RESEARCH Revista Mexicana de Física **67** (3) 465–470 MAY-JUNE 2021

Conjugate spinor field equation for massless spin-$\frac{3}{2}$ field in de Sitter ambient space

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Received 2 October 2020; accepted 6 January 2021

The quantum field theory in de Sitter ambient space provides us with a comprehensive description of massless gravitational fields. Using the gauge-covariant derivative in the de Sitter ambient space, the gauge invariant Lagrangian density has been found. In this paper, the equation of the conjugate spinor for massless spin-3/2 field is obtained using the Euler-Lagrange equation. Then the field equation is written in terms of the Casimir operator of the de Sitter group. Finally, the gauge invariant field equation is presented.

Keywords: De Sitter space-time; conjugate spinor; field equation; massless spin-3/2.

PACS: 04.62.+v; 11.15.-q; 11.10.-z

DOI: https://doi.org/10.31349/RevMexFis.67.465

1. Introduction

The experimental data and cosmological observations show that the universe is expanding with a constant positive acceleration, *i.e*., the space-time can be non-flat [1–6]. Since the simplest curved space-time that corresponds to these observations is de Sitter space-time and this space-time has maximal symmetry, the group of ten parametric SO (1,4) is the kinematic group of the de Sitter space. Therefore, quantum field theory, gauge theory and quantum cosmology are investigated in this space-time [7–16].

Supersymmetry has been introduced as one of the fundamental principles of all the efforts to achieve the grand unified theory. It acts in such a way that the relationship between the bosons (integer-valued spin) and the fermions (half-integer spin) is established. In the supersymmetry, all the particles have a partner. For example, for a graviton (gravity-carrier particle with a spin-2), a partner with a spin-$\frac{3}{2}$ called gravitino can be considered. With the development of quantum field theory, gauge theories and their very successful results, everyone began to think about quantization and gauge everything. As far as gauge theories are concerned, it should be considered that the structure of a particular symmetry called gauge symmetry, should remain invariant. Quantum gravity theories that use supersymmetry are called supergravity and seek to unify the gravitational interaction with other fundamental interactions. It should be noted that supergravity is a local supersymmetry theory [17]. Therefore, gauge theory can be extended to gravity [18].

To understand physical systems, we must obtain the equation of motion of physical quantities or, equivalently, the system's Lagrangian. In Lagrangian mechanics, changes in a physical system are described through solving the Euler-Lagrange equation for that system's behavior. The spin-$\frac{3}{2}$ field equation was first introduced by William Rarita and Julian Schwinger in 1941 using the Euler Lagrange method [19].

The field quantization and the gauge theory are reformulated in the ambient space formalism [12]. In the de Sitter ambient space formalism the spinor fields can be written in terms of the de Sitter plane wave. The dS plane wave cannot be defined properly since the plane wave solution has the singularity in the limit $x \longrightarrow \infty$ (x is the ambient space coordinate) [14]. Therefore, in ambient space formalism only the massless fields with spin $0, \frac{1}{2}, 1, \frac{3}{2}, 2$ can propagate. We assume that the interactions between the elementary systems in the universe are governed by the gauge principle and formulated through the gauge-covariant derivative which is defined as a quantity that preserves the gauge invariant transformation of the Lagrangian. In quantum field theory, the Gupta-Bleuler quantization method is used to eliminate the infrared divergence of the two-point function [20–22]. According to this method, the quantum field affects the space including all states with positive and negative norm. In this formalism, of all the degrees of freedom, only two are physical [12, 13].

In the previous work [13], we introduced Lagrangian for massless spin-$\frac{3}{2}$ field in de Sitter space-time. Due to the complex and difficult calculations, we intend to present more details of the calculations in this article . The notation of ambient space is briefly reviewed in Sec. 2. In Sec. 3, through this notation, we calculate the conjugate spinor field equation also the equation of this field is invariant. A brief conclusion is presented in Sec. 4. Finally, details of mathematical calculations are given in two appendices.

2. Notations

It has been discovered today that the universe is accelerating, with a small, but non-zero and positive cosmological constant, Therefore, it can be concluded that the shape and geometry of the universe is curved. So, in the first-order approximation, we can use the de Sitter space-time to explain the curved universe. This space-time is a 4-dimensional hyperboloid that can be embedded in a Minkowski 5-dimensional space-time [14, 15]:

$$X_H = \{x \in \mathbb{R}^5 | \; x \cdot x = \eta_{\alpha\beta} x^\alpha x^\beta = -H^{-2}\}, \;\; \alpha, \beta = 0, 1, 2, 3, 4, \tag{1}$$

where $\eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1, -1)$ and H is the Hubble parameter. The metric is defined as follows:

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta|_{x^2=-H^{-2}} = g^{dS}_{\mu\nu} dX^\mu dX^\nu, \quad \mu = 0, 1, 2, 3, \tag{2}$$

where X^μ are the components of the coordinate four-vector in a system of intrinsic coordinates on a hyperboloid and x^α is the five dimensional Minkowski space-time (de Sitter ambient space). Two Casimir operators of the group include:

$$Q^{(1)} = -\frac{1}{2} L_{\alpha\beta} L^{\alpha\beta}, \;\; \alpha, \beta = 0, 1, 2, 3, 4, \tag{3}$$

$$Q^{(2)} = -W_\alpha W^\alpha \;\;, \;\; W_\alpha = \frac{1}{8} \epsilon_{\alpha\beta\gamma\delta\eta} L^{\beta\gamma} L^{\delta\eta}, \tag{4}$$

where $\epsilon_{\alpha\beta\gamma\delta\eta}$ is an anti-symmetric tensor, and $L_{\alpha\beta}$ are ten infinitismal generators in de Sitter space. They can be written as a linear combination: $L_{\alpha\beta} = M_{\alpha\beta} + S_{\alpha\beta}$. Where $M_{\alpha\beta}$ is the orbital part and $S_{\alpha\beta}$ is the spinoral part. In this formalism, the space $M_{\alpha\beta}$ is represented as

$$M_{\alpha\beta} = -i(x_\alpha \partial_\beta - x_\beta \partial_\alpha) = -i(x_\alpha \partial^\top_\beta - x_\beta \partial^\top_\alpha), \tag{5}$$

where $\partial^\top_\beta = \theta_\beta{}^\alpha \partial_\alpha$ is the transverse derivative ($x.\partial^\top = 0$) and $\theta_{\alpha\beta} = \eta_{\alpha\beta} + H^2 x_\alpha x_\beta$ is the projection tensor on de sitter hyperboloid. For half-integer spin fields $s = l + 1/2$, the spinoral part is defined as:

$$S^{(s)}_{\alpha\beta} = S^{(l)}_{\alpha\beta} + S^{(\frac{1}{2})}_{\alpha\beta}, \tag{6}$$

where $S_{\alpha\beta}$for spin $(1/2)$ field is:

$$S_{\alpha\beta} = -\frac{i}{4}[\gamma_\alpha, \gamma_\beta], \tag{7}$$

and the γ-matrices satisfy following relation:

$$\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}\mathbb{I}. \tag{8}$$

A proper display for them is [14, 15]:

$$\gamma^0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \qquad \gamma^4 = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix},$$
$$\gamma^1 = \begin{pmatrix} 0 & i\sigma^1 \\ i\sigma^1 & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix},$$
$$\gamma^3 = \begin{pmatrix} 0 & i\sigma^3 \\ i\sigma^3 & 0 \end{pmatrix}, \tag{9}$$

$$\gamma^{\alpha\dagger} = \gamma^0 \gamma^\alpha \gamma^0 \qquad (\gamma^4)^2 = -1 \qquad (\gamma^0)^2 = 1, \tag{10}$$

where $\mathbb{I}$ is unit 2×2 matrix and σ^i are the Pauli matrices. It should be noted that for massless spin-$(3/2)$ field $Q^{(1)}_{(3/2)}$ is [13]:

$$Q^{(1)}_{\frac{3}{2}} \Psi_\alpha = Q^{(1)}_0 \Psi_\alpha + \not{x} \not{\partial}^\top \Psi_\alpha + 2x_\alpha \partial^\top \cdot \Psi - \frac{11}{2} \Psi_\alpha + \gamma_\alpha \not{\Psi}, \tag{11}$$

$Q^{(1)}_0 = -\partial^\top_\alpha \partial^{\alpha\top}$ is the ”scalar” Casimir operator.

3. Conjugate spinor field equation for massless spin-$3/2$ field

It is believed that the gauge theory is the basis of fundamental particle interactions. The Lagrangian for massless spin-$(3/2)$ field is peresented, in the linear approximation, by using the gauge theory and defining the gauge covariant derivative. In the ambient space notation the gauge covariant derivative can be defined as $D^\Psi_\beta = \nabla^\top_\beta + i(\Psi_\beta{}^A)^\dagger \gamma^0 \mathcal{Q}_A$, with $A = 1, .., N$ [13]. The vector-spinor field equation $\Psi_\alpha(x)$ is obtained from the usual Euler-Lagrange equations. This Lagrangian is invariant under the gauge transformation: $\Psi_\alpha \rightarrow \Psi^g_\alpha = \Psi_\alpha + \nabla^\top_\alpha \psi$, and $\tilde{\Psi}_\alpha \rightarrow \tilde{\Psi}^g_\alpha = \tilde{\Psi}_\alpha + \partial^\top_\alpha \tilde{\psi}$ (see more details [23]):

$$\mathcal{L} = \left(\tilde{\nabla}^\top_\alpha \tilde{\Psi}_\beta - \tilde{\nabla}^\top_\beta \tilde{\Psi}_\alpha\right) \left(\nabla^{\top\alpha} \Psi^\beta - \nabla^{\top\beta} \Psi^\alpha\right), \tag{12}$$

$\nabla^\top_\alpha$ is a transverse-covariant derivative which is defined to obtain an invariant Lagrangian according to the following equation:

$$\nabla^\top_\beta \Psi_{\alpha_1 \ldots . \alpha_l} \equiv (\partial^\top_\beta + \gamma^\top_\beta \not{x}) \Psi_{\alpha_1 \ldots . \alpha_l} - \sum_{n=1}^{l} x_{\alpha_n} \Psi_{\alpha_1 .. \alpha_{n-1} \beta \alpha_{n+1} .. \alpha_l}, \tag{13}$$

also for the conjugate spinor$\tilde{\Psi}_\alpha$:

$$\tilde{\nabla}^\top_\beta \tilde{\Psi}_{\alpha_1 \ldots . \alpha_l} \equiv \partial^\top_\beta \tilde{\Psi}_{\alpha_1 \ldots . \alpha_l} - \sum_{n=1}^{l} x_{\alpha_n} \tilde{\Psi}_{\alpha_1 .. \alpha_{n-1} \beta \alpha_{n+1} .. \alpha_l}, \tag{14}$$

where $\not{x} = \gamma_\alpha x^\alpha$and $\gamma^\top_\alpha = \theta^\beta_\alpha \gamma_\beta$. The above equations are specifically designed for our calculations:

$$\nabla^\top_\alpha \Psi_\beta = \partial^\top_\alpha \Psi_\beta + \gamma^\top_\alpha \not{x} \Psi_\beta - x_\beta \Psi_\alpha, \tag{15}$$

$$\tilde{\nabla}^\top_\beta \tilde{\Psi}_\alpha = \partial^\top_\beta \tilde{\Psi}_\alpha - x_\alpha \tilde{\Psi}_\beta. \tag{16}$$

Now we want to obtain the field equation for the conjugate spinor($\tilde{\Psi}_\alpha = \Psi^\dagger_\alpha \gamma^0$) by using the Euler-Lagrange equation in the linear approximation. The Euler-Lagrange equation is:

$$\frac{\delta \mathcal{L}}{\delta \Psi^m} - \partial^{l^\top} \frac{\delta \mathcal{L}}{\delta(\partial^{l\top} \Psi^m)} = 0. \quad (17)$$

First, we extend each terms of the Eq. (11):

$$A = \left(\nabla^{\top\alpha}\Psi^\beta - \nabla^{\top\beta}\Psi^\alpha\right)$$
$$= \left(\partial^{\top\alpha}\Psi^\beta + \gamma^\alpha \not{x}\Psi^\beta - \partial^{\top\beta}\Psi^\alpha - \gamma^\beta \not{x}\Psi^\alpha\right), \quad (18)$$

$$B = \left(\tilde{\nabla}^\top_\alpha \tilde{\Psi}_\beta - \tilde{\nabla}^\top_\beta \tilde{\Psi}_\alpha\right)$$
$$= \left(\partial^\top_\alpha \tilde{\Psi}_\beta - x_\beta \tilde{\Psi}_\alpha - \partial^\top_\beta \tilde{\Psi}_\alpha + x_\alpha \tilde{\Psi}_\beta\right). \quad (19)$$

By use the Euler-Lagrange equation, we consider the following terms:

$$\frac{\delta \mathcal{L}}{\delta \Psi^m} = \left(\gamma^\alpha \not{x}\delta^\beta_m - \gamma^\beta \not{x}\delta^\alpha_m\right)\left(\tilde{\nabla}^\top_\alpha \tilde{\Psi}_\beta - \tilde{\nabla}^\top_\beta \tilde{\Psi}_\alpha\right), \quad (20)$$

and

$$\frac{\delta \mathcal{L}}{\delta(\partial^{l\top}\Psi^m)} = \left(\delta^\alpha_l \delta^\beta_m - \delta^\beta_l \delta^\alpha_m\right)\left(\tilde{\nabla}^\top_\alpha \tilde{\Psi}_\beta - \tilde{\nabla}^\top_\beta \tilde{\Psi}_\alpha\right), \quad (21)$$

if $\beta = m$, then, one obtains:

$$\frac{\delta \mathcal{L}}{\delta \Psi^m} = \gamma^\alpha \not{x}\left(\tilde{\nabla}^\top_\alpha \tilde{\Psi}_\beta - \tilde{\nabla}^\top_\beta \tilde{\Psi}_\alpha\right), \quad (22)$$

$$\frac{\delta \mathcal{L}}{\delta(\partial^{l\top}\Psi^m)} = \delta^\alpha_l \left(\tilde{\nabla}^\top_\alpha \tilde{\Psi}_\beta - \tilde{\nabla}^\top_\beta \tilde{\Psi}_\alpha\right). \quad (23)$$

Now, by placing the above expressions in the Euler-Lagrange equation, the field equation is obtained as follows:

$$\gamma^\alpha \not{x}\left(\tilde{\nabla}^\top_\alpha \tilde{\Psi}_\beta - \tilde{\nabla}^\top_\beta \tilde{\Psi}_\alpha\right)$$
$$- \partial^{\top\alpha}\left(\tilde{\nabla}^\top_\alpha \tilde{\Psi}_\beta - \tilde{\nabla}^\top_\beta \tilde{\Psi}_\alpha\right) = 0, \quad (24)$$

this equation can be written in the summarized form:

$$\left(\partial^{\top\alpha} - \gamma^\alpha \not{x}\right)\left(\tilde{\nabla}^\top_\alpha \tilde{\Psi}_\beta - \tilde{\nabla}^\top_\beta \tilde{\Psi}_\alpha\right) = 0. \quad (25)$$

In the Appendix A, the field equations for the conjugate spinor is obtained by using the second order Casimir operator:

$$\left(Q^{(1)}_{\frac{3}{2}} + \frac{5}{2}\right)\tilde{\Psi}_\alpha + \partial^\top_\alpha\left(\not{x}\tilde{\not{\Psi}} + \partial^\top \cdot \tilde{\Psi}\right)$$
$$- 2\left(\gamma_\alpha \tilde{\not{\Psi}} + \not{x}\not{\partial}^\top \tilde{\Psi}_\alpha - \tilde{\Psi}_\alpha\right) = 0. \quad (26)$$

In this here we present the gauge invariant field equation for the conjugate spinor. Given the definition of $Q^{(1)}_{\frac{3}{2}}$, we rewrite Eq. (25) as follows:

$$-Q_0\tilde{\Psi}_\beta + \tilde{\Psi}_\beta - 2x_\beta(\partial^\top \cdot \tilde{\Psi}) - \partial^\top_\beta(\partial^\top \cdot \tilde{\Psi})$$
$$+ \not{x}\not{\partial}^\top \tilde{\Psi}_\beta - x_\beta \not{x}\tilde{\not{\Psi}} - \not{x}\partial^\top_\beta \tilde{\not{\Psi}} = 0. \quad (27)$$

We show that the above equation is invariant under the following gauge transformation:

$$\tilde{\Psi}_\alpha \longrightarrow \tilde{\Psi}^g_\alpha = \tilde{\Psi}_\alpha + \partial^\top_\alpha \tilde{\psi}, \quad (28)$$

with Ψ as an arbitrary spinor field. Therefore, Eq. (26) comes as follows:

$$-Q_0(\tilde{\Psi}_\beta + \partial^\top_\beta \tilde{\psi}) + \tilde{\Psi}_\beta + \partial^\top_\beta \tilde{\psi} - 2x_\beta \partial^{\alpha\top}(\tilde{\Psi}_\alpha + \partial^\top_\alpha \tilde{\psi})$$
$$- \partial^\top_\beta \partial^{\alpha\top}(\tilde{\Psi}_\alpha + \partial^\top_\alpha \tilde{\psi}) + \not{x}\not{\partial}^\top(\tilde{\Psi}_\beta + \partial^\top_\beta \tilde{\psi})$$
$$- x_\beta \not{x}(\tilde{\not{\Psi}} + \not{\partial}^\top \tilde{\psi}) - \not{x}\partial^\top_\beta(\tilde{\not{\Psi}} + \not{\partial}^\top \tilde{\psi}) = 0. \quad (29)$$

Consequently, we can derive:

$$\Longrightarrow = -Q_0\tilde{\Psi}_\beta + \tilde{\Psi}_\beta - 2x_\beta(\partial^\top \cdot \tilde{\Psi}) - \partial^\top_\beta(\partial^\top \cdot \tilde{\Psi}) + \not{x}\not{\partial}^\top \tilde{\Psi}_\beta - x_\beta \not{x}\tilde{\not{\Psi}} - \not{x}\partial^\top_\beta \tilde{\not{\Psi}}$$
$$\underbrace{-Q_0\partial^\top_\beta \tilde{\psi} + \partial^\top_\beta \tilde{\psi} - 2x_\beta \partial^{\alpha\top}\partial^\top_\alpha \psi - \partial^\top_\beta \partial^{\alpha\top}\partial^\top_\alpha \psi + \not{x}\not{\partial}^\top \partial^\top_\beta \tilde{\psi} - x_\beta \not{x}\not{\partial}^\top \tilde{\psi} - \not{x}\partial^\top_\beta \not{\partial}^\top \tilde{\psi}}_{=0} = 0. \quad (30)$$

For a gauge invariant, the last terms must be zero. In order to prove this, we use the following auxiliary relationships:

$$[\partial^\top_\alpha, \not{\partial}^\top] = \not{x}\partial^\top_\alpha - x_\alpha \not{\partial}^\top, \quad (31)$$

$$[\partial^\top_\alpha, Q_0] = -6\partial^\top_\alpha - 2(Q_0 + 4)x_\alpha. \quad (32)$$

Thus, the last terms of equation (29) are

$$-Q_0\partial^\top_\beta \tilde{\psi} + \partial^\top_\beta \tilde{\psi} - 2x_\beta \partial^{\alpha\top}\partial^\top_\alpha \psi - \partial^\top_\beta \partial^{\alpha\top}\partial^\top_\alpha \psi + \not{x}\not{\partial}^\top \partial^\top_\beta \tilde{\psi} - x_\beta \not{x}\not{\partial}^\top \tilde{\psi} - \not{x}\partial^\top_\beta \not{\partial}^\top \tilde{\psi}$$
$$= 2x_\beta Q_0 \tilde{\psi} + \partial^\top_\beta \tilde{\psi} - x_\beta \not{x}\not{\partial}^\top \tilde{\psi} - 6\partial^\top_\beta \tilde{\psi} - 2Q_0 x_\beta \tilde{\psi} - 8x_\beta \tilde{\psi} + x_\beta \not{x}\not{\partial}^\top \tilde{\psi} + \partial^\top_\beta \tilde{\psi}. \quad (33)$$

After some simplification, the equation (32) becomes:

$$\Longrightarrow 2x_\beta Q_0 \tilde{\psi} - 4\partial^\top_\beta \tilde{\psi} - 2Q_0 x_\beta \tilde{\psi} - 8x_\beta \tilde{\psi} = 2(x_\beta Q_0 - Q_0 x_\beta)\tilde{\psi} - 4\partial^\top_\beta \tilde{\psi} - 8x_\beta \tilde{\psi}. \quad (34)$$

Finally, using the auxiliary relationship $[x_\alpha, Q_0] = 2\partial^\top_\alpha + 4x_\alpha$, we obtain:

$$= 2(2\partial^\top_\beta + 4x_\beta)\tilde{\psi} - 4\partial^\top_\beta\tilde{\psi} - 8x_\beta\tilde{\psi} = 0, \tag{35}$$

proving that the field equation is invariant.

4. Conclusions

In order to better understand the evolution of the universe, it is necessary to extend the theory of quantum fields, field interactions, or gauge theory, supersymmetry and supergravity in the de Sitter space-time. We studied the conjugate spinor field equation for massless gravitational field by the Euler-Lagrange equation in the de Sitter ambient space formalism. The field equation in terms of the Casimir operator is obtained. We have shown that the field equation of the conjugate spinor for massless spin-$(3/2)$ field is gauge invariant. Studies of this kind are of particular interest given the recent observations of gravitational waves (LIGO Collaboration); the graviton is a particle that is believed to carry the force of gravity, which would be accompanied by the gravitino in a supersymmetric theory in curved space.

Appendix

A. The field equation in terms of Casimir operator

Here we want to show how the field equation is written in terms of the Casimir operator:

$$\left(\partial^{\top\alpha} - \gamma^\alpha \not{x}\right)\left(\partial^\top_\alpha\tilde{\Psi}_\beta - x_\beta\tilde{\Psi}_\alpha - \partial^\top_\beta\tilde{\Psi}_\alpha + x_\alpha\tilde{\Psi}_\beta\right) = 0, \tag{A.1}$$

$$\underbrace{\partial^{\top\alpha}\left(\partial^\top_\alpha\tilde{\Psi}_\beta - x_\beta\tilde{\Psi}_\alpha - \partial^\top_\beta\tilde{\Psi}_\alpha + x_\alpha\tilde{\Psi}_\beta\right)}_{1}\underbrace{-\gamma^\alpha \not{x}\left(\partial^\top_\alpha\tilde{\Psi}_\beta - x_\beta\tilde{\Psi}_\alpha - \partial^\top_\beta\tilde{\Psi}_\alpha + x_\alpha\tilde{\Psi}_\beta\right)}_{2} = 0. \tag{A.2}$$

We first consider the expression 1:

$$\partial^{\top\alpha}\left(\partial^\top_\alpha\tilde{\Psi}_\beta - x_\beta\tilde{\Psi}_\alpha - \partial^\top_\beta\tilde{\Psi}_\alpha + x_\alpha\tilde{\Psi}_\beta\right) = \partial^{\top\alpha}(\partial^\top_\alpha\tilde{\Psi}_\beta) - \partial^{\top\alpha}(x_\beta\tilde{\Psi}_\alpha) - \partial^{\top\alpha}(\partial^\top_\beta\tilde{\Psi}_\alpha) + \partial^{\top\alpha}(x_\alpha\tilde{\Psi}_\beta), \tag{A.3}$$

$$= -Q_0\tilde{\Psi}_\beta - (\partial^{\top\alpha}x_\beta)\tilde{\Psi}_\alpha - x_\beta(\partial^{\top\alpha}\tilde{\Psi}_\alpha) - \partial^{\top\alpha}(\partial^\top_\beta\tilde{\Psi}_\alpha) + 4\tilde{\Psi}_\beta, \tag{A.4}$$

$$= -Q_0\tilde{\Psi}_\beta - (\delta^\alpha_\beta + x_\beta x^\alpha)\tilde{\Psi}_\alpha - x_\beta(\partial^\top\cdot\tilde{\Psi}) - \partial^{\top\alpha}(\partial^\top_\beta\tilde{\Psi}_\alpha) + 4\tilde{\Psi}_\beta, \tag{A.5}$$

$$= -Q_0\tilde{\Psi}_\beta - \tilde{\Psi}_\beta - x_\beta(\partial^\top\cdot\tilde{\Psi}) - (\partial^\top_\beta\partial^{\top\alpha}\tilde{\Psi}_\alpha + x_\beta\partial^{\top\alpha}\tilde{\Psi}_\alpha - x^\alpha\partial^\top_\beta\tilde{\Psi}_\alpha) + 4\tilde{\Psi}_\beta, \tag{A.6}$$

$$= -Q_0\tilde{\Psi}_\beta + 3\tilde{\Psi}_\beta - 2x_\beta(\partial^\top\cdot\tilde{\Psi}) - \partial^\top_\beta(\partial^\top\cdot\tilde{\Psi}) + x^\alpha\partial^\top_\beta\tilde{\Psi}_\alpha, \tag{A.7}$$

$$= -Q_0\tilde{\Psi}_\beta + 2\tilde{\Psi}_\beta - 2x_\beta(\partial^\top\cdot\tilde{\Psi}) - \partial^\top_\beta(\partial^\top\cdot\tilde{\Psi}). \tag{A.8}$$

We first consider the expression 2:

$$-\gamma^\alpha \not{x}\left(\partial^\top_\alpha\tilde{\Psi}_\beta - x_\beta\tilde{\Psi}_\alpha - \partial^\top_\beta\tilde{\Psi}_\alpha + x_\alpha\tilde{\Psi}_\beta\right), \tag{A.9}$$

$$= -(2x^\alpha - \not{x}\gamma^\alpha)\left(\partial^\top_\alpha\tilde{\Psi}_\beta - x_\beta\tilde{\Psi}_\alpha - \partial^\top_\beta\tilde{\Psi}_\alpha + x_\alpha\tilde{\Psi}_\beta\right), \tag{A.10}$$

$$= 2x^\alpha\partial^\top_\beta\tilde{\Psi}_\alpha - 2x^\alpha x_\alpha\tilde{\Psi}_\beta + \not{x}\,\not{\partial}^\top\tilde{\Psi}_\beta - \not{x}\gamma^\alpha x_\beta\tilde{\Psi}_\alpha - \not{x}\gamma^\alpha\partial^\top_\beta\tilde{\Psi}_\alpha + \not{x}x_\alpha\tilde{\Psi}_\alpha, \tag{A.11}$$

$$= \not{x}\,\not{\partial}^\top\tilde{\Psi}_\beta - x_\beta\,\not{x}\,\tilde{\not{\Psi}} - \not{x}\partial^\top_\beta\,\tilde{\not{\Psi}} - \tilde{\Psi}_\beta, \tag{A.12}$$

in the above calculations, we have used the terms $x\cdot\Psi = 0$and $x\cdot\partial^\top = 0$. According to 1 and 2 we can write the equation of motion as follow:

$$\Longrightarrow = -Q_0\tilde{\Psi}_\beta + \tilde{\Psi}_\beta - 2x_\beta(\partial^\top\cdot\tilde{\Psi}) - \partial^\top_\beta(\partial^\top\cdot\tilde{\Psi}) + \not{x}\,\not{\partial}^\top\tilde{\Psi}_\beta - x_\beta\,\not{x}\,\tilde{\not{\Psi}} - \not{x}\partial^\top_\beta\,\tilde{\not{\Psi}} = 0. \tag{A.13}$$

Finally, given $Q^{(1)}_{\frac{3}{2}}$ definition, we have:

$$\left(Q^{(1)}_{\frac{3}{2}} + \frac{5}{2}\right)\tilde{\Psi}_\alpha + \partial^\top_\alpha\left(\not{x}\tilde{\not{\Psi}} + \partial^\top\cdot\tilde{\Psi}\right) - 2\left(\gamma_\alpha\tilde{\not{\Psi}} + \not{x}\,\not{\partial}^\top\tilde{\Psi}_\alpha - \tilde{\Psi}_\alpha\right) = 0. \tag{A.14}$$

B. The auxiliary relationships

Here are some of the auxiliary relationships used in this article:

$$[\partial^\top_\alpha, \partial^\top_\beta] = x_\beta \partial^\top_\alpha - x_\alpha \partial^\top_\beta, \qquad [\partial^\top_\alpha, x_\beta] = \theta_{\alpha\beta},$$

$$[x_\alpha, \not{\partial}^\top] = -\gamma^\top_\alpha, \qquad [\gamma^\top_\alpha, \partial^\top_\alpha] = -4\ \not{x},$$

$$[Q_0, \not{x}] = -4\ \not{x} - 2\ \not{\partial}^\top, \qquad [x_\alpha, Q_0] = 2\partial^\top_\alpha + 4x_\alpha,$$

$$[\not{x}, \not{\partial}^\top] = 4 - 2\ \not{\partial}^\top\ \not{x}, \qquad \gamma^\top_\alpha = \gamma_\alpha + x_\alpha x \cdot \gamma,$$

$$[\not{x}, \partial^\top_\alpha] = -\gamma^\top_\alpha, \qquad [\not{x}, \gamma^\top_\alpha] = 2x_\alpha - 2\gamma_\alpha\ \not{x},$$

$$[\partial^\top_\alpha, \not{\partial}^\top] = \not{x}\partial^\top_\alpha - x_\alpha\ \not{\partial}^\top, \qquad [\gamma^\top_\alpha, \partial^\top_\alpha] = -4\ \not{x},$$

$$[\partial^\top_\alpha, Q_0] = -6\partial^\top_\alpha - 2(Q_0 + 4)x_\alpha, \qquad [\not{\partial}^\top, \gamma^\top_\alpha] = -2\gamma^\top_\alpha\ \not{\partial}^\top + 2\partial^\top_\alpha + \gamma^\top_\alpha\ \not{x} + 4x_\alpha,$$

$$[Q_0, \gamma^\top_\alpha] = -8x_\alpha\ \not{x} - 2\ \not{x}\partial^\top_\alpha - 2\gamma^\top_\alpha - 2x_\alpha\ \not{\partial}^\top.$$

Acknowlegements

We would like to express our heartfelt thank and sincere gratitude to Professor M.V. Takook and E. Yusofi for their helpful discussions. This work has been supported by the Islamic Azad University, Kermanshah Branch, Kermanshah, Iran.

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