the deformed commutation relation (1) are in general not Hermitian $\hat{x}^\dagger = \hat{x} + 2i\beta\hbar\hat{p}$ and $\hat{p}^\dagger = \hat{p}$, albeit the simplified version (3) still allows one operator to remain Hermitian [20]. According to the works of Bagchia *et al.* [21] the positivity of β becomes important, as it ensures the absence of singularities in the metric.

According to the Kempf's prescription [10–12, 19], the position and momentum operators satisfying Eq. (1) can be also represented by

$$\hat{x} = i\hbar\left(1 + \beta p^2\right) x + \gamma p; \qquad \hat{p} = p, \tag{4}$$

where the operators x and p satisfy the canonical commutation relation$[x, p] = i\hbar$. In this case, the internal product in Fourier space becomes

$$< \phi\left(p\right)^* \psi\left(p\right) >= \int \frac{\phi\left(p\right)^* \psi\left(p\right)}{\left(1 + \beta p^2\right)^{1-\frac{\gamma}{\beta}}} dp. \tag{5}$$

The parameter γ appears in both Eqs. (4) and (5) is an arbitrary constant which does not affect the observable quantities; its choice determines only the weight function in the definition of the scalar product [22]. In this work, we have opted with the Kempf method, and so we chose $\gamma = 0$.

Nowadays, the reconsideration of the relativistic quantum mechanics in the presence of a minimal measurable length have been studied extensively. In this context, many papers were published where a different quantum system in space with Heisenberg algebra was studied. They are: the Abelian Higgs model [19], the thermostatics with minimal length [23], the one-dimensional Hydrogen atom [24], the Casimir effect in minimal length theories [25], the effect of minimal lengths on electron magnetism [26–28], the solutions of a two-dimensional Dirac equation in presence of an external magnetic field [29], the non-commutative phase space Schrödinger equation [30], and the Schrödinger equation with Harmonic potential in the presence of a magnetic field [31].

The study of NC spaces and their implications in physics is an extremely active area of research. It has been argued in various instances that non-commutativity should be considered as a fundamental feature of space-time at the Planck scale. On the other side, the study of quantum systems in an NC space has been the subject of much interest in past years, assuming that non-commutativity may be, in fact, a result of quantum gravity effects. In these studies, some attention has been paid to the models of NC quantum mechanics (NCQM). The interest in this approach lies on the fact that NCQM is a fruitful theoretical laboratory where we can get some insight on the consequences of non-commutativity in field theory by using standard calculation techniques of quantum mechanics. Various NC field theory models have been discussed as well as many extensions of quantum mechanics. Of particular interest is the so-called phase space noncommutativity, which has been investigated in the context of quantum cosmology, black holes physics, and the singularity problem. This specific formulation is necessary to implement the Bose–Einstein statistics in the context of NCQM (see [32–38]).

In addition, investigating the influence of the minimal length assumption on the energy spectrum of quantum systems has become an interesting issue primarily for two reasons. First, this may help to set some upper bounds on the value of the minimal length. In this context, we can cite some studies of the hydrogen atom and a two dimensional Dirac equation in an external magnetic field. Moreover, the classical limit has also provided some interesting insights into some cosmological problems. Second, it has been argued that quantum mechanics with a minimal length may also be useful to describe non-point-like particles, such as quasi- particles and various collective excitations in solids, or composite particles (see Ref [25] and references therein).

The purpose of this work is to investigate the formulation of a two-dimensional Klein Gordon oscillator (KGO) in the presence of a magnetic field by solving fundamental equations in the framework of relativistic quantum mechanics with minimal length in the NC space. To do this we first mapped the problem in question into a commutative space by using an appropriate transformations. Then, we solved it in the presence of a minimal length.

The paper is organized as follows. In Sec. 2, we solve the KGO in the presence of magnetic field in noncommutative phase space. Then, in Sec. 3, we study this problem in the framework of relativistic quantum mechanics with minimal length. Finally, Sec. 4 will be a conclusion.

2. The solutions within habitual quantum mechanics in NC space

To begin with, we note that the non-commutative phase space is characterized by the fact that their coordinate operators satisfy the equation [33–38]

$$\left[x_\nu^{(NC)}, x_\mu^{(NC)}\right] = i\Theta_{\mu\nu},\ \left[p_\mu^{(NC)}, p_\nu^{(NC)}\right] = 0,$$
$$\left[x_\mu^{(NC)}, p_\nu^{(NC)}\right] = i\hbar\delta_{\mu\nu}, \quad (6)$$

where $\Theta_{\mu\nu}$ is an antisymmetric tensor of space dimension. In order to obtain a theory which includes the aspects of being unitary and causal, we choose $\Theta_{0\nu} = 0$, which implies that the time remains as a parameter and the non-commutativity affects only the physical space. The noncommutative models specified by Eq. (1) can be realized in terms of a $\star$-product: the commutative algebra of functions with the usual product $f(x)g(x)$ is replaced by the $\star$-product Moyal algebra [37]:

$$(f \star g)(x) = \exp\left[\frac{i}{2}\tilde{\theta}_{ab}\partial_{x_a}\partial_{y_b}\right] f(x)\, g(y)\,|_{x=y}. \quad (7)$$

Since the system in which we study is two-dimensional, we limit our analysis to the xy plane, where the noncommutative algebra is written by

$$\left[x_i^{(NC)}, x_j^{(NC)}\right] = i\Theta\epsilon_{ij},\ \left[p_i^{(NC)}, p_j^{(NC)}\right] = 0,$$
$$\left[x_i^{(NC)}, p_j^{(NC)}\right] = i\hbar\delta_{ij},\ (i,j=1,2), \quad (8)$$

where ϵ_{ij} is the two-dimensional Levi-civita tensor. Instead of solving the NC Klein-Gordon equation by using the star product procedure, we use Bopp's shift method, that is, we replace the star product by the usual product by making a Bopp's shift

$$x_i^{(NC)} = x_i - \frac{1}{2\hbar}\Theta_{\mu\nu}\epsilon_{ij}p_j, \qquad p_i^{(NC)} = p_i. \quad (9)$$

Hence, in the two dimensional non-commutative phase-space, Eq. (9) becomes

$$x^{(NC)} = x - \frac{\Theta}{2\hbar}p_y,\ y^{(NC)} = y + \frac{\Theta}{2\hbar}p_x,\ p_x^{(NC)} = p_x,\ p_y^{(NC)} = p_y. \quad (10)$$

In this case, the two-dimensional Klein-Gordon oscillator , in commutative space, which is written by

$$\left\{(p_x+im_0\omega x)(p_x-im_0\omega x) + (p_y+im_0\omega y)(p_y-im_0\omega y) - \frac{E^2-m_0^2c^4}{c^2}\right\}\psi_{KG} = 0, \quad (11)$$

is modified and transformed into

$$\left\{\underbrace{(\hat{p}_x + im_0\omega\hat{x})(\hat{p}_x - im_0\omega\hat{x})}_{\cup} + \underbrace{(\hat{p}_y + im_0\omega\hat{y})(\hat{p}_y - im_0\omega\hat{y})}_{\cap} - \frac{E^2-m_0^2c^4}{c^2}\right\}\psi_{KG} = 0. \quad (12)$$

with

$$\cup = (p_x + im_0\omega x)(p_x - im_0\omega x) = p_x^2 + m_0^2\omega^2x^2 + im_0\omega(xp_x - p_x x) - m_0^2\omega^2\left(\frac{\Theta}{\hbar}\right)xp_y, \quad (13)$$

$$\cap = (p_y + im_0\omega y)(p_y - im_0\omega y) = p_y^2 + m_0^2\omega^2y^2 + im_0\omega(yp_y - p_y y) + m_0^2\omega^2\left(\frac{\Theta}{\hbar}\right)yp_x. \quad (14)$$

Here we have used that

$$\hat{x} = x - \frac{\Theta}{2\hbar}p_y, \quad \hat{y} = y + \frac{\Theta}{2\hbar}p_x,$$
$$\hat{p}_x = p_x, \quad \hat{p}_y = p_y. \quad (15)$$

In the literature (see Ref [6] and references therein), the upper bound on the value of the coordinate commutator found is $\Theta \leq 4\times10^{-40}\,\mathrm{m}^2$. So, as an approximation, all terms with the square of Θ have been neglected.

Now, by putting Eqs. (13) and (14) in (12), we obtain

$$\left[p_x^2 + p_y^2 + m_0^2\omega^2\left(x^2+y^2\right) + im_0\omega\,[x,p_x] + im_0\omega\,[y,p_y] - m_0^2\omega^2\frac{\Theta}{\hbar}(xp_y - yp_x) - \frac{E^2-\left(m_0c^2\right)^2}{c^2}\right]\Psi_{KG} = 0 \quad (16)$$

In order to solve the last equation, and for the sake of simplicity, we bring the problem into the momentum space.

Recalling that

$$\hat{x} = i\hbar\frac{\partial}{\partial\hat{p}_x}, \qquad \hat{y} = i\hbar\frac{\partial}{\partial\hat{p}_x}, \quad (17)$$
$$\hat{p}_x = p_x, \qquad \hat{p}_y = p_y, \quad (18)$$

and when passing onto polar coordinates [9]

$$p_x = p\cos\theta,\ \ p_y = p\sin\theta,\ \ \text{with}\ \ p^2 = p_x^2 + p_y^2, \quad (19)$$

Eqs. (17) and (18) become

$$x = i\hbar\frac{\partial}{\partial p_x} = i\hbar\left(\cos\theta\frac{d}{dp} - \frac{\sin\theta}{p}\frac{d}{d\theta}\right), \quad (20)$$

$$y = i\hbar\frac{\partial}{\partial p_y} = i\hbar\left(\sin\theta\frac{d}{dp} + \frac{\cos\theta}{p}\frac{d}{d\theta}\right). \quad (21)$$

Putting Eqs. (20) and (21) in Eq. (16), the last equation can be written explicitly in polar coordinates by

$$\left[p^2 - \lambda^2\left(\frac{\partial^2}{\partial p^2} + \frac{1}{p}\frac{\partial}{\partial p} + \frac{1}{p^2}\frac{\partial^2}{\partial\theta^2}\right) + m_0^2\omega^2\Theta\left(i\frac{\partial}{\partial\theta}\right) - 2\lambda - \varsigma\right]\Psi_{KG} = 0, \quad (22)$$

where

$$\lambda = m_0\omega\hbar, \tag{23}$$

$$\varsigma = \frac{E^2 - m_0^2c^4}{c^2}. \tag{24}$$

With the help of the following relation [39]

$$\psi_{KG}(p,\theta) = f(p)\, e^{i|l|\theta}, \tag{25}$$

Eq. (22) is transformed into

$$\left(\frac{d^2f(p)}{dp^2} + \frac{1}{p}\frac{df(p)}{dp} - \frac{l^2}{p^2}f(p)\right) + \left(\kappa^2 - k^2p^2\right) f(p) = 0, \tag{26}$$

with

$$\kappa^2 = \frac{m_0^2\omega^2\Theta l + 2\lambda + \varsigma}{\lambda^2},\ k^2 = \frac{1}{\lambda^2}. \tag{27}$$

Now, noting that [39]

$$f(p) = p^{|l|}e^{-\frac{k}{2}p^2}F(p), \tag{28}$$

then the differential equation

$$F'' + \left(\frac{2|l|+1}{p} - 2kp\right)F' - \left[2k(|l|+1) - \kappa^2\right]F = 0, \tag{29}$$

is transformed into the Kummer equation

$$t\frac{d^2F}{dt^2} + \{|l|+1-t\}\frac{dF}{dt} - \frac{1}{2}\left\{|l|+1-\frac{\kappa^2}{4k}\right\}F = 0, \tag{30}$$

which by using, instead of p, the variable $xt = kp^2$. The solution of this equation is the confluent series ${}_1F_1(a;|l|+1;t)$, with

$$a = \frac{1}{2}(|l|+1) - \frac{\kappa^2}{4k}. \tag{31}$$

The confluent series becomes a polynomial if and only if $a = -n,\ (n = 0, 1, 2,)$.

Thus, following this, we have the solutions

$$\psi_{KG}(p,\theta) = C_{n,|l|}p^{|l|}e^{-\frac{k}{2}p^2} \times {}_1F_1\left(-n;|l|+1;kp^2\right)e^{i|l|\theta}, \tag{32}$$

$$E_n = \pm m_0c^2\sqrt{1 + 2rN - \frac{\omega^2\Theta}{c^2}l}, \tag{33}$$

with $N = 2n + |l|$ is the principal quantum number, and $r = (\hbar\omega/m_0c^2)$ the parameter that controls the non-relativistic limit.

We can see that the presence of the parameter Θ, in the spectrum of energy, breaks the degeneracy of the energy levels. Furthermore, by taking that $\Theta = 0$, we reach the exact result of the two-dimensional Klein-Gordon oscillator in Minkowski space-time

$$E_n = \pm m_0c^2\sqrt{1+2rN}. \tag{34}$$

This form of energy is in a good agreement with that obtained in the literature (see Ref. [37]).

3. The solutions in the presence of a minimal length

In the minimal length formalism, the Heisenberg algebra is given by [11–22]

$$[\hat{x}_i, \hat{p}_i] = i\hbar\delta_{ij}\left(1+\beta p^2\right), \tag{35}$$

where $0 < \beta \leq 1$ is minimal length and p is magnitude of the momentum. When the energy is much smaller than the Planck mass, β goes to zero and we recover Heisenberg uncertainty principle.

A representation of $\hat{x}_i$ and $\hat{p}_i$ which satisfies Eq. (35), may be taken as

$$\hat{x} = i\hbar\left(1+\beta p^2\right)\frac{d}{dp_x},\ \hat{p}_x = p_x, \tag{36}$$

$$\hat{y} = i\hbar\left(1+\beta p^2\right)\frac{d}{dp_y},\ \hat{p}_y = p_y. \tag{37}$$

By using the Eqs. (36) and (37), Eq. (16) becomes

$$\Bigg[p^2 - \lambda^2\left(1+\beta p^2\right)^2\left(\frac{\partial^2}{\partial p^2} + \frac{1}{p}\frac{\partial}{\partial p} + \frac{1}{p^2}\frac{\partial^2}{\partial\theta^2}\right) - 2\lambda\left(1+\beta p^2\right) - 2\lambda^2\left(1+\beta p^2\right)\beta p\frac{\partial}{\partial p} + \left(1+\beta p^2\right)m_0^2\omega^2\Theta\left(i\frac{\partial}{\partial\theta}\right) - \varsigma\Bigg]\Psi_{KG} = 0. \tag{38}$$

Now, when we put that

$$\psi_{KG} = h(p)e^{i|j|\theta}, \tag{39}$$

with $j = 0, \pm1, \pm2, \ldots$, the Eq. (38) is transformed into

$$\left\{-a(p)\frac{\partial^2}{\partial p^2} + b(p)\frac{\partial}{\partial p} + c(p) - \varsigma\right\}h(p) = 0, \tag{40}$$

with

$$a(p) = a(p) = \lambda^2\left(1+\beta p^2\right)^2,$$

$$b(p) = -\frac{\lambda^2\left(1+\beta p^2\right)^2}{p} - 2\beta\lambda^2\left(1+\beta p^2\right)p$$

$$c(p) = p^2 + \lambda^2\frac{\left(1+\beta p^2\right)^2j^2}{p^2} - 2\lambda\left(1+\beta p^2\right) - \left(1+\beta p^2\right)m_0^2\omega^2\Theta j. \tag{41}$$

In order to solve Eq. (40), we adopt the following substitution as used by Jana *et al.*, [38]

$$h(p)=\rho(p)\varphi(p),\ q=\int\frac{1}{\sqrt{a(p)}}dp, \tag{42}$$

$$\rho(p)=\exp\left(\int\chi(p)dp\right),\ \chi\left(p\right)=\frac{2b+a'}{4a}=-\frac{1}{2p}. \tag{43}$$

After these susbsitutions, Eq. (40) becomes

$$\left[-\frac{d^2\varphi(p)}{dq^2}+V(p)\right]\varphi(p)=\varsigma\varphi(p), \tag{44}$$

with

$$\begin{aligned}V(p)&=p^2-2\lambda\left(1+\beta p^2\right)+\beta\lambda^2\left(1+\beta p^2\right)\\&+\frac{\lambda^2\left(1+\beta p^2\right)^2}{p^2}\left(j^2-\frac{1}{4}\right)\\&-\left(1+\beta p^2\right)m_0^2\omega^2\Theta j,\end{aligned} \tag{45}$$

with $\rho(p)=p^{-1/2}$.

In order to simplify the function $V(p)$, we use the change of variable

$$p=\frac{1}{\sqrt{\beta}}\tan\left(q\lambda\sqrt{\beta}\right). \tag{46}$$

In this case, the form $V\left(p\right)$ with the new variable q becomes

$$\begin{aligned}V\left(q\right)&=-\frac{1}{\beta}+\underbrace{\beta\lambda^2}_{U_0}\\&\times\left(\frac{\frac{1}{\beta^2\lambda^2}-\frac{2}{\beta\lambda}+j^2+\frac{3}{4}-m_0^2\omega^2\Theta\frac{j}{\beta\lambda^2}}{\cos^2\alpha q}+\frac{j^2-\frac{1}{4}}{\sin^2\alpha q}\right).\end{aligned} \tag{47}$$

Consequently, the final form of our differential equation is

$$\begin{aligned}&\left\{-\frac{d^2\varphi(p)}{dq^2}+\frac{U_0}{2}\right.\\&\times\left(\frac{\frac{1}{\beta^2\lambda^2}-\frac{2}{\beta\lambda}+j^2+\frac{3}{4}-m_0^2\omega^2\Theta\frac{j}{\beta\lambda^2}}{\cos^2\alpha q}\right.\\&\left.\left.+\frac{j^2-\frac{1}{4}}{\sin^2\alpha q}\right)\right\}\varphi(p)=\bar{\varsigma}\varphi(p),\end{aligned} \tag{48}$$

where

$$\bar{\varsigma}=\varsigma+\frac{1}{\beta}. \tag{49}$$

Thus, Eq. (48) is brought to

$$\begin{aligned}&\left\{-\frac{d^2\varphi(p)}{dq^2}+\frac{U_0}{2}\right.\\&\left.\times\left(\frac{\zeta_1\left(\zeta_1-1\right)}{\cos^2\alpha q}+\frac{\zeta_2\left(\zeta_2-1\right)}{\sin^2\alpha q}\right)\right\}\varphi(q)=\bar{\varsigma}\varphi(q),\end{aligned} \tag{50}$$

with

$$V\left(q\right)=-\frac{1}{\beta}+\beta\lambda^2\left\{\frac{\zeta_1\left(\zeta_1-1\right)}{\sin^2\left(\alpha q\right)}+\frac{\zeta_2\left(\zeta_2-1\right)}{\cos^2\left(\alpha q\right)}\right\}, \tag{51}$$

and where

$$\zeta_1\left(\zeta_1-1\right)=j^2-\frac{1}{4}, \tag{52}$$

$$\zeta_2\left(\zeta_2-1\right)=\frac{1}{\beta^2\lambda^2}-\frac{2}{\beta\lambda}+j^2+\frac{3}{4}-m_0^2\omega^2\Theta\frac{j}{\beta\lambda^2}. \tag{53}$$

Finally, we have

$$\begin{aligned}&\left(-\frac{d^2}{dq^2}+\frac{1}{2}U_0\left\{\frac{\zeta_1\left(\zeta_1-1\right)}{\sin^2\left(\alpha q\right)}+\frac{\zeta_2\left(\zeta_2-1\right)}{\cos^2\left(\alpha q\right)}\right\}\right)\\&\times\varphi\left(q\right)=\bar{\xi}^2\varphi\left(q\right),\end{aligned} \tag{54}$$

where $U_0=\alpha^2$ with $\alpha=\lambda\sqrt{\beta}$.

In this stage, we can see that Eq. (54) is the well-known Schrödinger equation in a Pöschl-Teller potential with the following potential [39]

$$U=\frac{1}{2}U_0\left\{\frac{\zeta_1\left(\zeta_1-1\right)}{\sin^2\left(\alpha q\right)}+\frac{\zeta_2\left(\zeta_2-1\right)}{\cos^2\left(\alpha q\right)}\right\}, \tag{55}$$

and where we have the conditions that $(\zeta_1,\zeta_2)>1$.

By comparing Eq. (48) with Eq. (54), we obtain

$$\zeta_1=|j|\pm\frac{1}{2}, \tag{56}$$

$$\zeta_2=\frac{1}{2}\pm\sqrt{\left(1+\frac{1}{\beta^2\lambda^2}-\frac{2}{\beta\lambda}+j^2-m_0^2\omega^2\Theta\frac{j}{\beta\lambda^2}\right)} \tag{57}$$

Now, in order to solve Eq. (48), we introduce the new variable

$$z=\sin^2\left(\alpha q\right). \tag{58}$$

In this case, Eq. (48) can be rewritten as

$$\begin{aligned}&z\left(1-z\right)\varphi^{''}+\left(\frac{1}{2}-z\right)\varphi^{'}+\frac{1}{4}\left\{\frac{\bar{\xi}^2}{\alpha^2}\right.\\&\left.-\frac{\zeta_1\left(\zeta_1-1\right)}{z}-\frac{\zeta_2\left(\zeta_2-1\right)}{1-z}\right\}\varphi=0.\end{aligned} \tag{59}$$

With the new wave function φ, defined by

$$\varphi=z^{\frac{\zeta_1}{2}}\left(1-z\right)^{\frac{\zeta_2}{2}}\Psi\left(z\right), \tag{60}$$

we arrive at

$$\begin{aligned}&z\left(1-z\right)\Psi^{''}+\left[\left(\zeta_1+\frac{1}{2}\right)-z\left(\zeta_1+\zeta_2+1\right)\right]\Psi^{'}\\&+\frac{1}{4}\left\{\frac{\bar{\xi}}{\alpha}-\left(\zeta_1+\zeta_2\right)^2\right\}\Psi=0.\end{aligned} \tag{61}$$

The general solution of this equation is

$$\begin{aligned}\Psi&=C_1\,{}_2F_1\left(a';b';c';z\right)\\&+C_2\,z^{1-c}\,{}_2F_1\left(a'+1-c';b'+1-c';2-c;z\right),\end{aligned} \tag{62}$$

with

$$a' = \frac{1}{2}\left(\zeta_1 + \zeta_2 + \frac{\bar{\xi}}{\alpha}\right),$$
$$b' = \frac{1}{2}\left(\zeta_1 + \zeta_2 - \frac{\bar{\xi}}{\alpha^2}\right), \; c' = \zeta_1 + \frac{1}{2}. \tag{63}$$

With the condition $a' = -n$, we obtain

$$\bar{\xi}^2 = \alpha^2 \left(\zeta_1 + \zeta_2 + 2n\right)^2. \tag{64}$$

In order to obtain the energy spectrum, it should be noted that in the limit $\beta \to 0$, the energy spectrum should regenerate to the no-GUP result.

Thus, the exact form of ζ_1 and ζ_2 are

$$\zeta_1 = |j| + \frac{1}{2}, \tag{65}$$

$$\zeta_2 = \frac{1}{2} + \sqrt{\left(1 + \frac{1}{\beta^2\lambda^2} - \frac{2}{\beta\lambda} + j^2 - m_0^2\omega^2\Theta\frac{j}{\beta\lambda^2}\right)}. \tag{66}$$

where $j \neq 0$.

With the aid of Eqs. (56), (57) and (64), we obtain the final form of the spectrum of energy: it is expressed explicitly by

$$E^2 = m_0^2c^4\left\{1 - \frac{1}{\left(m_0c\right)^2\beta} + \frac{\beta\lambda^2}{\left(m_0c\right)^2}\left(|j| + 1 + 2n + \sqrt{\left(1 + \frac{1}{\beta^2\lambda^2} - \frac{2}{\beta\lambda} + j^2 - m_0^2\omega^2\Theta\frac{j}{\beta\lambda^2}\right)}\right)^2\right\} \tag{67}$$

From Eq. (67), the presence of both β and Θ breaks the degeneracy of our spectrum of energy. In addition, the presence of the spectrum of energy with the parameter β, contrarily to the case of Θ, leads to the dependence of the energy levels on n^2: as mentioned by Nouicer [27], this dependence on n^2 is a feature of hard confinement. Finally, in the limit where $\beta \to 0$, we reach the following equation

$$E_n = \pm m_0c^2\sqrt{1 + 2rN - \frac{\omega^2\Theta}{c^2}l}, \tag{68}$$

which has the same form as the Eq. (33). The corresponding wave function is

$$\psi_{KG} = \mathrm{N}_{\mathrm{norm}}e^{i|j|\theta}p^{-\frac{1}{2}}z^{\frac{\zeta_1}{2}}\left(1-z\right)^{\frac{\zeta_2}{2}}\,{}_2F_1\left(-n; b'; c'; z\right). \tag{69}$$

with $\mathrm{N}_{\mathrm{norm}}$ is the constant of normalization.

4. The solutions in the presence of a magnetic field

In a magnetic field, when we choose the potential vectors as

$$\boldsymbol{A} = \left(-\frac{B}{2}y, \frac{B}{2}x, 0\right), \tag{70}$$

the two-dimensional Klein-Gordon oscillator in non-commutative space is

$$\left[\left(\hat{p}_x - \frac{eA_x}{c} + im_0\omega\hat{x}\right)\left(\hat{p}_x - \frac{eA_x}{c} - im_0\omega\hat{x}\right) + \left(\hat{p}_y - \frac{eA_y}{c} + im_0\omega\hat{y}\right)\left(\hat{p}_y - \frac{eA_y}{c} - im_0\omega\hat{y}\right) - \frac{E^2 - \left(m_0c^2\right)^2}{c^2}\right]\Psi_{KG} = 0 \tag{71}$$

or in the explicit form as

$$\left[\underbrace{\left(\hat{p}_x + \frac{eB}{2c}\hat{y} + im_0\omega\hat{x}\right)\left(\hat{p}_x + \frac{eB}{2c}\hat{y} - im_0\omega\hat{x}\right)}_{\sqcup} + \underbrace{\left(\hat{p}_y - \frac{eB}{2c}\hat{x} + im_0\omega\hat{y}\right)\left(\hat{p}_y - \frac{eB}{2c}\hat{x} - im_0\omega\hat{y}\right)}_{\sqcap} - \frac{E^2 - \left(m_0c^2\right)^2}{c^2}\right]\Psi_{KG} = 0. \tag{72}$$

By using Bopp's shift method, Eq. (72) is transformed into

$$\left\{\left(1 + \frac{eB\Theta}{4\hbar c}\right)^2\left(p_x^2 + p_y^2\right) + \left(m_0^2\omega^2 + \left(\frac{eB}{2c}\right)^2\right)\left(x^2 + y^2\right) + im_0\omega\left(1 + \frac{eB\Theta}{2\hbar c}\right)[x, p_x] +\right.$$

$$+im_0\omega\left(1+\frac{eB\Theta}{2\hbar c}\right)[y,p_y]-\left\{\left(1+\frac{eB\Theta}{4\hbar c}\right)\frac{eB}{c}+m_0^2\omega^2\frac{\Theta}{\hbar}\right\}\underbrace{(xp_y-yp_x)}_{L_z}-\frac{E^2-\left(m_0c^2\right)^2}{c^2}\Bigg\}\Psi_{KG}=0. \tag{73}$$

Putting Eqs. (36), (37) and (39) in (73), we obtain

$$\left\{\left(1+\frac{eB\Theta}{4\hbar c}\right)^2p^2-\left(\lambda^2+\hbar^2\left(\frac{eB}{2c}\right)^2\right)\left(1+\beta p^2\right)^2\left(\frac{\partial^2}{\partial p^2}+\frac{1}{p}\frac{\partial}{\partial p}-\frac{j^2}{p^2}\right)-2\left(\lambda^2+\hbar^2\left(\frac{eB}{2c}\right)^2\right)\left(1+\beta p^2\right)\beta p\frac{\partial}{\partial p}\right.$$
$$\left.-2\lambda\left(1+\beta p^2\right)\left(1+\frac{eB\Theta}{2\hbar c}\right)-\left(1+\beta p^2\right)\left\{\hbar\left(1+\frac{eB\Theta}{4\hbar c}\right)\frac{eB}{c}+m_0^2\omega^2\Theta\right\}|j|-\varsigma^2\right\}\Psi_{KG}=0. \tag{74}$$

with $\varsigma^2=(E^2-\left(m_0c^2\right)^2/c^2)$. Following the same procedure as in the above section, we have

$$\left[-\frac{d^2\varphi(p)}{dq^2}+V(p)\right]\varphi(p)=\varsigma'\varphi(p), \tag{75}$$

or with the new variable q,

$$\left\{-\frac{d^2\varphi(p)}{dq^2}+\frac{U_0}{2}\left(\frac{\zeta_1'\left(\zeta_1'-1\right)}{\cos^2\alpha q}+\frac{\zeta_2'\left(\zeta_2'-1\right)}{\sin^2\alpha q}\right)\right\}\varphi(q)=\bar{\varsigma}'\varphi(q).$$

In this case, the form of V is

$$V(p)=\left(1+\frac{eB\Theta}{4\hbar c}\right)^2p^2+\left\{\lambda^2+\hbar^2\left(\frac{eB}{2c}\right)^2\right\}\left(1+\beta p^2\right)^2\frac{1}{p^2}\left(j^2-\frac{1}{4}\right)$$
$$-2\lambda\left(1+\beta p^2\right)\left(1+\frac{eB\Theta}{2\hbar c}\right)-\hbar\left(1+\beta p^2\right)\left\{\left(1+\frac{eB\Theta}{4\hbar c}\right)\frac{eB}{c}+m_0^2\omega^2\Theta\right\}|j|+\left\{\lambda^2+\hbar^2\left(\frac{eB}{2c}\right)^2\right\}\left(1+\beta p^2\right)\beta,$$

or with the variable q

$$V(q)=-\frac{1}{\beta}+\beta\lambda^2\left\{\frac{\zeta_1'\left(\zeta_1'-1\right)}{\sin^2(\alpha q)}+\frac{\zeta_2'\left(\zeta_2'-1\right)}{\cos^2(\alpha q)}\right\},$$

where ζ_1' and ζ_2' are defined by

$$\zeta_1'\left(\zeta_1'-1\right)=j^2-\frac{1}{4}, \tag{76}$$

$$\zeta_2'\left(\zeta_2'-1\right)=\frac{\left(1+\frac{eB\Theta}{4\hbar c}\right)^2}{\beta^2\left(\lambda^2+\hbar^2\left(\frac{eB}{2c}\right)^2\right)}+\left(j^2+\frac{3}{4}\right)-\frac{2\lambda\left(1+\frac{eB\Theta}{2\hbar c}\right)}{\beta\left(\lambda^2+\hbar^2\left(\frac{eB}{2c}\right)^2\right)}-\frac{\left\{\hbar\left(1+\frac{eB\Theta}{4\hbar c}\right)\frac{eB}{c}+m_0^2\omega^2\Theta\right\}|j|}{\beta\left(\lambda^2+\hbar^2\left(\frac{eB}{2c}\right)^2\right)}. \tag{77}$$

Finally, our differential equation can be written as

$$\left(-\frac{d^2}{dq^2}+\frac{1}{2}U_0\left\{\frac{\zeta_1'\left(\zeta_1'-1\right)}{\sin^2(\alpha q)}+\frac{\zeta_2'\left(\zeta_2'-1\right)}{\cos^2(\alpha q)}\right\}\right)\varphi(q)=\bar{\xi}'^2\varphi(q), \tag{78}$$

where $U_0=\alpha^2$ and $\bar{\xi}'^2=\varsigma'^2+\left(1+[eB\Theta/4\hbar c]\right)^2(1/\beta)$ with $\alpha=\sqrt{\beta\left(\lambda^2+\hbar^2[eB/2c]^2\right)}$.

As in the above case, Eq. (78) is the well-known Schrödinger equation in a Pöschl-Teller potential [39]. with the constraint that $(\zeta_1',\zeta_2')>1$, the exact form of ζ_1 and ζ_2 are

$$\zeta_1'=|j|+\frac{1}{2}, \tag{79}$$

$$\zeta_2'=\frac{1}{2}+\sqrt{\left(1+\frac{\left(1+\frac{eB\Theta}{4\hbar c}\right)^2}{\beta^2\left(\lambda^2+\hbar^2\left(\frac{eB}{2c}\right)^2\right)}+j^2-\frac{2\lambda\left(1+\frac{eB\Theta}{2\hbar c}\right)}{\beta\left(\lambda^2+\hbar^2\left(\frac{eB}{2c}\right)^2\right)}-\frac{\left\{\hbar\left(1+\frac{eB\Theta}{4\hbar c}\right)\frac{eB}{c}+m_0^2\omega^2\Theta\right\}|j|}{\beta\left(\lambda^2+\hbar^2\left(\frac{eB}{2c}\right)^2\right)}\right)}. \tag{80}$$

So, the general solution of Eq. (78) is

$$\Psi = C_1\, {}_2F_1\left(a';b';c';z\right) + C_2\, z^{1-c}\, {}_2F_1\left(a'+1-c';b'+1-c';2-c;z\right), \tag{81}$$

with $z=\sin^2(\alpha q)$ and where

$$a' = \frac{1}{2}\left(\zeta_1+\zeta_2+\frac{\bar{\xi}}{\alpha}\right),\; b' = \frac{1}{2}\left(\zeta_1+\zeta_2-\frac{\bar{\xi}}{\alpha^2}\right),\; c' = \zeta_1+\frac{1}{2}. \tag{82}$$

With the condition $a'=-n$, and by using Eq. (64), we obtain.

$$\bar{\xi}^2 = \beta\left(\lambda^2+\hbar^2\left(\frac{eB}{2c}\right)^2\right)\left(\zeta_1+\zeta_2+2n\right)^2 = \frac{E^2-\left(m_0c^2\right)^2}{c^2}+\left(1+\frac{eB\Theta}{4\hbar c}\right)^2\frac{1}{\beta}. \tag{83}$$

Now, from Eq. (83) the final form of the spectrum of energy is

$$E_n = \pm m_0c^2\sqrt{1-\frac{\left(1+\frac{eB\Theta}{4\hbar c}\right)^2}{\left(m_0c\right)^2\beta}+\frac{\beta\left(\lambda^2+\hbar^2\left(\frac{eB}{2c}\right)^2\right)}{\left(m_0c\right)^2}\left\{|j|+1+2n+\Xi'\right\}^2}, \tag{84}$$

with

$$\Xi' = \sqrt{\left(1+\frac{\left(1+\frac{eB\Theta}{4\hbar c}\right)^2}{\beta^2\left(\lambda^2+\hbar^2\left(\frac{eB}{2c}\right)^2\right)}+j^2-\frac{2\lambda\left(1+\frac{eB\Theta}{2\hbar c}\right)}{\beta\left(\lambda^2+\hbar^2\left(\frac{eB}{2c}\right)^2\right)}-\frac{\left\{\hbar\left(1+\frac{eB\Theta}{4\hbar c}\right)\frac{eB}{c}+m_0^2\omega^2\Theta\right\}|j|}{\beta\left(\lambda^2+\hbar^2\left(\frac{eB}{2c}\right)^2\right)}\right)} \tag{85}$$

Also, we can see that the presence of both parameters β and Θ with a magnetic field breaks the degeneracy of the spectrum of energy. In addition, the presence of a term on n^2 shows the existence of hard confinement. This dependence is only clear when we introduce minimal length in the problem in question.

Finally, the corresponding wave function is

$$\psi_{KG} = \mathrm{N}_{\mathrm{norm}} e^{i|j|\theta} p^{-\frac{1}{2}} z^{\frac{\zeta_1}{2}}\left(1-z\right)^{\frac{\zeta_2}{2}}\, {}_2 \times F_1\left(-n;b';c';z\right). \tag{86}$$

with $\mathrm{N}_{\mathrm{norm}}$ the constant of normalization.

Now, to achieve a deeper understanding of our spectrum of energy presented by Eq. (84), we have plotted the variation of the positive energy E with quantum number n and magnetic field B for different various situations. As the spectrum of energy is symmetric, any comments on the case of the positive energy can be extended to the case of the negative energy.

Figure 1 shows the behavior of the spectrum of energy versus n by changing the three parameters appearing in the expression for the energy spectrum, *i.e*, the magnetic field B, the NC parameter θ and the minimal length β. This figure has been depicted for two cases: $j=0$ and $j=1$. Here, three remarks can be made

- When varying only the magnetic field, one observes that the allure of the spectrum of energy tends to become linear in the high magnetic field regime.
- From the variation of β only, we can see that for each value of β we have two regions: first region, the values of the energy are zero until a specific number where the spectrum begin grows. According to this figure, this specific number depends inversely with β.
- Now, when varying θ, we see that all curves coincide.

In Fig. 2, we represent the variation of the positive energy levels with magnetic field for four levels $n=0,1,2,3$. For each level of energy, we have chosen $j=0,1,2,3$. With the exception of the case $j=0$, every curve show approximately a linear form with a magnetic field B.

Finally, in order to show that the problem in question has relevance from a physical point of view, we make the following remarks. Firstly, we can mention some studies that showed the existence of a close relation between graphene and the Dirac oscillator model. [6, 40–42].

Bastos *et al.*, [33] consider a non-commutative description of graphene. This description consists of a Dirac equation for massless Dirac fermions plus non-commutative corrections, which are treated in the presence of an external magnetic field. They argue that, being a two-dimensional Dirac system, graphene is particularly interesting to test non-ommutativity. They find that momentum non-commutativity affects the energy levels of graphene and they obtain a bound for the momentum non-commutative parameter. Their results show that momentum non-commutativity yields interesting results

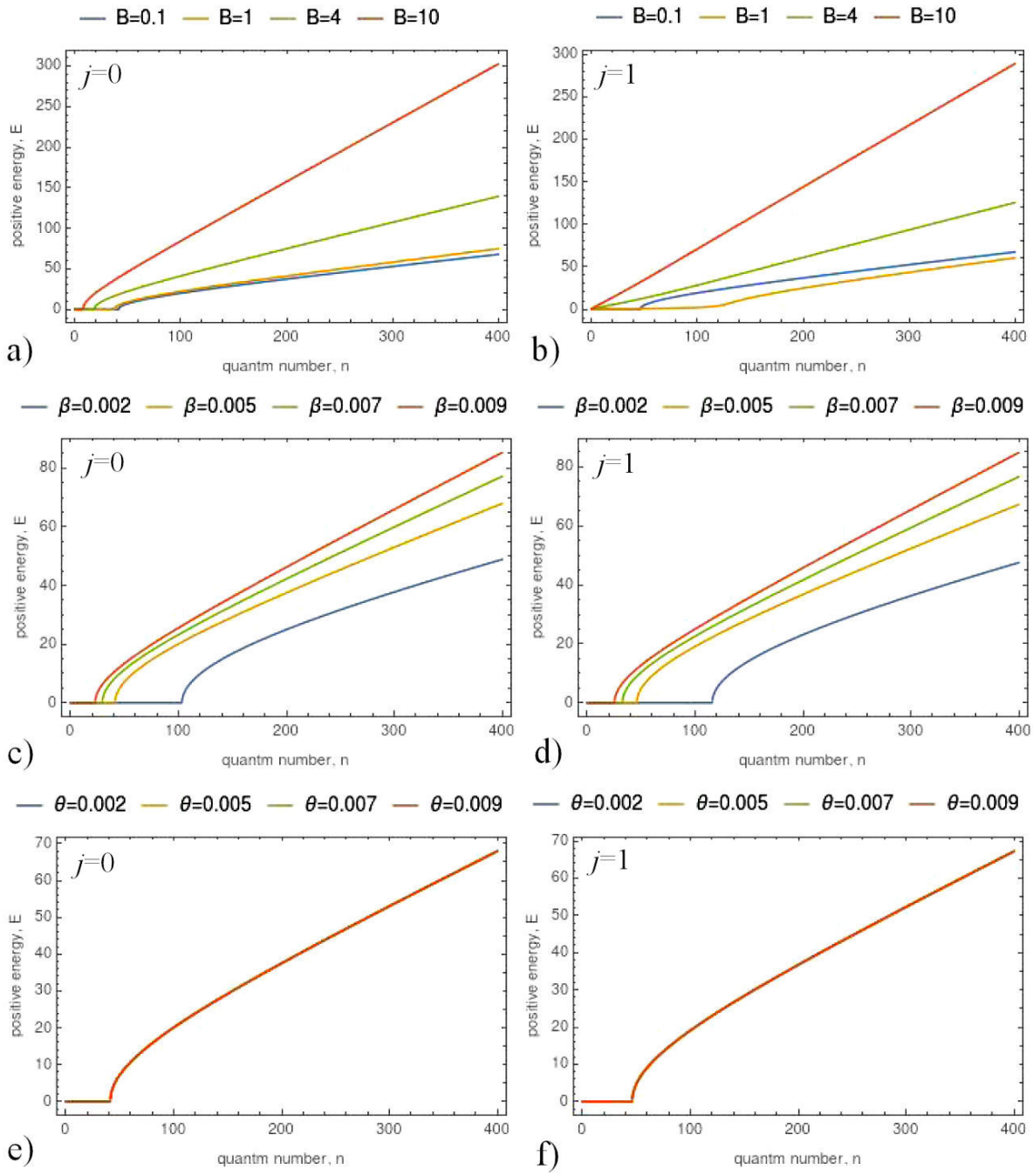

FIGURE 1. The positive energy eigenvalues as a function of the quantum number n. a, b) Variation with respect of B for fixed value of $\theta = 0.00$ and $\beta = 0.005$; c, d) variation with respect of β for fixed values of $B =$ and $\theta = 0.005$; e, f) variation with respect of θ for fixed value of $B = 0.1$ and $\beta = 0.005$.

also at low-energies and that its implications are not restricted to quantum cosmology and black holes physics.

In the same context, Falomir *et al.* [43] employ a simple nonrelativistic model to describe the low-energy excitation of graphene. The model is based on a deformation of the Heisenberg algebra which makes the commutator of momenta proportional to the pseudo-spin. As a result, the resulting Hall conductivity is consistent with the anomalous integer quantum Hall effect found in graphene.

Jallel [44] describes the lattice deformation in graphene under strain effect by considering the spacial-moment coordinates do not commute. Within such framework, he (i) build a new model describing Dirac fermions interacting with an external source that is non commutative parameter dependent, and (ii) the eigenvalues are showing Landau levels in similar way to the case of a real magnetic field applied to graphene.

Iorio and Pais [45], in their paper "Generalized uncertainty principle in graphene" showed that, by going beyond

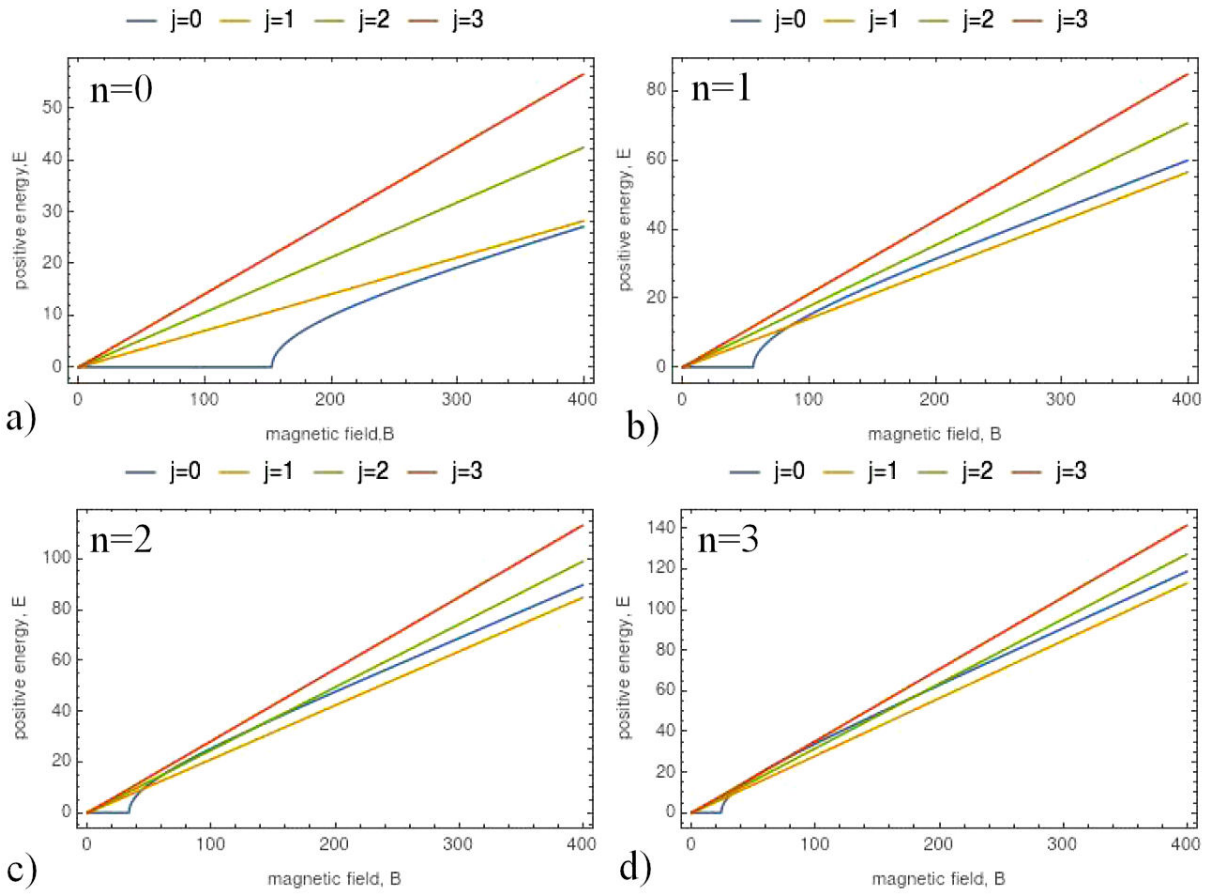

FIGURE 2. The positive energy eigenvalues as a function of the magnetic field B for $n = 0, 1, 2, 3$.

the low-energy approximation for which the dispersion relations of graphene are linear, the corresponding emergent field theory is a specific generalization a Dirac field theory. The generalized Dirac Hamiltonians one obtains are those compatible with specific generalizations of the uncertainty principle. They also comment on the compatibility of the latter with non-commuting positions, and on their possible physical realization.

Thus, following these works and maybe others which are not cited here, we can expect that our study can be regarded as a contribution to the case of graphene by introducing the minimal length in the commutations rules of Heisenberg.

5. Conclusion

In this paper, we have exactly solved the KGO in two dimensions in the presence of an external magnetic field in the framework of relativistic quantum mechanics with minimal length and in the NC space. Firstly, by adopting the same procedure used by Menculini *et al.* [9], we have solved the problem only in the case of non-commutative space. The results found are in good agreement with those obtained in the literature. After that, we have introduced the minimal length in the problem in question. This introduction has been making as follows: (i) we write the coordinates of the non-commutative space with those in commutative space by using the Bopp shift approximation, and (ii) then we introduce the minimal length in our equation. By these, the problem in question is identified with a Pöschl-Teller potential.

The eigensolutions of the problem in question in a magnetic field are obtained in the presence of a minimal length in non-commutative space. The dependence on theses eigensolutions with the minimum length and the non-commutative parameters is very clear. Furthermore, by comparing the spectrum of energy obtained in our case with those of the same problem in the flat space-time, we can see that the presence of both parameters breaks the degeneracy of the spectrum of energy. In addition, the presence of a term on n^2 in the form of the spectrum of energy shows the existence of hard confinement. This dependence is only clear when we introduce minimal length in the problem in question. Finally, In the limit where $\beta \to 0$ and $\Theta \to 0$ tend to zero, we recover well the results obtained in the literature.

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