Solutions of generalized fractional perturbed Zakharov-Kuznetsov equation arising in a magnetized dusty plasma

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The generalized fractional perturbed $(3+1)$-dimensional Zakharov-Kuznetsov (PZK) equation, which appears in the magnetized two-ion-temperature dusty plasma, and quantum physics is considered. The sub-equation method in the conformable sense is proposed to obtain exact solutions to this equation. The new solutions obtained by the proposed method are dark soliton, multi-soliton, solitary wave, kink-shape, bell-shaped soliton, and periodic solutions that are substantial in the field of mathematical physics and can be of relevance in the field of plasma physics, also for future research.

Keywords: Perturbed $(3+1)$-dimensional Zakharov-Kuznetsov equation; conformable derivative; sub-equation method; Riccati equation; mathematical physics.

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1. Introduction

The nonlinear evolution equations have played a fundamental role in the field of mathematical physics and many other areas of applied science. As an instance, in signal processing, dusty plasma, fluid dynamics, nonlinear optics, quantum mechanics, biology, and so forth [1–6]. Examining the exact solutions of these nonlinear models help us to understand the mechanism, application and gives better knowledge of the model. With the virtue of these solutions, a better vision can be captured into the physical feature of the considered model. In recent years, series of methods have been developed to obtain the exact and approximate solutions of the nonlinear evolution equations in mathematical physics, as an instance, the Jacobi elliptic function method [7], Adomian decomposition method [8], sine-cosine method [9, 10], first integral method [11, 12], variational iteration method [13, 14], extended tanh method [15, 16], q-homotopy analysis method [17–19], exp-function method [20–22], q-homotopy analysis transform method [23–25], tanh-sech method [26, 27], homotopy perturbation method [28], (G'/G)-expansion method [29, 30], fractional reduced differential transform method [31, 32], homogeneous balance method [33, 34], inverse scattering method [35], iterative shehu transform method [36], homotopy analysis method [37, 38], Jacobi elliptic expansion method [39, 40], residual power series method [41–43], perturbation-iteration algorithm [44, 45], modified Kudryashov method [46], new extended direct algebraic method [47], Sardar sub-equation method [48], Sine-cosine and sinh-cosh techniques [49], simple equation method [50] and so on.

The $(3+1)$-dimensional Zakharov-Kuznetsov (ZK) equation which comprises of the nonlinear term "PP_x" and third-order dispersion term "P_{xxx}" is restricted to the waves of small amplitudes only is presented by

$$\Delta := P_t + A\,PP_x + B\,P_{xxx} + C\big(P_{xyy} + P_{xzz}\big) = 0, \quad (1)$$

where P represents electrostatic potential, the physical quantities A, B, and C are constants. Seadawy *et al*. [51, 52] and Zhen *et al*. [53] have outlined these physical quantities. The width of the soliton and its velocity deviate from the predictions of this equation when the amplitude of the wave increases. As a result, an additional fifth-order dispersion term which is a higher-order dispersion term, "$\mathcal{E}P_{xxxxx}$", is added to Eq. (1) to overcome this problem (see [51–54], for more detailed). The new perturbed $(3+1)$-dimensional ZK equation reads

$$\Delta : = P_t + A\,PP_x + B\,P_{xxx} + C\big(P_{xyy} + P_{xzz}\big) + \mathcal{E}\,P_{xxxxx} = 0. \quad (2)$$

In this present study, our objective is to further complement the previous studies conducted on the perturbed $(3+1)$-dimensional ZK equation by introducing the more general form by replacing the nonlinear term "PP_x" with "P^kP_x". We now consider the generalized fractional form of this equation as

$$\mathcal{D}_t^{\gamma}P + A\,P^kP_x + B\,P_{xxx} + C\big(P_{xyy} + P_{xzz}\big) + \mathcal{E}\,P_{xxxxx} = 0, \quad 0<\gamma\leq 1, \;\; t>0, \quad (3)$$

where k is a positive number, γ is the fractional-order and $\mathcal{E}$ is a smallness parameter. For this purpose, we have carefully proposed the sub-equation method of conformable type to find analytic closed-form solutions of Eq. (3). The solutions consist of the dark soliton, multi-soliton, kink-shape, solitary wave, bell-shaped solitons, and periodic solutions, which are all substantial in the field of mathematical physics.

When $\gamma = 1$, $k = 1$, and $\mathcal{E} = 0$, Eq. (3) reduces to the standard Zakharov?Kuznetsov (ZK) equation (see Eq. (1). For $\gamma = 1$, $k = 2$, and $\mathcal{E} = 0$, Eq. (3) reduces to the modified KdV-ZK equation developed for a plasma comprised of cool and hot electrons and a species of fluid ions [55]. The case when $k = 1, 2$ and 4 are considered in this present study. It is worth mentioning that the case when $\gamma = 1$ and $k = 1$ in Eq. (3) have been investigated by the following authors. Elwakil *et al*. in Ref. [56] have studied the electron-acoustic solitary waves in a magnetized collisionless plasma consisting of a cold electron fluid and non-thermal hot electrons obeying a non-thermal distribution, and stationary. Ali *et al.* in Ref. [57] have investigated the exact solutions by using a sine-cosine method and modified Kudryashov methods and constructed six Lie point symmetries and nonlocal conservation laws for this equation. Recently, in Ref. [58], using two methods, Lie symmetry analysis and generalized exponential rational function. The authors present new exact solutions through optimal systems of one-dimensional Lie subalgebras; some solitary waves depict single soliton, multi-soliton, and annihilation profiles and six other closed-form solutions. To our knowledge, the case when $k = 2$ and $k = 4$ have not been studied before.

The rest of the paper is organized as follows: Section 2 gives a brief discussion of conformable derivatives which includes the definitions, basic properties, lemmas, and theorems. Section 3 presents the general idea of the sub-equation method. In Sec. 4, the application of sub-equation to time-fractional perturbed $(3+1)$-dimensional ZK equation of conformable type is demonstrated. In Sec. 5, the graphical representation of some solutions is depicted in 3D for different fractional orders. Finally, Sec. 6 gives the conclusion.

2. Preliminaries

This section contains a brief discussion of conformable derivatives, which includes the definitions, basic properties, lemmas, and theorems. Most of the concepts presented in this section have been introduced in [59, 60]

Definition 2.1. *Let $P : [0,\infty) \rightarrow \mathbb{R}$. The conformable derivative of P of order γ is given by*

$$\mathcal{D}^{\gamma}P(t) = \lim_{\varepsilon \to 0} \frac{P(t + \varepsilon t^{1-\gamma}) - P(t)}{\varepsilon}, \ \forall\, t > 0, \ \gamma \in (0, 1). \tag{4}$$

Furthermore, If P is γ-differentiable in some interval $(0, \zeta)$ where $\zeta > 0$, and $\lim_{t\to 0} P^{(\gamma)}(t)$ exists. We define

$$P^{(\gamma)}(0) = \lim_{t \to 0^+} P^{(\gamma)}(t). \tag{5}$$

Lemma 2.2. [59] *Let $\gamma \in (0, 1]$ and P, Q be γ-differentiable at a point $t > 0$. Then*

(i.) $\mathcal{D}^{\gamma}(\delta_1 P + \delta_2 Q) = \delta_1 \mathcal{D}^{\gamma}P + \delta_2 \mathcal{D}^{\gamma}Q, \ \forall\ \delta_1, \delta_2 \in \mathbb{R}$.

(ii.) $\mathcal{D}^{\gamma}(t^{\sigma}) = \sigma t^{\sigma-\gamma}, \ \forall\ \sigma \in \mathbb{R}$.

(iii.) $\mathcal{D}^{\gamma}(PQ) = P\mathcal{D}^{\gamma}Q + Q\mathcal{D}^{\gamma}P$.

(iv.) $\mathcal{D}^{\gamma}\left(\frac{P}{Q}\right) = \frac{Q\mathcal{D}^{\gamma}P - P\mathcal{D}^{\gamma}Q}{Q^2}$, *provided* $Q \neq 0$.

(v.) $\mathcal{D}^{\gamma}(C) = 0$, *where C is a constant.*

Lemma 2.3. [59] *Let P be a differentiable and γ-differentiable function. Then*

$$\mathcal{D}^{\gamma}P(t) = t^{1-\gamma}\frac{\partial P(t)}{\partial t}. \tag{6}$$

Theorem 2.4. [61, 62] *Let $P : (0,\infty) \rightarrow \mathcal{R}$ be a differentiable and γ-differentiable function. Let Q be a differentiable function defined in the range of P. Then*

$$\mathcal{D}_t^{\gamma}\big(P \circ Q\big)(t) = t^{1-\gamma}Q(t)^{\gamma-1} \times Q'(t)\mathcal{D}_t^{\gamma}\big(P(\Phi)\big)|_{\{\Phi = Q(t)\}}. \tag{7}$$

3. Algorithm of the proposed method

The main concept of the sub-equation method [63, 64] for solving FPDEs is presented below:

$$F\big(P, P_x, P_y, P_z, P_{xx}, P_{yy}, P_{zz}, \cdots, \mathcal{D}_t^{\gamma}P\big) = 0. \tag{8}$$

Here, $\mathcal{D}_t^{\gamma}$ is defined according to Sec. 2 with fractional order γ and an unknown function $P = P(x, y, z, t)$.

Step 1: Let $P(x, y, z, t) = P(\xi)$, $\xi = px + qy + rz + s\,\frac{t^{\gamma}}{\gamma}$, where p, q, r and s are constants to be calculated respectively. With the help of the chain rule, we can transform Eq. (8) to an integer order nonlinear ODE given as

$$G\big(P(\xi), P'(\xi), P''(\xi), P'''(\xi), \cdots\big) = 0. \tag{9}$$

Step 2: We suppose that Eq. (9) has the solution:

$$P(\xi) = e_0 + \sum_{j=1}^{N} e_j \vartheta^j(\xi), \ \ e_N \neq 0, \tag{10}$$

where $p,\ q,\ r,\ s,$ and $e_j(j = 0, 1, \cdots, N)$, are constants to be obtained accordingly. Balancing the highest-order derivative and biggest nonlinear term in Eq. (9) provides value for the integer N. The function $\vartheta(\xi)$ satisfies the Riccati equation:

$$\vartheta'(\xi) = \sigma + \vartheta^2(\xi), \tag{11}$$

where σ is a constant and the solutions are given as

$$\vartheta(\xi) = \begin{cases} -\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\,\xi), & \sigma < 0, \\ -\sqrt{-\sigma}\coth(\sqrt{-\sigma}\,\xi), & \sigma < 0, \\ \sqrt{\sigma}\tan(\sqrt{\sigma}\,\xi), & \sigma > 0, \\ -\sqrt{\sigma}\cot(\sqrt{\sigma}\,\xi), & \sigma > 0, \\ -\frac{1}{\xi+\phi},\ \ \phi \quad \text{is a constant,} & \sigma = 0. \end{cases} \tag{12}$$

Step 3: Substituting Eqs. (10) and (11) into Eq. (9), we have some polynomial in $\vartheta^j(\xi),\ (j = 0, 1, 2\cdots)$. Furthermore, setting the coefficients of $\vartheta^j(\xi)$ to zero, results set of nonlinear algebraic equations in $p,\ q,\ r,\ s,$ and $e_j(j = 0, 1, \cdots, N)$.

Step 4: Finally, by solving the equations found in Step 3. Then, substitute these constants $p,\ q,\ r,\ s,\ \sigma$ and e_j into Eq. (10) in addition to Eq. (12), we conclude the desirable solutions for Eq. (8) immediately.

4. Sub-Equation Method to PZK Equations of Comformable Type

Here, the application of the sub-equation method to the time-fractional perturbed $(3 + 1)$-dimensional ZK equation of conformable type is presented. Consider

$$\mathcal{D}_t^{\gamma} P + A\,P^k P_x + B\,P_{xxx} + C\big(P_{xyy} + P_{xzz}\big) + \mathcal{E}\,P_{xxxxx} = 0. \tag{13}$$

Application of chain rule on $P(x, y, z, t) = P(\xi),\ \xi = px + qy + rz + s\frac{t^{\gamma}}{\gamma}$ reduces Eq. (13) to a nonlinear ODE given as

$$\big(s + pAP^k\big)P' + \big(p^3 B + C(p\,q^2 + p\,r^2)\big)P''' + p^5\mathcal{E}\,P''''' = 0. \tag{14}$$

Balance principle to the terms P^kP' and P''''' yields $N = 4/k$. To obtain closed-form solutions, N must be an integer; therefore, we choose $k = 1,\ 2$ and 4.

4.1. Solutions for $k = 1$:

For $k = 1$, we have $N = 4$, then Eq. (10) gives

$$P(\xi) = e_0 + e_1\vartheta(\xi) + e_2\vartheta^2(\xi) + e_3\vartheta^3(\xi) + e_4\vartheta^4(\xi). \tag{15}$$

By substituting Eqs. (11) and (15) into Eq. (14), then equating the coefficients of $\vartheta^r(\xi)$ to zero yields

$$\begin{aligned}
\vartheta^0(\xi) &: Ae_0e_1p\sigma + 6e_3\sigma^3\left(Bp^3+Cpq^2+Cpr^2\right) + 2e_1\sigma^2\left(Bp^3+Cpq^2+Cpr^2\right) + 120\mathcal{E}e_3p^5\sigma^4 \\
&\quad + 16\mathcal{E}e_1p^5\sigma^3 + e_1s\sigma = 0, \\
\vartheta^1(\xi) &: Ae_1^2p\sigma + 2Ae_0e_2p\sigma + 24e_4\sigma^3\left(Bp^3+Cpq^2+Cpr^2\right) + 16e_2\sigma^2\left(Bp^3+Cpq^2+Cpr^2\right) \\
&\quad + 960\mathcal{E}e_4p^5\sigma^4 + 272\mathcal{E}e_2p^5\sigma^3 + 2e_2s\sigma = 0, \\
\vartheta^2(\xi) &: 3Ae_1e_2p\sigma + 3Ae_0e_3p\sigma + Ae_0e_1p + 60e_3\sigma^2\left(Bp^3+Cpq^2+Cpr^2\right) \\
&\quad + 8e_1\sigma\left(Bp^3+Cpq^2+Cpr^2\right) + 1848\mathcal{E}e_3p^5\sigma^3 + 136\mathcal{E}e_1p^5\sigma^2 + 3e_3s\sigma + e_1s = 0, \\
\vartheta^3(\xi) &: 2Ae_2^2p\sigma + 4Ae_1e_3p\sigma + 4Ae_0e_4p\sigma + Ae_1^2p + 2Ae_0e_2p \\
&\quad + 152e_4\sigma^2\left(Bp^3+Cpq^2+Cpr^2\right) + 40e_2\sigma\left(Bp^3+Cpq^2+Cpr^2\right) \\
&\quad + 7744\mathcal{E}e_4p^5\sigma^3 + 1232\mathcal{E}e_2p^5\sigma^2 + 4e_4s\sigma + 2e_2s = 0, \\
\vartheta^4(\xi) &: 5Ae_2e_3p\sigma + 5Ae_1e_4p\sigma + 3Ae_1e_2p + 3Ae_0e_3p + 114e_3\sigma\left(Bp^3+Cpq^2+Cpr^2\right) \\
&\quad + 6e_1\left(Bp^3+Cpq^2+Cpr^2\right) + 5808\mathcal{E}e_3p^5\sigma^2 + 240\mathcal{E}e_1p^5\sigma + 3e_3s = 0, \\
\vartheta^5(\xi) &: 3Ae_3^2p\sigma + 6Ae_2e_4p\sigma + 2Ae_2^2p + 4Ae_1e_3p + 4Ae_0e_4p + 248e_4\sigma\left(Bp^3+Cpq^2+Cpr^2\right) \\
&\quad + 24e_2\left(Bp^3+Cpq^2+Cpr^2\right) + 19264\mathcal{E}e_4p^5\sigma^2 + 1680\mathcal{E}e_2p^5\sigma + 4e_4s = 0, \\
\vartheta^6(\xi) &: 7Ae_3e_4p\sigma + 5Ae_2e_3p + 5Ae_1e_4p + 60e_3\left(Bp^3+Cpq^2+Cpr^2\right) + 6600\mathcal{E}e_3p^5\sigma + 120\mathcal{E}e_1p^5 = 0, \\
\vartheta^7(\xi) &: 4Ae_4^2p\sigma + 3Ae_3^2p + 6Ae_2e_4p + 120e_4\left(Bp^3+Cpq^2+Cpr^2\right) + 19200\mathcal{E}e_4p^5\sigma + 720\mathcal{E}e_2p^5 = 0, \\
\vartheta^8(\xi) &: 7Ae_3e_4p + 2520\mathcal{E}e_3p^5 = 0, \\
\vartheta^9(\xi) &: 4Ae_4^2p + 6720\mathcal{E}e_4p^5 = 0.
\end{aligned} \tag{16}$$

With the aid of Mathematica, we solve the above equations as
Case 1.

$$\begin{gathered}
e_0 = e_0, \quad e_1 = 0, \quad e_2 = -\frac{3360\mathcal{E}p^4\sigma}{A}, \quad e_3 = 0, \quad e_4 = -\frac{1680\mathcal{E}p^4}{A}, \\
s = -Ae_0p - 1104\mathcal{E}p^5\sigma^2, \quad r = \pm\sqrt{\frac{-Bp^2 - Cq^2 + 52\mathcal{E}p^4\sigma}{C}}.
\end{gathered} \tag{17}$$

By substituting Eq. (17) into Eq. (15) with solutions defined in Eq. (12), we have the required solutions of Eq. (13) as:

$$P_1 = e_0 + \frac{3360\mathcal{E}p^4\sigma^2}{A}\tanh^2\left(\sqrt{-\sigma}\left[px+qy+rz+\frac{s}{\gamma}t^\gamma\right]\right) - \frac{1680\mathcal{E}p^4\sigma^2}{A}\tanh^4\left(\sqrt{-\sigma}\left[px+qy+rz+\frac{s}{\gamma}t^\gamma\right]\right), \quad \sigma > 0,$$

$$P_2 = e_0 + \frac{3360\mathcal{E}p^4\sigma^2}{A}\coth^2\left(\sqrt{-\sigma}\left[px+qy+rz+\frac{s}{\gamma}t^\gamma\right]\right) - \frac{1680\mathcal{E}p^4\sigma^2}{A}\coth^4\left(\sqrt{-\sigma}\left[px+qy+rz+\frac{s}{\gamma}t^\gamma\right]\right), \quad \sigma > 0,$$

$$P_3 = e_0 - \frac{3360\mathcal{E}p^4\sigma^2}{A}\tan^2\left(\sqrt{\sigma}\left[px+qy+rz+\frac{s}{\gamma}t^\gamma\right]\right) - \frac{1680\mathcal{E}p^4\sigma^2}{A}\tan^4\left(\sqrt{\sigma}\left[px+qy+rz+\frac{s}{\gamma}t^\gamma\right]\right), \quad \sigma < 0,$$

$$P_4 = e_0 - \frac{3360\mathcal{E}p^4\sigma^2}{A}\cot^2\left(\sqrt{\sigma}\left[px+qy+rz+\frac{s}{\gamma}t^\gamma\right]\right) - \frac{1680\mathcal{E}p^4\sigma^2}{A}\cot^4\left(\sqrt{\sigma}\left[px+qy+rz+\frac{s}{\gamma}t^\gamma\right]\right), \quad \sigma < 0,$$

$$P_5 = e_0 - \frac{1680\mathcal{E}p^4}{A\left(px+qy\pm\sqrt{\frac{-Bp^2-Cq^2}{C}}z - \frac{Ae_0p}{\gamma}t^\gamma + \phi\right)^4}, \quad \sigma = 0,$$

where ϕ is a constant, $s = -Ae_0p - 1104\mathcal{E}p^5\sigma^2$ and $r = \pm\sqrt{(-Bp^2 - Cq^2 + 52\mathcal{E}p^4\sigma)/C}$.

Case 2.

$$e_0 = e_0, \quad e_1 = 0, \quad e_2 = \frac{1680i\left(\sqrt{31}\mathcal{E}p^4\sigma + 31i\mathcal{E}p^4\sigma\right)}{31A}, \quad e_3 = 0, \quad e_4 = -\frac{1680\mathcal{E}p^4}{A},$$
$$s = \frac{1}{31}\big(-31Ae_0p - 3720\mathcal{E}p^5\sigma^2 + 1368i\sqrt{31}\mathcal{E}p^5\sigma^2\big), \quad r = \pm\sqrt{-\frac{Bp^2}{C} - \frac{26\mathcal{E}p^4\sigma}{C} - \frac{78i\mathcal{E}p^4\sigma}{\sqrt{31}C} - q^2}. \tag{18}$$

The required solutions for Eq. (13 are

$$\begin{aligned}
P_6 &= e_0 - \frac{1680i\sigma\left(\sqrt{31}\mathcal{E}p^4\sigma + 31i\mathcal{E}p^4\sigma\right)}{31A}\tanh^2\left(\sqrt{-\sigma}\Big[px+qy+rz+\frac{s}{\gamma}t^\gamma\Big]\right) \\
&\quad - \frac{1680\mathcal{E}p^4\sigma^2}{A}\tanh^4\left(\sqrt{-\sigma}\Big[px+qy+rz+\frac{s}{\gamma}t^\gamma\Big]\right), \qquad \sigma<0, \\
P_7 &= e_0 - \frac{1680i\sigma\left(\sqrt{31}\mathcal{E}p^4\sigma + 31i\mathcal{E}p^4\sigma\right)}{31A}\coth^2\left(\sqrt{-\sigma}\Big[px+qy+rz+\frac{s}{\gamma}t^\gamma\Big]\right) \\
&\quad - \frac{1680\mathcal{E}p^4\sigma^2}{A}\coth^4\left(\sqrt{-\sigma}\Big[px+qy+rz+\frac{s}{\gamma}t^\gamma\Big]\right), \qquad \sigma<0, \\
P_8 &= e_0 + \frac{1680i\sigma\left(\sqrt{31}\mathcal{E}p^4\sigma + 31i\mathcal{E}p^4\sigma\right)}{31A}\tan^2\left(\sqrt{\sigma}\Big[px+qy+rz+\frac{s}{\gamma}t^\gamma\Big]\right) \\
&\quad - \frac{1680\mathcal{E}p^4\sigma^2}{A}\tan^4\left(\sqrt{\sigma}\Big[px+qy+rz+\frac{s}{\gamma}t^\gamma\Big]\right), \qquad \sigma>0, \\
P_9 &= e_0 + \frac{1680i\sigma\left(\sqrt{31}\mathcal{E}p^4\sigma + 31i\mathcal{E}p^4\sigma\right)}{31A}\cot^2\left(\sqrt{\sigma}\Big[px+qy+rz+\frac{s}{\gamma}t^\gamma\Big]\right) \\
&\quad - \frac{1680\mathcal{E}p^4\sigma^2}{A}\cot^4\left(\sqrt{\sigma}\Big[px+qy+rz+\frac{s}{\gamma}t^\gamma\Big]\right), \qquad \sigma>0, \\
P_{10} &= e_0 - \frac{1680\mathcal{E}p^4}{A\left(px+qy\pm z\sqrt{-\frac{Bp^2}{C}-q^2} - \frac{Ae_0p}{\gamma}t^\gamma+\phi\right)^4}, \qquad \sigma=0,
\end{aligned}$$

where $s = \frac{1}{31}\big(-31Ae_0p - 3720\mathcal{E}p^5\sigma^2 + 1368i\sqrt{31}\mathcal{E}p^5\sigma^2\big)$ and $r = \pm\sqrt{-\frac{Bp^2}{C} - \frac{26\mathcal{E}p^4\sigma}{C} - \frac{78i\mathcal{E}p^4\sigma}{\sqrt{31}C} - q^2}$.

Case 3.

$$e_0 = e_0, \quad e_1 = 0, \quad e_2 = -\frac{1680i\left(\sqrt{31}\mathcal{E}p^4\sigma - 31i\mathcal{E}p^4\sigma\right)}{31A}, \quad e_3 = 0, \quad e_4 = -\frac{1680\mathcal{E}p^4}{A},$$
$$s = \frac{1}{31}\Big(-31Ae_0p - 1368i\sqrt{31}\mathcal{E}p^5\sigma^2 - 3720\mathcal{E}p^5\sigma^2\Big), \quad r = \pm\sqrt{-\frac{Bp^2}{C} + \frac{78i\mathcal{E}p^4\sigma}{\sqrt{31}C} - \frac{26\mathcal{E}p^4\sigma}{C} - q^2}. \tag{19}$$

The required solutions for Eq. (13 are

$$P_{11}=e_0+\frac{1680i\sigma\left(\sqrt{31}\mathcal{E}p^4\sigma-31i\mathcal{E}p^4\sigma\right)}{31A}\tanh^2\left(\sqrt{-\sigma}\Big[px+qy+rz+\frac{s}{\gamma}t^{\gamma}\Big]\right)$$
$$-\frac{1680\mathcal{E}p^4\sigma^2}{A}\tanh^4\left(\sqrt{-\sigma}\Big[px+qy+rz+\frac{s}{\gamma}t^{\gamma}\Big]\right),\qquad \sigma<0,$$
$$P_{12}=e_0+\frac{1680i\sigma\left(\sqrt{31}\mathcal{E}p^4\sigma-31i\mathcal{E}p^4\sigma\right)}{31A}\coth^2\left(\sqrt{-\sigma}\Big[px+qy+rz+\frac{s}{\gamma}t^{\gamma}\Big]\right)$$
$$-\frac{1680\mathcal{E}p^4\sigma^2}{A}\coth^4\left(\sqrt{-\sigma}\Big[px+qy+rz+\frac{s}{\gamma}t^{\gamma}\Big]\right),\qquad \sigma<0,$$
$$P_{13}=e_0-\frac{1680i\sigma\left(\sqrt{31}\mathcal{E}p^4\sigma-31i\mathcal{E}p^4\sigma\right)}{31A}\tan^2\left(\sqrt{\sigma}\Big[px+qy+rz+\frac{s}{\gamma}t^{\gamma}\Big]\right)$$
$$-\frac{1680\mathcal{E}p^4\sigma^2}{A}\tan^4\left(\sqrt{\sigma}\Big[px+qy+rz+\frac{s}{\gamma}t^{\gamma}\Big]\right),\qquad \sigma>0,$$
$$P_{14}=e_0-\frac{1680i\sigma\left(\sqrt{31}\mathcal{E}p^4\sigma-31i\mathcal{E}p^4\sigma\right)}{31A}\cot^2\left(\sqrt{\sigma}\Big[px+qy+rz+\frac{s}{\gamma}t^{\gamma}\Big]\right)$$
$$-\frac{1680\mathcal{E}p^4\sigma^2}{A}\cot^4\left(\sqrt{\sigma}\Big[px+qy+rz+\frac{s}{\gamma}t^{\gamma}\Big]\right),\qquad \sigma>0,$$
$$P_{15}=e_0-\frac{1680\mathcal{E}p^4}{A\left(px+qy\pm z\sqrt{-\frac{Bp^2}{C}-q^2}-\frac{Ae_0p}{\gamma}t^{\gamma}+\phi\right)^4},\qquad \sigma=0,$$

where $s=\frac{1}{31}\left(-31Ae_0p-1368i\sqrt{31}\mathcal{E}p^5\sigma^2-3720\mathcal{E}p^5\sigma^2\right)$ and $r=\pm\sqrt{-\frac{Bp^2}{C}+\frac{78i\mathcal{E}p^4\sigma}{\sqrt{31}C}-\frac{26\mathcal{E}p^4\sigma}{C}-q^2}$.

4.2. Solutions for $k=2$:

When $k=2$, we have $N=2$, then Eq. (10) gives

$$P(\xi)=e_0+e_1\vartheta(\xi)+e_1\vartheta^2(\xi). \tag{20}$$

By substituting Eqs. (20) and (11) into Eq. (14), then equating the coefficients of $\vartheta^r(\xi)$ to zero yields

$$\begin{aligned}
&\vartheta^0(\xi): Ae_0^2e_1p\sigma+2e_1\sigma^2\left(Bp^3+Cpq^2+Cpr^2\right)+16\mathcal{E}e_1p^5\sigma^3+e_1s\sigma=0,\\
&\vartheta^1(\xi): 2Ae_0e_1^2p\sigma+2Ae_0^2e_2p\sigma+16e_2\sigma^2\left(Bp^3+Cpq^2+Cpr^2\right)+272\mathcal{E}e_2p^5\sigma^3+2e_2s\sigma=0,\\
&\vartheta^2(\xi): Ae_1^3p\sigma+6Ae_0e_1e_2p\sigma+Ae_0^2e_1p+8e_1\sigma\left(Bp^3+Cpq^2+Cpr^2\right)+136\mathcal{E}e_1p^5\sigma^2+e_1s=0,\\
&\vartheta^3(\xi): 4Ae_0e_2^2p\sigma+4Ae_1^2e_2p\sigma+2Ae_0e_1^2p+2Ae_0^2e_2p+\ 40e_2\sigma\left(Bp^3+Cpq^2+Cpr^2\right)\\
&\qquad +1232\mathcal{E}e_2p^5\sigma^2+2e_2s=0,\\
&\vartheta^4(\xi): 5Ae_1e_2^2p\sigma+Ae_1^3p+6Ae_0e_1e_2p+6e_1\left(Bp^3+Cpq^2+Cpr^2\right)+240\mathcal{E}e_1p^5\sigma=0,\\
&\vartheta^5(\xi): 2Ae_2^3p\sigma+4Ae_0e_2^2p+4Ae_1^2e_2p+24e_2\left(Bp^3+Cpq^2+Cpr^2\right)+1680\mathcal{E}e_2p^5\sigma=0,\\
&\vartheta^6(\xi): 5Ae_1e_2^2p+120\mathcal{E}e_1p^5=0,\\
&\vartheta^7(\xi): 2Ae_2^3p+720\mathcal{E}e_2p^5=0.
\end{aligned} \tag{21}$$

With the help of Mathematica, we solve the above equations as

Case 1.

$$
\begin{aligned}
&e_0 = e_0, \quad e_1 = 0, \quad e_2 = -6ip^2\sqrt{\frac{10\mathcal{E}}{A}}, \quad s = -8i\sqrt{10A\mathcal{E}}e_0p^3\sigma - Ae_0^2p + 184\mathcal{E}p^5\sigma^2, \\
&r = \pm\sqrt{\frac{i\sqrt{10A\mathcal{E}}e_0p^2 - Bp^2 - Cq^2 - 40\mathcal{E}p^4\sigma}{C}}.
\end{aligned} \tag{22}
$$

By substituting Eq. (22) into Eq. (20) and using the solutions defined in Eq. (12), we obtain the required solutions to Eq. (13) as:

$$
\begin{aligned}
&P_{16} = e_0 + 6ip^2\sigma\sqrt{\frac{10\mathcal{E}}{A}}\tanh^2\left(\sqrt{-\sigma}\Big(px + qy + rz + \frac{s}{\gamma}t^\gamma\Big)\right), \qquad \sigma < 0, \\
&P_{17} = e_0 + 6ip^2\sigma\sqrt{\frac{10\mathcal{E}}{A}}\coth^2\left(\sqrt{-\sigma}\Big(px + qy + rz + \frac{s}{\gamma}t^\gamma\Big)\right), \qquad \sigma < 0, \\
&P_{18} = e_0 - 6ip^2\sigma\sqrt{\frac{10\mathcal{E}}{A}}\tan^2\left(\sqrt{\sigma}\Big(px + qy + rz + \frac{s}{\gamma}t^\gamma\Big)\right), \qquad \sigma > 0, \\
&P_{19} = e_0 - 6ip^2\sigma\sqrt{\frac{10\mathcal{E}}{A}}\cot^2\left(\sqrt{\sigma}\Big(px + qy + rz + \frac{s}{\gamma}t^\gamma\Big)\right), \qquad \sigma > 0, \\
&P_{20} = e_0 - \frac{6ip^2\sqrt{10\mathcal{E}}}{\sqrt{A}\left(px + qy \pm \sqrt{\frac{i\sqrt{10A\mathcal{E}}e_0p^2 - Bp^2 - Cq^2}{C}}z - \frac{Ae_0^2p}{\gamma}t^\gamma + \phi\right)^2}, \qquad \sigma = 0,
\end{aligned}
$$

where $s = -8i\sqrt{10}\sqrt{A}\sqrt{\mathcal{E}}e_0p^3\sigma - Ae_0^2p + 184\mathcal{E}p^5\sigma^2$ and $r = \pm\sqrt{\frac{i\sqrt{10A\mathcal{E}}e_0p^2 - Bp^2 - Cq^2 - 40\mathcal{E}p^4\sigma}{C}}$.

Case 2.

$$
\begin{aligned}
&e_0 = e_0, \quad e_1 = 0, \quad e_2 = 6ip^2\sqrt{\frac{10\mathcal{E}}{A}}, \quad s = 8i\sqrt{10A\mathcal{E}}e_0p^3\sigma - Ae_0^2p + 184\mathcal{E}p^5\sigma^2, \\
&r = \pm\sqrt{\frac{-i\sqrt{10A\mathcal{E}}e_0p^2 - Bp^2 - Cq^2 - 40\mathcal{E}p^4\sigma}{C}}.
\end{aligned} \tag{23}
$$

By substituting Eq. (23) into Eq. (20) and using the solutions defined in Eq. (12), we obtain the required solutions to Eq. (13) as:

$$
\begin{aligned}
&P_{21} = e_0 - 6ip^2\sigma\sqrt{\frac{10\mathcal{E}}{A}}\tanh^2\left(\sqrt{-\sigma}\Big(px + qy + rz + s\frac{t^\gamma}{\gamma}\Big)\right), \qquad \sigma < 0, \\
&P_{22} = e_0 - 6ip^2\sigma\sqrt{\frac{10\mathcal{E}}{A}}\coth^2\left(\sqrt{-\sigma}\Big(px + qy + rz + s\frac{t^\gamma}{\gamma}\Big)\right), \qquad \sigma < 0, \\
&P_{23} = e_0 + 6ip^2\sigma\sqrt{\frac{10\mathcal{E}}{A}}\tan^2\left(\sqrt{\sigma}\Big(px + qy + rz + s\frac{t^\gamma}{\gamma}\Big)\right), \qquad \sigma > 0, \\
&P_{24} = e_0 + 6ip^2\sigma\sqrt{\frac{10\mathcal{E}}{A}}\cot^2\left(\sqrt{\sigma}\Big(px + qy + rz + s\frac{t^\gamma}{\gamma}\Big)\right), \qquad \sigma > 0, \\
&P_{25} = e_0 + \frac{6ip^2\sqrt{10\mathcal{E}}}{\sqrt{A}\left(px + qy \pm \sqrt{\frac{-i\sqrt{10A\mathcal{E}}e_0p^2 - Bp^2 - Cq^2}{C}}z - \frac{Ae_0^2p}{\gamma}t^\gamma + \phi\right)^2}, \qquad \sigma = 0,
\end{aligned}
$$

where $s = 8i\sqrt{10A\mathcal{E}}e_0p^3\sigma - Ae_0^2p + 184\mathcal{E}p^5\sigma^2$ and $r = \pm\sqrt{\frac{-i\sqrt{10A\mathcal{E}}e_0p^2 - Bp^2 - Cq^2 - 40\mathcal{E}p^4\sigma}{C}}$.

4.3. Solutions for $k = 4$:

When $k = 4$, we have $N = 1$, then Eq. (10) yields

$$P(\xi) = e_0 + e_1 \vartheta(\xi). \tag{24}$$

By substituting Eqs. (24) and (11) into Eq. (14), then equating the coefficients of $\vartheta^r(\xi)$ to zero gives

$$\begin{aligned}
&\vartheta^0(\xi): Ae_0^4 e_1 p\sigma + 2e_1\sigma^2\left(Bp^3 + Cpq^2 + Cpr^2\right) + 16\mathcal{E}e_1 p^5\sigma^3 + e_1 s\sigma = 0,\\
&\vartheta^1(\xi): 4Ae_0^3 e_1^2 p\sigma = 0,\\
&\vartheta^2(\xi): 6Ae_0^2 e_1^3 p\sigma + Ae_0^4 e_1 p + 8e_1\sigma\left(Bp^3 + Cpq^2 + Cpr^2\right) + 136\mathcal{E}e_1 p^5\sigma^2 + e_1 s = 0,\\
&\vartheta^3(\xi): 4Ae_0 e_1^4 p\sigma + 4Ae_0^3 e_1^2 p = 0,\\
&\vartheta^4(\xi): Ae_1^5 p\sigma + 6Ae_0^2 e_1^3 p + 6e_1\left(Bp^3 + Cpq^2 + Cpr^2\right) + 240\mathcal{E}e_1 p^5\sigma = 0,\\
&\vartheta^5(\xi): 4Ae_0 e_1^4 p = 0,\\
&\vartheta^6(\xi): Ae_1^5 p + 120\mathcal{E}e_1 p^5 = 0.
\end{aligned} \tag{25}$$

By utilizing Mathematica, we solve the above equations as

Case 1:

$$e_0 = e_0, \quad e_1 = \pm(1+i)p\sqrt[4]{\frac{30\mathcal{E}}{A}}, \quad s = 24\mathcal{E}p^5\sigma^2, \quad r = \pm\sqrt{\frac{-Bp^2 - Cq^2 - 20\mathcal{E}p^4\sigma}{C}}. \tag{26}$$

By substituting Eq. (26) into Eq. (24) and using the solutions defined in Eq. (12), we obtain the required solutions to Eq. (13) as:

$$\begin{aligned}
P_{26} &= \pm(1+i)p\sqrt{-\sigma}\sqrt[4]{\frac{30\mathcal{E}}{A}}\tanh\left(\sqrt{-\sigma}\left[px + qy \pm \sqrt{\frac{-Bp^2 - Cq^2 - 20\mathcal{E}p^4\sigma}{C}}z + \frac{24\mathcal{E}p^5\sigma^2}{\gamma}t^\gamma\right]\right), \quad \sigma < 0,\\
P_{27} &= \pm(1+i)p\sqrt{-\sigma}\sqrt[4]{\frac{30\mathcal{E}}{A}}\coth\left(\sqrt{-\sigma}\left[px + qy \pm \sqrt{\frac{-Bp^2 - Cq^2 - 20\mathcal{E}p^4\sigma}{C}}z + \frac{24\mathcal{E}p^5\sigma^2}{\gamma}t^\gamma\right]\right), \quad \sigma < 0,\\
P_{28} &= \mp(1+i)p\sqrt{\sigma}\sqrt[4]{\frac{30\mathcal{E}}{A}}\tan\left(\sqrt{\sigma}\left[px + qy \pm \sqrt{\frac{-Bp^2 - Cq^2 - 20\mathcal{E}p^4\sigma}{C}}z + \frac{24\mathcal{E}p^5\sigma^2}{\gamma}t^\gamma\right]\right), \quad \sigma > 0,\\
P_{29} &= \pm(1+i)p\sqrt{\sigma}\sqrt[4]{\frac{30\mathcal{E}}{A}}\cot\left(\sqrt{\sigma}\left[px + qy \pm \sqrt{\frac{-Bp^2 - Cq^2 - 20\mathcal{E}p^4\sigma}{C}}z + \frac{24\mathcal{E}p^5\sigma^2}{\gamma}t^\gamma\right]\right), \quad \sigma > 0,\\
P_{30} &= \pm\frac{(1+i)p\sqrt[4]{30\mathcal{E}}}{\sqrt[4]{A}\left(px + qy \pm \sqrt{\frac{-Bp^2}{C} - q^2}\,z + \phi\right)}, \quad \sigma = 0.
\end{aligned}$$

Case 2:

$$e_0 = e_0, \quad e_1 = \pm(-1+i)p\sqrt[4]{\frac{30\mathcal{E}}{A}}, \quad s = 24\mathcal{E}p^5\sigma^2, \quad r = \pm\sqrt{\frac{-Bp^2 - Cq^2 - 20\mathcal{E}p^4\sigma}{C}}. \tag{27}$$

By substituting Eq. (27) into Eq. (24) and using the solutions defined in Eq. (12), we obtain the required solutions to Eq. (13)

as:

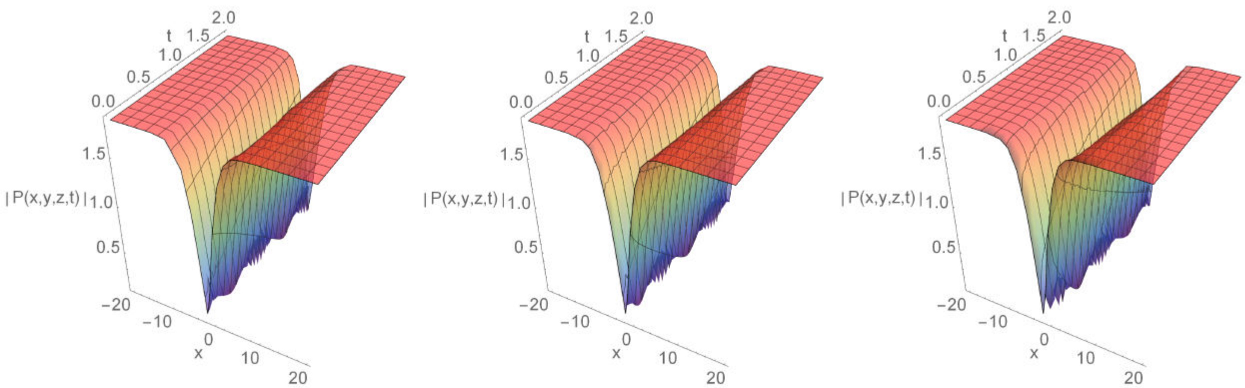
$$P_{31} = \pm(-1+i)p\sqrt{-\sigma}\sqrt[4]{\frac{30\mathcal{E}}{A}}\tanh\left(\sqrt{-\sigma}\left[px+qy\pm\sqrt{\frac{-Bp^2-Cq^2-20\mathcal{E}p^4\sigma}{C}}z+\frac{24\mathcal{E}p^5\sigma^2}{\gamma}t^\gamma\right]\right), \quad \sigma<0,$$

$$P_{32} = \pm(-1+i)p\sqrt{-\sigma}\sqrt[4]{\frac{30\mathcal{E}}{A}}\coth\left(\sqrt{-\sigma}\left[px+qy\pm\sqrt{\frac{-Bp^2-Cq^2-20\mathcal{E}p^4\sigma}{C}}z+\frac{24\mathcal{E}p^5\sigma^2}{\gamma}t^\gamma\right]\right), \quad \sigma<0,$$

$$P_{33} = \mp(-1+i)p\sqrt{\sigma}\sqrt[4]{\frac{30\mathcal{E}}{A}}\tan\left(\sqrt{\sigma}\left[px+qy\pm\sqrt{\frac{-Bp^2-Cq^2-20\mathcal{E}p^4\sigma}{C}}z+\frac{24\mathcal{E}p^5\sigma^2}{\gamma}t^\gamma\right]\right), \quad \sigma>0,$$

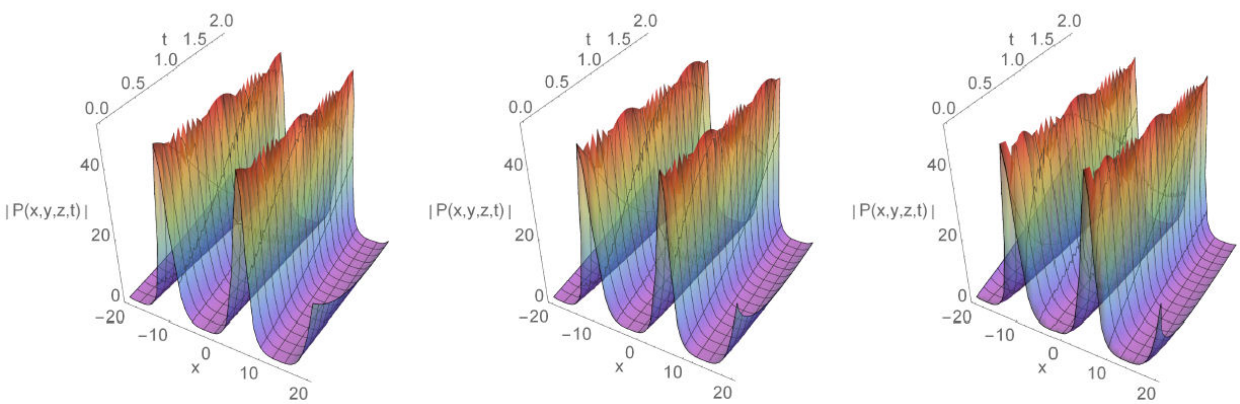
$$P_{34} = \pm(-1+i)p\sqrt{\sigma}\sqrt[4]{\frac{30\mathcal{E}}{A}}\cot\left(\sqrt{\sigma}\left[px+qy\pm\sqrt{\frac{-Bp^2-Cq^2-20\mathcal{E}p^4\sigma}{C}}z+\frac{24\mathcal{E}p^5\sigma^2}{\gamma}t^\gamma\right]\right), \quad \sigma>0,$$

$$P_{35} = \pm\frac{(-1+i)p\sqrt[4]{30\mathcal{E}}}{\sqrt[4]{A}\left(px+qy\pm\sqrt{\frac{-Bp^2}{C}-q^2}\,z+\phi\right)}, \quad \sigma=0.$$



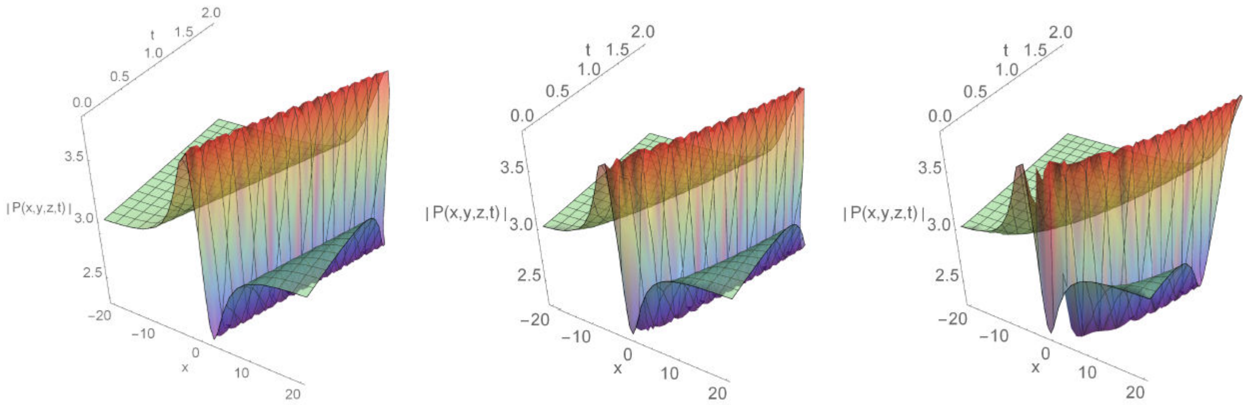
(a) $\gamma = 1$ (b) $\gamma = 0.75$ (c) $\gamma = 0.5$

FIGURE 1. The plots of P_1 solution for $A = 1,\ B = 2,\ C = -20,\ y = z = 2,\ \mathcal{E} = 0.5, \sigma = -1,\ e_0 = 0.5,\ p = 0.2$ and $q = 0.2$.



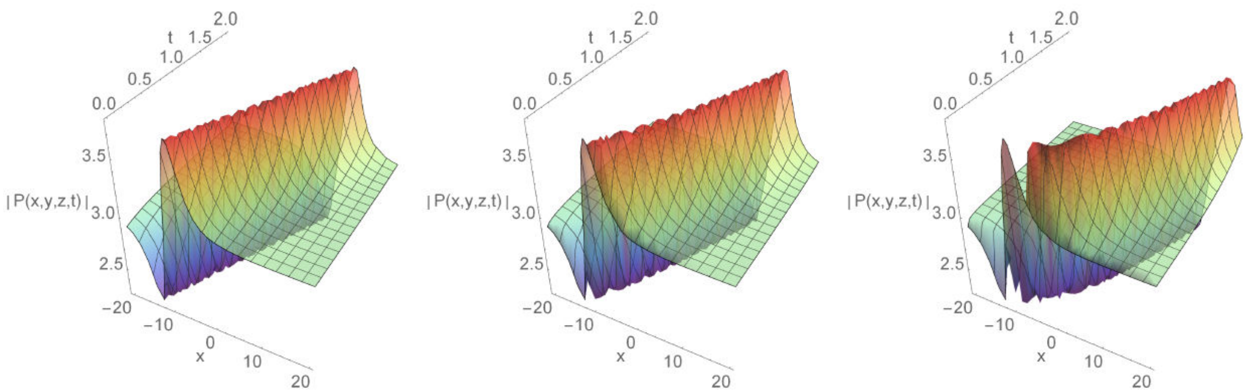
(a) $\gamma = 1$ (b) $\gamma = 0.75$ (c) $\gamma = 0.5$

FIGURE 2. The plots of P_3 solution for $A = 1,\ B = 2,\ C = -20,\ y = z = 2,\ \mathcal{E} = 0.5, \sigma = 1,\ e_0 = 0.5,\ p = 0.2$ and $q = 0.2$.



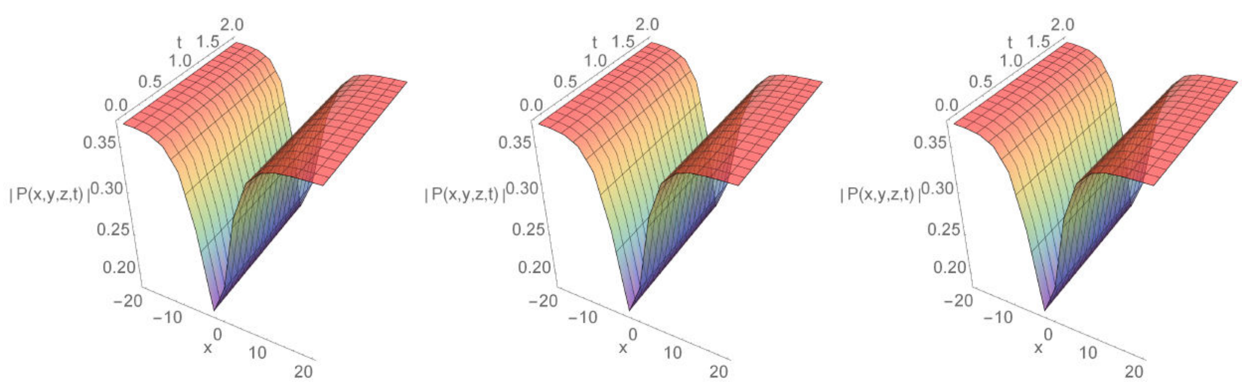
(a) $\gamma = 1$ (b) $\gamma = 0.75$ (c) $\gamma = 0.5$

FIGURE 3. The plots of P_{20} solution for $A = 1,\ B = 2,\ C = -1,\ y = z = 2,\ \mathcal{E} = 0.5,\ \phi = 1,\ e_0 = 3,\ p = 0.2$ and $q = 0.2$.



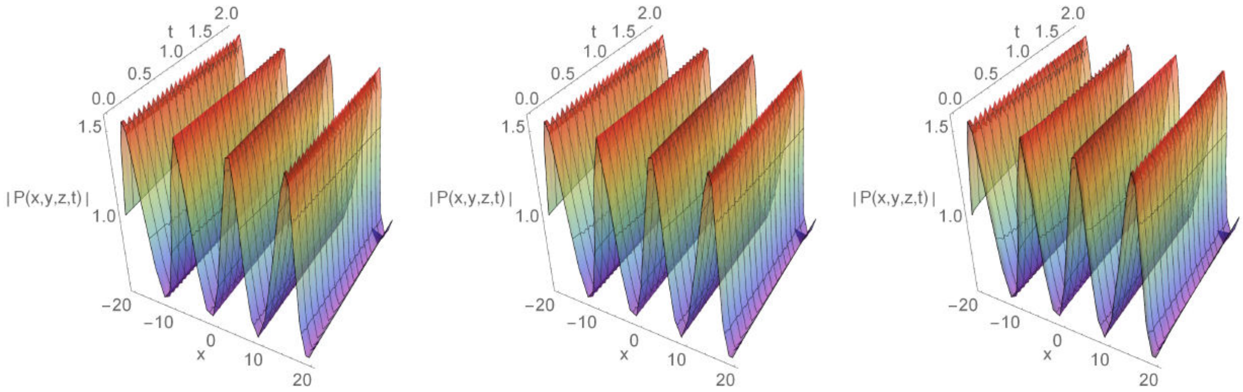
(a) $\gamma = 1$ (b) $\gamma = 0.75$ (c) $\gamma = 0.5$

FIGURE 4. The plots of P_{25} solution for $A = 1,\ B = 2,\ C = -1,\ y = z = 2,\ \mathcal{E} = 0.5,\ \phi = 1,\ e_0 = 3,\ p = 0.2$ and $q = 0.2$.



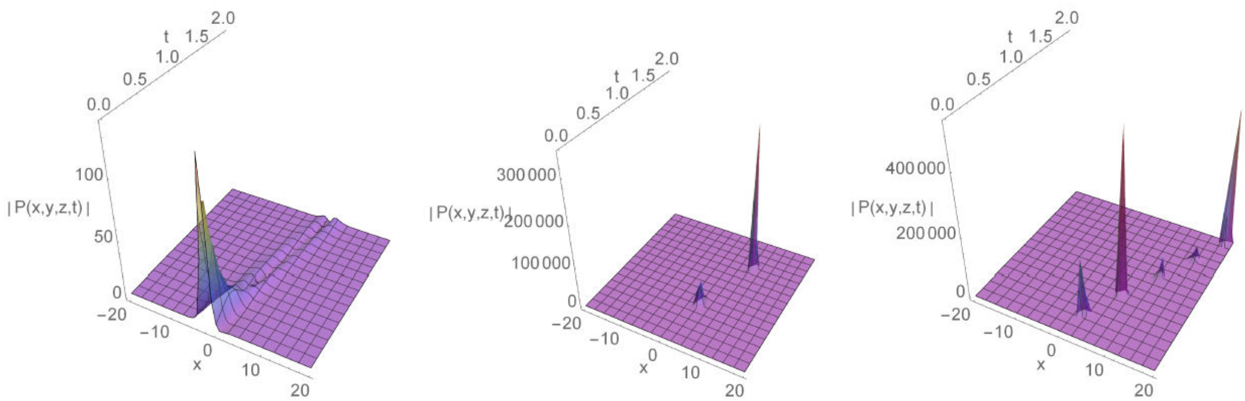
(a) $\gamma = 1$ (b) $\gamma = 0.75$ (c) $\gamma = 0.5$

FIGURE 5. The plots of P_{26} solution for $A = 1,\ B = 2,\ C = 10,\ y = z = 2,\ \mathcal{E} = 0.1, \sigma = -1,\ e_0 = 3,\ p = 0.2$ and $q = 0.2$.



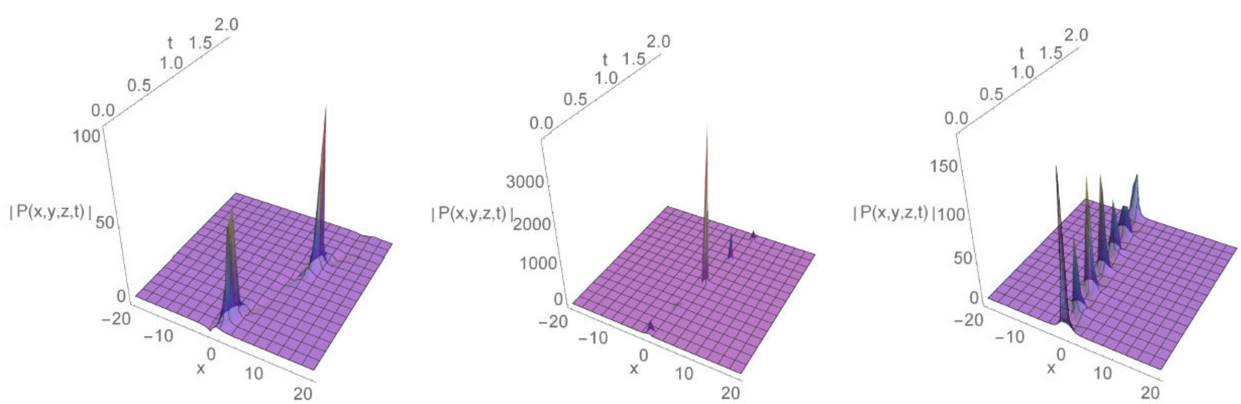
(a) $\gamma = 1$ (b) $\gamma = 0.75$ (c) $\gamma = 0.5$

FIGURE 6. The plots of P_{28} solution for $A = 1,\ B = 2,\ C = 10,\ y = z = 2,\ \mathcal{E} = 0.8, \sigma = 1,\ e_0 = 3,\ p = 0.3$ and $q = 0.3$.



(a) $\mathcal{E} = 0.1$ (b) $\mathcal{E} = 0.5$ (c) $\mathcal{E} = 1$

FIGURE 7. The effect of the parameter $\mathcal{E}$ on solution profile of P_{11} for $\gamma = 1,\ A = 1,\ B = 2,\ C = 10,\ y = z = 2,\ \sigma = -1,\ e_0 = 3,\ p = 0.5$ and $q = 0.5$.



(a) $\mathcal{E} = 0.1$ (b) $\mathcal{E} = 0.5$ (c) $\mathcal{E} = 1$

FIGURE 8. The effect of the parameter $\mathcal{E}$ on solution profile of P_{16} for $\gamma = 1,\ A = 1,\ B = 2,\ C = -5,\ y = z = 2, \sigma = -1,\ e_0 = 3,\ p = 0.5$ and $q = 0.5$.

5. Graphical representation of some solutions

The absolute behavior in 3D plots with integer and fractional order respectively $\gamma = 1$, $\gamma = 0.75$, and 0.5 are presented for some solutions in Fig. 1-8. These plots reveal different structures such as the dark soliton, multi-soliton, solitary wave, bell-shaped soliton, periodic and kink-type solutions which give the readers a better vision of the behavior of these solutions and captured some physical features of the considered model. Furthermore, Fig. 7 and 8 display the effect of adding a new perturbation term on the profile of the solution.

6. Conclusion

In this paper, we introduced the generalized time-fractional perturbed $(3+1)$ Zakharov-Kuznetsov (PZK) equations which describe the nonlinear dust-ion-acoustic waves in the magnetized two-ion-temperature dusty plasmas. We investigate the exact solutions by the use the of sub-equation method in the conformable sense. The use of conformable derivative in this study gives flexibility when applying to the proposed model and satisfies the power rule, product rule, quotient rule, integration by parts, chain rule, linearity, and the derivatives with constant is zero. The newly obtained solutions by the proposed method are, respectively, the dark soliton multi-soliton, kink-shape, solitary wave, periodic and bell-shaped soliton solutions that are significant in the field of mathematical physics. Graphical representation (see Fig. 1 to 8) of obtained solutions are plotted in 3D for particular values of parameters. Figures 7 and 8, demonstrate the effect of adding a higher-order dispersion term "$\mathcal{E}P_{xxxxx}$" to Eq. (1). The performance of this method is reliable and effective and the obtained results are in a more general form. Finally, through Mathematica, we have authenticated the obtained solutions by substituting them back into the proposed equation.

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