

# Approximate solutions of the Schrödinger equation with Hulthén-Hellmann potentials for a quarkonium system

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Hulthén plus Hellmann potentials are adopted as the quark-antiquark interaction potential for studying the mass spectra of heavy mesons. We solved the radial Schrödinger equation analytically using the Nikiforov-Uvarov method. The energy eigenvalues and corresponding wave function in terms of Laguerre polynomials were obtained. The present results are applied for calculating the mass of heavy mesons such as charmonium  $c\bar{c}$  and bottomonium  $b\bar{b}$ . Four special cases were considered when some of the potential parameters were set to zero, resulting into Hellmann potential, Yukawa potential, Coulomb potential, and Hulthén potential, respectively. The present potential provides satisfying results in comparison with experimental data and the work of other researchers.

**Keywords:** Schrödinger equation; Nikiforov-Uvarov method; Hulthén potential; Hellmann potential; heavy mesons.

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## 1. Introduction

The study of the fundamental or constituent blocks of matter has been for long time a fascinating field in physics. In the nineteenth century, the atom was considered to be the fundamental particle, the one from which all matter was composed. This idea was used to explain the basic structure of all elements [1].

The problem of what were considered to be fundamental particles was resolved by the quarks. Because of the heavy masses of the constituent quarks, a good description of many features of these systems can be obtained using non-relativistic models where the quark-antiquark strong interaction is described by a phenomenological potential [2]. Heavy quarkonium systems have turned out to provide extremely useful probes for the deconfined state of matter because the force between a heavy quark and anti-quark is weakened due to the presence of gluons which lead to the dissociation of quarkonium bound states [3]. The quarkonia with heavy quark and antiquark and their interaction are well described by the Schrödinger equation (SE) [4]. The solution of the spectral problem for the SE with spherically symmetric potentials is of major concern in describing the spectra of quarkonia [5]. Potential models offer a rather good description of the mass spectra of systems such as a bottomonium, and charmonium [6]. In simulating the interaction potentials for these systems, confining-type potentials are generally used. The holding potential is the so-called Cornell potential with two terms, one of which is responsible for the Coulomb interaction of the quarks and the other corresponds to a confining term [7].

The solutions to the SE can be established only if we know the confining potential for a particular physical system. Till now, there are only a few confining potentials, like the harmonic oscillator and the hydrogen atom, for which solutions to the SE are found exactly [8].

The Hulthén potential takes the form [9]

$$V(r) = -\frac{A_0 e^{-\alpha r}}{1 - e^{-\alpha r}}, \quad (1)$$

where  $\alpha$  is the screening parameter and  $A_0$  is the potential strength which is sometimes identified with the atomic number when the potential is used for atomic phenomena [10]. It is a short-range potential which is applied in many branches of physics, such as nuclear and particle physics, atomic physics, solid state physics, and chemical physics [11,12].

The Hellmann potential which is a superposition of an attraction Coulomb potential and a Yukawa potential can be expressed as [13].

$$V(r) = -\frac{A_1}{r} + \frac{A_2 e^{-\alpha r}}{r}, \quad (2)$$

where the parameters  $A_1$  and  $A_2$  denote the strength of Coulomb and Yukawa potentials respectively,  $\alpha$  denotes the screening parameter, and  $r$  is the distance between two particles. These potentials have been used to study bound state problems by many researchers [14-20]. Recently, Inyang *et al.* [21] obtained the Klein-Gordon equation solutions for the Yukawa potential using the Nikiforov-Uvarov (NU) method. The energy eigenvalues were obtained both in a relativistic and non-relativistic regime. They applied the results to calculate heavy-meson masses of charmonium  $c\bar{c}$  and bottomonium  $b\bar{c}$ . Apart from that, many researchers have provided approximate solutions to SE using different methods with Cornell potential. For instance, Vega and Flores [22] obtained the approximate solutions of the Schrödinger equation with the Cornell potential using variational method and super symmetric quantum mechanics (SUSYQM). Abu-Shady *et al.* [23] studied the N-dimensional radial Schrödinger equation using the analytical exact iteration method (AEIM), in which the Cornell potential is generalized to finite temperature and chemical potential. In addition, Ciftci and Kisoglu

[24], solved non-relativistic arbitrary l-states of quarkonium through asymptotic iteration method (AIM). An analytic solution of the N-dimensional radial Schrödinger equation with the mixture of vector and scalar potentials via the Laplace transformation method (LTM) were studied in Ref. [25]. Al-Jamel and Widyan [26] studied heavy quarkonium mass spectra in a Coulomb field plus quadratic potential using NU method. Ibekwe *et al.* [27] solved the radial SE with an exponential, generalized, harmonic Cornell potential using the series expansion method. Their results were used to calculate the mass spectra of heavy-mesons. Al-Oun *et al.* [28] examine heavy quarkonia characteristics properties in the general framework of non-relativistic potential model consisting of a Coulomb plus quadratic potential. Chouikh *et al.* [29] proposed an approach to achieve quantum computation with atomics qubits in a cavity QED. Recently, researchers have shown great interest in the combination of two or more potentials in both the relativistic and non relativistic approach. The fundamental nature of combining two or more physical potential models is to have a wider range of application [30]. For example, the Cornell potential which is the combination of the Coulomb potential with linear terms is used in studying the mass spectra for coupled states and for the electromagnetic characteristics of meson [31]. For instance, William *et al.* [32] obtained bound state solutions of the radial Schrödinger equation by the combination of Hulthén and Hellmann potential within the framework of Nikiforov-Uvarov method. Also, Edet *et al.* [33] obtained an approximate solution of the SE for the modified Kratzer potential plus screened Coulomb potential model using the Nikiforov-Uvarov method. In this present work, we aim to study the SE with the combination of Hulthén and Hellmann potential analytically by using the NU method and apply the results to calculate the mass spectra of heavy quarkonium particles such as bottomonium and charmonium, in which the quarks are considered as spinless particles for easiness, which have not been considered before using this potentials to the best of our knowledge. The adopted potential is of the form [32]

$$V(r) = -\frac{A_0 e^{-\alpha r}}{1 - e^{-\alpha r}} - \frac{A_1}{r} + \frac{A_2 e^{-\alpha r}}{r}, \quad (3)$$

where  $A_0, A_1,$  and  $A_2$  are potential strength parameters and  $\alpha$  is the screening parameter. In other to make Eq. (3) temperature dependent, the screening parameter is replaced with the Debye mass  $m_D(T)$ , which is temperature-dependent and vanishes at  $T \rightarrow 0$  and we have,

$$V(r, T) = -\frac{A_0 e^{-m_D(T)r}}{1 - e^{-m_D(T)r}} - \frac{A_1}{r} + \frac{A_2 e^{-m_D(T)r}}{r}. \quad (4)$$

We carry out a series expansion of the exponential terms in Eq. (4) up to order three in order to model the potential to interact in the quark-antiquark system and this yields,

$$\frac{e^{-m_D(T)r}}{r} = \frac{1}{r} - m_D(T) + \frac{m_D^2(T)r^2}{6} + \dots \quad (5)$$

$$\frac{e^{-m_D(T)r}}{1 - e^{-m_D(T)r}} = \frac{1}{m_D(T)r} - \frac{1}{2} + \frac{m_D(T)r}{12} + \dots \quad (6)$$

We substitute Eqs. (5) and (6) into Eq. (4) and obtain

$$V(r, T) = -\frac{\beta_0}{r} + \beta_1 r - \beta_2 r^2 + \beta_3, \quad (7)$$

where

$$\begin{aligned} -\beta_0 &= A_2 - A_1 - \frac{A_0}{m_D(T)}, \\ \beta_1 &= \frac{A_2 m_D^2(T)r}{2} - \frac{A_0 m_D(T)}{12}, \\ \beta_2 &= \frac{A_2 m_D^3(T)}{6}, \\ \beta_3 &= \frac{A_0}{2} - A_2 m_D(T). \end{aligned} \quad (8)$$

The first term in Eq. (7) is the Coulomb potential that describes the short distance between quarks, while the second term is a linear term featuring confinement.

## 2. Approximate solutions of the Schrödinger equation with Hulthén plus Hellmann potential

The Schrödinger equation (SE) for two particles interacting via potential  $V(r)$  in three dimensional space, is given by [34]

$$\frac{d^2 R(r)}{dr^2} + \left( \frac{2\mu}{\hbar^2} (E_{nl} - V(r)) - \frac{l(l+1)}{r^2} \right) R(r) = 0, \quad (9)$$

where  $l, \mu, r,$  and  $\hbar$  are the angular momentum quantum number, the reduced mass for the quarkonium particle, interparticle distance and reduced plank constant respectively. We substitute Eq. (7) into Eq. (9) and obtain

$$\begin{aligned} \frac{d^2 R(r)}{dr^2} + \left[ \frac{2\mu E_{nl}}{\hbar^2} + \frac{2\mu\beta_0}{\hbar^2 r} - \frac{2\mu\beta_1 r}{\hbar^2} + \frac{2\mu\beta_2 r^2}{\hbar^2} \right. \\ \left. - \frac{2\mu\beta_3}{\hbar^2} - \frac{l(l+1)}{r^2} \right] R(r) = 0. \end{aligned} \quad (10)$$

Let,

$$\begin{aligned} \zeta &= \frac{2\mu}{\hbar^2} (E_{nl} - \beta_3), \quad \alpha_0 = \frac{2\mu\beta_0}{\hbar^2}, \\ \alpha_1 &= \frac{2\mu\beta_1}{\hbar^2}, \quad \alpha_2 = \frac{2\mu\beta_2}{\hbar^2}, \quad \gamma = l(l+1). \end{aligned} \quad (11)$$

Substituting Eq. (11) into Eq. (10), we have

$$\frac{d^2 R(r)}{dr^2} + \left( \zeta + \frac{\alpha_0}{r} - \alpha_1 r + \alpha_2 r^2 - \frac{\gamma}{r^2} \right) R(r) = 0. \quad (12)$$

Transforming the coordinate of Eq. (12) we set

$$x = \frac{1}{r}. \tag{13}$$

Differentiating Eq. (13) and simplifying we have

$$\frac{d^2 R}{dr^2} + \frac{2}{r^3} \frac{dR}{dx} + \frac{1}{r^4} \frac{d^2 R}{dx^2}. \tag{14}$$

Substituting Eqs. (13) and (14) into Eq. (12) we have

$$\begin{aligned} \frac{d^2 R(x)}{dx^2} + \frac{2}{x} \frac{dR}{dx} + \frac{1}{x^4} \left[ \zeta + \alpha_0 x \right. \\ \left. + \frac{\alpha_1}{x} + \frac{\alpha_2}{x^2} - \gamma x^2 \right] R(x) = 0. \end{aligned} \tag{15}$$

Next, we propose the following approximation scheme on the term  $\alpha_1/x$  and  $\alpha_2/x^2$ .

Let us assume that there is a characteristic radius  $r_0$  of the meson. Then the scheme is based on the expansion of  $\alpha_1/x$  and  $\alpha_2/x^2$  in a power series around  $r_0$ , i.e. around  $\delta \equiv 1/r_0$ , in the  $x$ -space up to the second order. This is similar to the Pekeris approximation, which helps to deform the centrifugal term such that the modified potential can be solved by the NU method [35].

Setting  $y = x - \delta$  and around  $y = 0$ , it can be expanded into a series of powers as;

$$\frac{\alpha_1}{x} = \frac{\alpha_1}{y + \delta} = \frac{\alpha_1}{\delta} \left( 1 + \frac{y}{\delta} \right)^{-1}, \tag{16}$$

which yields

$$\frac{\alpha_1}{x} = \alpha_1 \left( \frac{3}{\delta} - \frac{3x}{\delta^2} + \frac{x^2}{\delta^3} \right). \tag{17}$$

Similarly,

$$\frac{\alpha_2}{x^2} = \alpha_2 \left( \frac{6}{\delta^2} - \frac{8x}{\delta^3} + \frac{3x^2}{\delta^4} \right). \tag{18}$$

By substituting Eqs. (18) and (17) into Eq. (15), we obtain

$$\frac{d^2 R(x)}{dx^2} + \frac{2x}{x^2} \frac{dR(x)}{dx} + \frac{1}{x^4} [-\varepsilon + \alpha x - \beta x^2] R(x) = 0, \tag{19}$$

where

$$\begin{aligned} -\varepsilon &= \left( \zeta + \frac{6\alpha_2}{\delta^2} - \frac{3\alpha_1}{\delta} \right), \\ \alpha &= \left( \frac{3\alpha_1}{\delta^2} + \alpha_0 - \frac{8\alpha_2}{\delta^3} \right), \\ \beta &= \left( \gamma + \frac{\alpha_1}{\delta^3} - \frac{3\alpha_2}{\delta^4} \right). \end{aligned} \tag{20}$$

Comparing Eq. (19) and Eq. (A1) we obtain

$$\begin{aligned} \tilde{\tau}(x) &= 2x, \quad \sigma(x) = x^2 \\ \tilde{\sigma}(x) &= -\varepsilon + \alpha x - \beta x^2 \\ \sigma' &= 2x, \quad \sigma''(x) = 2. \end{aligned} \tag{21}$$

We substitute Eq. (21) into Eq. (A9) and obtain

$$\pi(x) = \pm \sqrt{\varepsilon - \alpha x + (\beta + k)x^2}. \tag{22}$$

To determine  $k$ , we take the discriminant of the function under the square root, which yields

$$k = \frac{\alpha^2 - 4\beta\varepsilon}{4\varepsilon}. \tag{23}$$

We substitute Eq. (23) into Eq. (22) and have

$$\pi(x) = \pm \left( \frac{\alpha x}{2\sqrt{\varepsilon}} - \frac{\varepsilon}{\sqrt{\varepsilon}} \right). \tag{24}$$

For a physically acceptable solution, we take the negative part of Eq. (24) which is required for bound state problems and differentiate; this yields

$$\pi'(x) = -\frac{\alpha}{2\sqrt{\varepsilon}}. \tag{25}$$

Substituting Eqs. (21) and (25) into Eq. (A7) we have

$$\tau(x) = 2x - \frac{\alpha x}{\sqrt{\varepsilon}} + \frac{2\varepsilon}{\sqrt{\varepsilon}}. \tag{26}$$

Differentiating Eq. (26) we have

$$\tau'(x) = 2 - \frac{\alpha}{\sqrt{\varepsilon}}. \tag{27}$$

By using Eq. (A10), we obtain

$$\lambda = \frac{\alpha^2 - 4\beta\varepsilon}{4\varepsilon} - \frac{\alpha}{2\sqrt{\varepsilon}}, \tag{28}$$

and using Eq. (A11), we obtain

$$\lambda_n = \frac{n\alpha}{\sqrt{\varepsilon}} - n^2 - n. \tag{29}$$

Equating Eqs. (28) and (29), the energy eigenvalues of Eq. (10) are given

$$E_{nl} = A_0 \left( \frac{1}{2} - \frac{m_D(T)}{4\delta} \right) + A_2 m_D(T) \left( \frac{3m_D(T)}{2\delta} - m_D^2(T) - 1 \right),$$

$$-\frac{\hbar^2}{8\mu} \left( \frac{\frac{2\mu}{\hbar^2} \left[ A_2 - A_1 + \frac{A_0}{m_D(T)} \right] + \frac{\mu m_D(T)}{\delta^2 \delta^2} \left[ 3A_2 m_D(T) - \frac{A_0}{2} \right] - \frac{8\mu A_2 m_D^3(T)}{3\hbar^2 \delta^3}}{n + \frac{1}{2} + \sqrt{\left[ l + \frac{1}{2} \right]^2 + \frac{\mu A_2 m_D^2(T)}{\hbar^2 \delta^3} \left[ 1 - \frac{m_D(T)}{\delta} \right] - \frac{\mu A_0 m_D(T)}{6\hbar^2 \delta^3}} \right). \tag{30}$$

**2.1. Special cases**

In this subsection, we obtain the special case by setting some parameters to zero.

1. When we set  $A_0 = A_1 = 0$ , we obtain the energy eigenvalues for Yukawa potential

$$E_{nl} = A_2 m_D(T) \left( \frac{3m_D(T)}{2\delta} - m_D^2(T) - 1 \right) - \frac{\hbar^2}{8\mu} \left( \frac{\frac{-2\mu A_2}{\hbar^2} + \frac{3\mu A_2 m_D^2(T)}{\hbar^2 \delta^2} - \frac{8\mu A_2 m_D^3(T)}{3\hbar^2 \delta^3}}{n + \frac{1}{2} + \sqrt{\left[ l + \frac{1}{2} \right]^2 + \frac{\mu A_2 m_D^2(T)}{\hbar^2 \delta^3} \left[ 1 - \frac{m_D(T)}{\delta} \right]}} \right)^2. \tag{31}$$

2. When we set  $A_1 = A - 2 = 0$ , we obtain the energy eigenvalue for Hulthén potential

$$E_{nl} = A_0 \left( \frac{1}{2} - \frac{m_D(T)}{4\delta} \right) - \frac{\hbar^2}{8\mu} \left( \frac{\frac{2\mu A_0}{\hbar^2 m_D(T)} - \frac{A_0 \mu m_D(T)}{2\hbar^2 \delta^2}}{n + \frac{1}{2} + \sqrt{\left[ l + \frac{1}{2} \right]^2 - \frac{\mu A_0 m_D^2(T)}{6\hbar^2 \delta^3}}} \right)^2. \tag{32}$$

3. When we set  $A_0$ , we obtain the energy eigenvalue for Hellmann potential

$$E_{nl} = A_2 m_D(T) \left( \frac{3m_D(T)}{2\delta} - m_D^2(T) - 1 \right) - \frac{\hbar^2}{8\mu} \left( \frac{\frac{2\mu}{\hbar^2} (A_1 - A_2) + \frac{3A_2 \mu m_D^2(T)}{\hbar^2 \delta^2} - \frac{8\mu A_2 m_D^3(T)}{3\hbar^2 \delta^3}}{n + \frac{1}{2} + \sqrt{\left[ l + \frac{1}{2} \right]^2 + \frac{\mu A_2 m_D^2(T)}{\hbar^2 \delta^3} \left( 1 - \frac{m_D(T)}{\delta} \right)}} \right)^2. \tag{33}$$

4. When we set  $A_0 = A_2 = m_D(T) = 0$ , we obtain the energy eigenvalues for Coulomb potential

$$E_{nl} = \frac{\mu A_1^2}{2\hbar^2 (n + l + 1)^2}. \tag{34}$$

The result of Eq. (34) is very consistent with the result obtained in Eq. (36) of Ref. [33].

To determine the wavefunction, we substitute Eqs. (21) and (24) into Eq. (A4) and obtain

$$\frac{d\phi}{\phi} = \left( \frac{\varepsilon}{x^2 \sqrt{\varepsilon}} - \frac{\alpha}{2x \sqrt{\varepsilon}} \right) dx. \tag{35}$$

Integrating Eq. (35), we obtain

$$\phi(x) = x^{-(\alpha/2\sqrt{\varepsilon})} e^{(\varepsilon/x\sqrt{\varepsilon})}. \tag{36}$$

By substituting Eqs. (21) and (24) into Eq. (A6) and integrating, we obtain

$$\rho(x) = x^{-(\alpha/\sqrt{\varepsilon})} e^{-(2\varepsilon/x\sqrt{\varepsilon})}. \tag{37}$$

Substituting Eqs. (21) and (37) into Eq. (A5) we have

$$y_n(x) = B_n e^{(2\varepsilon/x\sqrt{\varepsilon})} x^{(\alpha/\sqrt{\varepsilon})} \times \frac{d^n}{dx^n} \left( e^{(2\varepsilon/x\sqrt{\varepsilon})} x^{(2n - [\alpha/\sqrt{\varepsilon}])} \right). \tag{38}$$

The Rodrigues' formula of the associated Laguerre polynomials is

$$L_n^{(\alpha/\sqrt{\varepsilon})} \left( \frac{2\varepsilon}{x\sqrt{\varepsilon}} \right) = \frac{1}{n!} e^{(2\varepsilon/x\sqrt{\varepsilon})} x^{\alpha/\sqrt{\varepsilon}} \times \frac{d^n}{dx^n} \left( e^{-(2\varepsilon/x\sqrt{\varepsilon})} x^{(2n - [\alpha/\sqrt{\varepsilon}])} \right). \tag{39}$$

where

$$\frac{1}{n!} = B_n. \tag{40}$$

Hence,

$$y_n(x) \equiv L_n^{(\alpha/\sqrt{\varepsilon})} \left( \frac{2\varepsilon}{x\sqrt{\varepsilon}} \right). \tag{41}$$

Substituting Eqs. (36) and (41) into Eq. (A2) we obtain the wavefunction of Eq. (10) in terms of Laguerre polynomial as

$$\psi(x) = B_{nl} x^{-(\alpha/2\sqrt{\epsilon})} e^{-(\epsilon/x\sqrt{\epsilon})} L_n^{(\alpha/\sqrt{\epsilon})} \left( \frac{2\epsilon}{x\sqrt{\epsilon}} \right) \tag{42}$$

where  $N_{nl}$  is normalization constant, which can be obtained from

$$\int_0^\infty |B_{nl}(r)|^2 dr = 1. \tag{43}$$

### 3. Results

We calculate the mass spectra of the heavy quarkonium system such as charmonium and bottomonium that have the quark and antiquark flavor, and apply the following relation [36,37]

$$M = 2m + E_{nl}, \tag{44}$$

where  $m$  is quarkonium bare mass and  $E_{nl}$  is energy eigenvalues. By substituting Eq. (30) into Eq. (44) we obtain the mass spectra for Hulthén plus Hellmann potential as

$$M = 2m + A_0 \left( \frac{1}{2} - \frac{m_D(T)}{4\delta} \right) + A_2 m_D(T) \left( \frac{3m_D(T)}{2\delta} - m_D^2(T) - 1 \right) - \frac{\hbar^2}{8\mu} \left( \frac{\frac{2\mu}{\hbar^2} \left[ A_2 - A_1 + \frac{A_0}{m_D(T)} \right] + \frac{\mu m_D(T)}{\hbar^2 \delta^2} \left[ 3A_2 m_D(T) - \frac{A_0}{2} \right] - \frac{8\mu A_2 m_D^3(T)}{3\hbar^2 \delta^3}}{n + \frac{1}{2} + \sqrt{\left( l + \frac{1}{2} \right)^2 + \frac{\mu A_2 m_D^2(T)}{\hbar^2 \delta^3} \left( 1 - \frac{m_D(T)}{\delta} \right) - \frac{\mu A_0 m_D(T)}{6\hbar^2 \delta^3}}} \right)^2. \tag{45}$$

#### 3.1. Discussion of results

We calculate the mass spectra of charmonium and bottomonium for states from 1S to 1F, by using Eq. (45). The free parameters of Eq. (45) are fitted with experimental data by solving two algebraic equations. Experimental data are obtained from [38,39]. For bottomonium  $b\bar{b}$  and charmonium  $c\bar{c}$  systems we adopt the numerical values of these masses as

$m_b = 4.823$  and  $m_c = 1.209$ , respectively [40]. Then, the corresponding reduced mass are  $\mu_b = 2.4115$  and  $\mu_c = 0.6045$ . The Debye mass  $m_D(T)$  is taken as 1.52 GeV by fitting with experimental data. We note that calculation of mass spectra of charmonium and bottomonium are in a good agreement with experimental data as well as the work of other re-

TABLE I. Mass spectra of charmonium in (GeV) for Hulthén plus Hellmann potential, ( $m_c = 1.209$  GeV,  $\mu = 0.6045$  GeV,  $A_0 = -1.693$  GeV,  $A_1 = 20.654$  GeV,  $A_2 = 0.018$  GeV,  $\delta = 0.2$  GeV,  $m_D(T) = 1.52$  GeV,  $\hbar = 1$ ).

State	Present work	[35]	[24]	Experiment [38,39]
1S	3.096	3.096	3.096	3.096
2S	3.686	3.686	3.672	3.686
1P	3.521	3.255	3.521	3.525
2P	3.772	3.779	3.951	3.773
3S	4.040	4.040	4.085	4.040
4S	4.262	4.269	4.433	4.263
1D	3.768	3.504	3.800	3.770
2D	4.146	-	-	4.159
1F	3.962	-	-	-

TABLE II. Mass spectra of bottomonium in (GeV) for Hulthén plus Hellmann potential, ( $m_b = 4.823$  GeV,  $\mu = 2.4115$  GeV,  $A_0 = -1.591$  GeV,  $A_1 = 9.649$  GeV,  $A_2 = 0.028$  GeV,  $\delta = 0.25$  GeV,  $m_D(T) = 1.52$  GeV,  $\hbar = 1$ ).

State	Present work	[35]	[24]	Experiment [38,39]
1S	9.460	9.460	9.462	9.460
2S	10.023	10.023	10.027	10.023
1P	9.861	9.619	9.9630	9.899
2P	10.238	10.114	10.299	10.260
3S	10.355	10.355	10.361	10.355
4S	10.579	10.567	10.624	10.580
1D	10.143	9.864	10.209	10.164
2D	10.306	-	-	-
1F	10.209	-	-	-

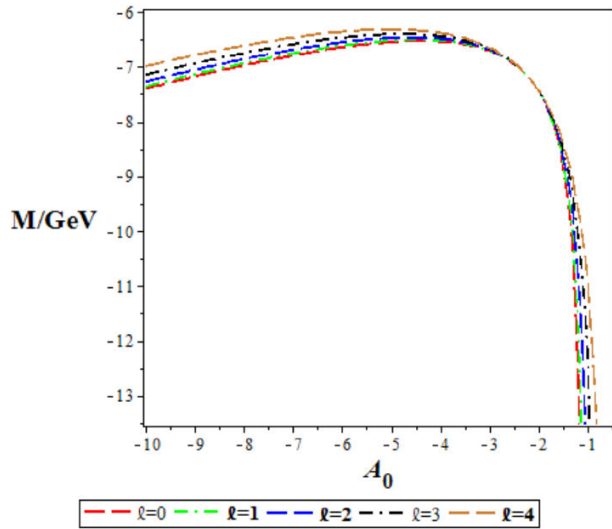


FIGURE 1. Mass spectra variation with potential parameter  $A_0$  for different quantum numbers.

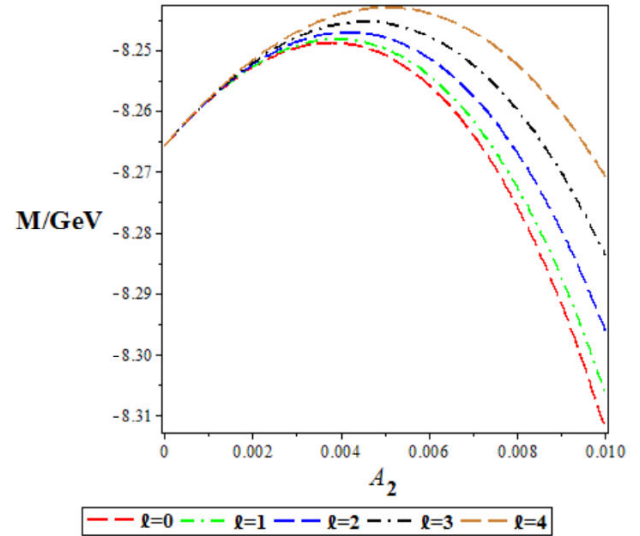


FIGURE 3. Mass spectra variation with potential parameter  $A_2$  for different quantum numbers.

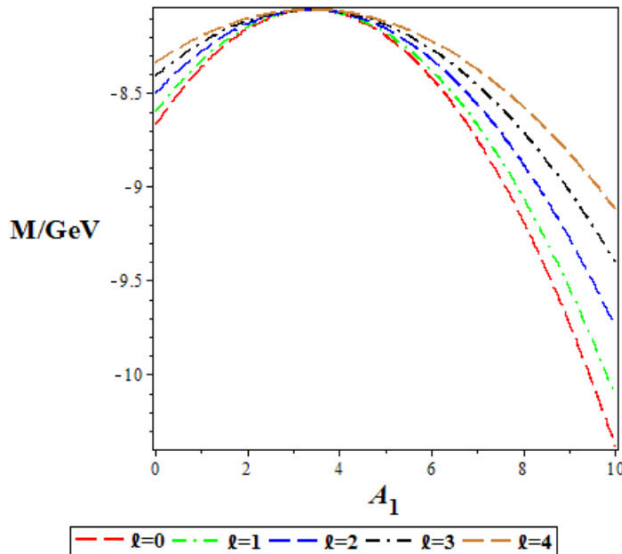


FIGURE 2. Mass spectra variation with potential parameter  $A_1$  for different quantum numbers

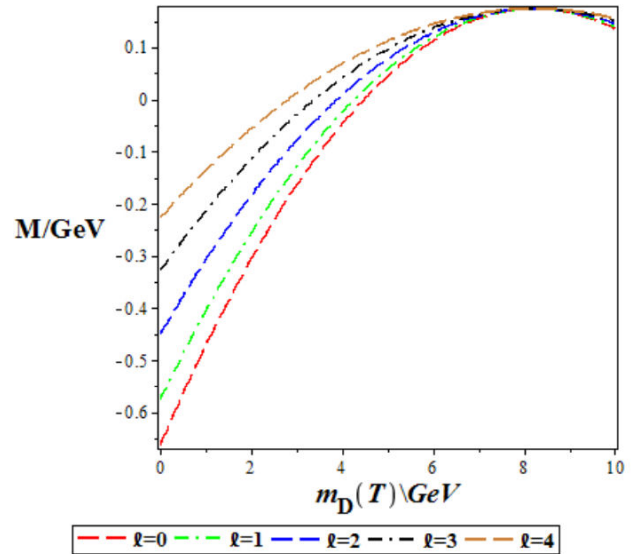


FIGURE 4. Mass spectra variation with the Debye mass  $m_D(T)$  for different quantum numbers.

searches as presented in Tables I and II. It is important to note that the values obtained are improved in comparison with works like that of Ref. [35], as shown in Tables I and II in which the author investigated the N- radial SE analytically when the Cornell potential was extended to finite temperature.

We also plotted the mass spectra energy as a function of potential parameters and Debye mass. In Figs. 1 and 2, the mass spectra energies increases to a peak and later decreases as potential parameters and increases, respectively. In Fig. 3 the mass spectra converges at the beginning, but spreads out and decrease monotonically with the increase in potential parameter  $A_2$ . Figure 4 show the increase in mass spectra as the Debye mass increases, for various angular quantum numbers.

#### 4. Conclusion

In this study, we adopted Hulthén plus Hellmann potential models for quark-antiquark interaction. The potential was made to be temperature dependent by replacing the screening parameter with Debye mass which vanishes at. The Schrödinger equation is analytically solved using the Nikiforov-Uvarov method. We obtained approximate solutions of the eigenvalues and eigenfunction in terms of Laguerre polynomials. We applied the present results to calculate heavy-meson masses such as charmonium  $c\bar{c}$ , and bottomonium  $b\bar{b}$  for states 1S to 1F which are in good agreement with experimental data and the work of others. Four special

cases were considered when some of the potential parameters were set to zero, resulting into Hellmann potential, Yukawa potential, Coulomb potential, and Hulthén potential, respectively. Different plots of mass spectra versus different potential parameters and Debye mass were analyzed and discussed.

## Appendix

### A. Review of Nikiforov-Uvarov (NU) method

The NU method was proposed by Nikiforov and Uvarov [41-43] to transform Schrödinger-like equations into a second-order differential equation via a coordinate transformation  $x = x(r)$ , of the form

$$\psi''(x) + \frac{\tilde{\tau}(x)}{\sigma(x)}\psi'(x) + \frac{\tilde{\sigma}(x)}{\sigma^2(x)}\psi(x) = 0, \quad (\text{A.1})$$

where  $\tilde{\sigma}(x)$ , and  $\sigma(x)$  are polynomials, at most second degree and  $\tilde{\tau}(x)$  is a first-degree polynomial. The exact solution of Eq. (A.1) can be obtained by using the transformation.

$$\psi(x) = \phi(x)y(x). \quad (\text{A.2})$$

This transformation reduces Eq. (A.1) into a hypergeometric-type equation of the form

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda y(x) = 0. \quad (\text{A.3})$$

The function  $\phi(x)$  can be defined as the logarithm derivative

$$\frac{\phi'(x)}{\phi(x)} = \frac{\pi(x)}{\sigma(x)}. \quad (\text{A.4})$$

With  $\pi(x)$  being at most a first-degree polynomial. The second part of  $\psi(x)$  being  $y(x)$  in Eq. (A.2) is the hypergeometric function with its polynomial solution given by Rodrigues relation as

$$y(x) = \frac{B_{nl}}{\rho(x)} \frac{d^n}{dx^n} [\sigma^n(x)\rho(x)], \quad (\text{A.5})$$

where  $B_{nl}$  is the normalization constant and  $\rho(x)$  the weight function which satisfies the condition below;

$$(\sigma(x)\rho(x))' = \tau(x)\rho(x), \quad (\text{A.6})$$

where also

$$\tau(x) = \tilde{\tau}(x) + 2\pi(x). \quad (\text{A.7})$$

For bound solutions, it is required that

$$\tau'(x) < 0. \quad (\text{A.8})$$

The eigenfunctions and eigenvalues can be obtained using the definition of the following function  $\pi(x)$  and parameter  $\lambda$ , respectively:

$$\pi(x) = \frac{\sigma'(x) - \tilde{\tau}(x)}{2} \pm \sqrt{\left(\frac{\sigma'(x) - \tilde{\tau}(x)}{2}\right)^2 - \tilde{\sigma}(x) + k\sigma(x)} \quad (\text{A.9})$$

and

$$\lambda = k_- + \pi'_-(x). \quad (\text{A.10})$$

The value of  $k$  can be obtained by setting the discriminant in the square root in Eq. (A.9) equal to zero. As such, the new eigenvalues equation can be given as

$$\lambda + n\tau'(x) + \frac{n(n-1)}{2}\sigma''(x) = 0, \quad (\text{A.11})$$

$$(n = 0, 1, 2, \dots).$$

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