Gravitation, Mathematical Physics and Field Theory Revista Mexicana de Física **67** 060702 1–18 NOVEMBER-DECEMBER 2021

Optical soliton solutions of nonlinear Davey-Stewartson equation using an efficient method

H. Günerhan

Department of Mathematics, Faculty of Education, Kafkas University, Kars TR-36040, Turkey.
**e-mail: hatira.gunerhan@kafkas.edu.tr*

Received 29 December 2020; accepted 30 June 2021

One of the most significant tools for expressing physical phenomena in the world around us is to express problems using differential equations with partial derivatives. The result of these considerations has been the invention and application of various analytical and numerical methods in solving this category of equations. In this work, we make use of a newly-developed technique called the generated exponential rational function method to compute the exact solution of the Davey-Stewartson equation. According to all the conducted research studies so far, results similar to those found in the present paper have not been published. The results attest to the efficiency of the proposed method. The method used in this paper has the ability to be implemented in other cases in solving equations with relative derivatives.

Keywords: Generalized exponential rational function method; exact soliton solutions; Davey-Stewartson equation; optical solutions; nonlinear partial differential equations.

DOI: https://doi.org/10.31349/RevMexFis.67.060702

1. Introduction

Many fields of physics benefit from the nonlinear partial differential equations (NPDEs) in a wide variety of applications to mechanics, electrostatics, quantum mechanics, and finance. In linear theory, solutions usually have representation formulas and conform to the superposition principle. Despite some equations governing nature were recognized to be linear since the 19th century, this advance was widely criticized in the 20th century. It is possible to observe NPDEs not only in non-Newtonian fluids, glaciology, rheology but also in stochastic game theory, nonlinear elasticity, flow through a porous medium, and image-processing. As a result, no available superposition can be found for nonlinear equations; therefore, there is a need for studying those equations. Over the last few years, a crucial number of new methods have been proposed to obtain exact solutions for NPDEs. The Davey-Stewartson equation (DSE) has been studied in many areas of research such as chemical engineering, nonlinear mechanics, biology, and physics. To define the evolution of a three-dimensional wave packet in finite depth water, Davey-Stewartson (1974) had presented the DSE in his research study about fluid dynamics [1].

In (2+1)-dimensions, the Davey-Stewartson equation is known as a solution equation that examines long and short wave resonances and other wave propagation patterns. For more details, we refer to the previous research works conducted in Refs. [2-3]. The Davey-Stewartson theory is an NPDE for a complex field (wave-amplitude) q and a real field (mean flow) ϕ which is described by the following nonlinear coupled system

$$iq_t + \frac{1}{2}\left(q_{xx} + \delta q_{yy}\right) + \lambda |q|^2 q - \phi_x q = 0, \tag{1}$$

$$\phi_{xx} - \delta^2 \phi_{yy} - 2\lambda\left(|q|^2\right)_x = 0. \tag{1.1}$$

This important system of equations has attracted the attention of many researchers. For example, the line soliton [4], the semi-inverse variational principle method (SIVPM), the improved $\tan(\phi/2)$-expansion method (ITEM) along with the generalized G'/G-expansion method (GGM) [5], the G'/G method and the 1-soliton solution [6], the Galerkin methods [7], the extended tanh-function method [8], the generalized Kudryashov method [9], the traveling wave solutions [10], the first integral method [11], the solitary wave solution [12], the dynamical system method [13], the traveling waves solution and the exponential function method [14], the inverse scattering transform method and the soliton solutions [15], the self-similar solutions [16], the bilinear method [17, 18], the single soliton and multi-soliton solutions [19], the G'/G -expansion method [20], the generalized $\tan(\phi/2)$ method and the He's semi-inverse variational method [21], and the bifurcation method [22, 23]. Others popular techniques can be found in Ref. [24, 43].

In Ref. [44], Ghanbari and his collaborator developed an efficient methodology for obtaining exact solutions to NPDEs which is known as a generalized exponential rational function method (GERFM). The authors applied the technique to solve the resonant nonlinear Schrödinger equation (R-NLSE). It has been proven over time that the method enables us to be implemented in many different NPDEs arising in mathematics, physics, and engineering [45-54]. The proposed method reproduces many types of precise solutions, and it is very useful for finding the exact solutions of the equation with relative ease. Recently, a new version of the method for solving partial differential equations with local fractional derivatives has been considered in Refs. [55, 56].

In this paper, the GERFM is used to solve the Davey-Stewartson equation. This paper consists of the following parts: In Sec. 2, we introduce the methodology of the GERFM. In Sec. 3, the results of using the method in determining the solutions of the equation (according to the main achievement of this article) will be presented. Finally, the article ends with some concluding remarks.

2. Methodology of the GERFM

The technique is a very efficient method in solving partial differential equations [44]. The basic steps of using this method are listed below.

1. NPDE will be accepted as follows:

$$\mathcal{F}\left(u(x,t), \frac{\partial u(x,t)}{\partial x}, \frac{\partial u(x,t)}{\partial t}, \frac{\partial^2 u(x,t)}{\partial x^2}, \dots\right) = 0. \quad (2.1)$$

   To abbreviate the NPDEs is the following ordinary differential equation (ODE), it will be used $\mathbf{u}(\xi) = u(x,t)$ and $\xi = kx - lt$.

$$\mathcal{F}(\mathbf{u}, \mathbf{u}', \mathbf{u}'', \dots) = 0, \quad (2.2)$$

2. The crucial part of the new methodology comes from the fact that Eq. (2.2) has the formal solution of

$$\mathbf{u}(\xi) = A_0 + \sum_{k=1}^{m} A_k \Xi(\xi)^k + \sum_{k=1}^{m} B_k \Xi(\xi)^{-k}, \quad (2.3)$$

   where

$$\Xi(\xi) = \frac{p_1 e^{q_1 \xi} + p_2 e^{q_2 \xi}}{p_3 e^{q_3 \xi} + p_4 e^{q_4 \xi}}. \quad (2.4)$$

   The real (or complex) unknown constants are $A_0, A_k, B_k (1 \leq k \leq m)$, and $p_k, q_k (1 \leq k \leq 4)$. These coefficients are determined in such a way that Eq. (2.3) satisfies the nonlinear ODE of Eq. (2.2).

   Also, it is essential to determine the positive integer m by the principle of balancing.

3. By adding all terms and inserting Eq. (2.3) into Eq. (2.2), the left-hand side of Eq. (2.2) is converted into the polynomial equation $P(Y_1, Y_2, Y_3, Y_4) = 0$ in terms of $Y_i = e^{q_i \xi}$ for $i = 1, \dots, 4$. With the help of symbolic calculation in Maple, we obtain a set of simultaneous algebraic equations for $p_n, q_n (1 \leq n \leq 4)$, and $k, \omega, \lambda, A_0, A_1, B_1$ by eliminating each coefficient of P.

4. Finally, exact solutions to Eq. (2.1) are derived through solving the algebraic nonlinear system of equations in step 3.

3. The results

The first step is the traveling wave transformation of Eq. (1.1) by utilizing the following new variables

$$q = \mathscr{U}(\xi) e^{i\theta}, \qquad \phi = \mathscr{V}(\xi) e^{i\theta}, \quad (3.1)$$

and

$$\xi = i\mu\left(x + y - \eta t\right), \qquad \theta = \alpha x + \beta y + \gamma t. \quad (3.2)$$

In addition, constants of μ, η, α, and β should be determined. Using the wave transformation of Eq. (3.1) and Eq. (1.1) together with $\eta = \alpha\delta^2 + \beta\delta^4$, the following system of nonlinear ODE is obtained [5]:

$$\frac{1}{2}\left(2\gamma + \alpha^2\delta^2 + \beta^2\delta^4\right)\mathscr{U} - \frac{1}{2}\mu^2\delta^2\left(1+\delta^2\right)\mathscr{U}'' + \lambda\mathscr{U}^3 - i\mu\mathscr{U}\mathscr{V}' = 0, \quad (3.3)$$

$$\mu\left(\delta^2 - 1\right)\mathscr{V}'' - 2i\lambda\left(\mathscr{U}^2\right)' = 0. \quad (3.4)$$

Integrating Eq. (3.4), we obtain

$$\mathscr{V} = \frac{2i\lambda}{\mu(\delta^2-1)}\int \mathscr{U}^2 d\xi. \quad (3.5)$$

Substituting Eq. (3.5) in Eq. (3.3) will turn into the following nonlinear differential equation:

$$\frac{1}{2}\mu^2\delta^2\left(\delta^4 - 1\right)\mathscr{U}'' + \frac{1}{2}\left(\delta^2 - 1\right)\left(2\gamma + \alpha^2\delta^2 + \beta^2\delta^4\right)\mathscr{U} - \lambda\left(1+\delta^2\right)\mathscr{U}^3 = 0, \quad (3.6)$$

where primes denote the derivatives with respect to ξ. By balancing terms of $\mathscr{U}''$ and $\mathscr{U}^3$ in Eq. (3.6) using homogenous principle yields $3m = m + 2$, so $m = 1$. Accordingly, the solution of Eq.(1.1) is expressed as follows:

$$\mathscr{U}(\xi) = A_0 + A_1\Xi(\xi) + \frac{B_1}{\Xi(\xi)}. \tag{3.7}$$

By following the described methodologies in section 2, we obtain several non-trivial solutions of (1.1).
Family 1: We attain the results for $p = [1, 1, -1, 1]$ and $q = [1, -1, 1, -1]$, which gives

$$\Xi\left(\xi\right) = \frac{\cosh\left(\xi\right)}{\sinh\left(\xi\right)}. \tag{3.8}$$

Case 1:

$$\alpha = \alpha, \qquad \beta = \frac{i\sqrt{\alpha^2\delta^4 + \left(2\,{A_1}^2\lambda - \alpha^2 + 2\,\gamma\right)\delta^2 + 2\,{A_1}^2\lambda - 2\,\gamma}}{\delta^2\sqrt{\delta^2 - 1}},$$
$$\mu = \frac{\sqrt{\lambda}A_1}{\delta\sqrt{\delta^2 - 1}}, \quad \delta = \delta, \quad \gamma = \gamma, \quad A_0 = 0, \quad A_1 = A_1, \quad B_1 = 0.$$

We have replaced the above values with Eqs. (3.7) and (3.8) together with (3.5)

$$\mathscr{U}_1(\xi) = \frac{A_1\cosh\left(\xi\right)}{\sinh\left(\xi\right)},$$
$$\mathscr{V}_1(\xi) = \frac{2\,i\lambda\,{A_1}^2\left(\xi - \coth\left(\xi\right)\right)}{\mu\,\left(\delta^2 - 1\right)}. \tag{3.9}$$

Hence, the following exact solution was reached for Eq. (1.1):

$$q_1\left(x, y, t\right) = \left(\frac{A_1\cosh\left[\xi\right]}{\sinh\left[\xi\right]}\right)e^{i(\alpha x+\beta y+\gamma t)},$$
$$\phi_1\left(x, y, t\right) = \left(\frac{2\,i\lambda\,{A_1}^2\left[\xi - \coth\left(\xi\right)\right]}{\mu\,\left[\delta^2 - 1\right]}\right)e^{i(\alpha x+\beta y+\gamma t)}. \tag{3.10}$$

Case 2:

$$\alpha = \alpha, \qquad \beta = \frac{i\sqrt{\alpha^2\delta^4 + \left(8\,{A_1}^2\lambda - \alpha^2 + 2\,\gamma\right)\delta^2 + 8\,{A_1}^2\lambda - 2\,\gamma}}{\delta^2\sqrt{\delta^2 - 1}},$$
$$\mu = \frac{\sqrt{\lambda}A_1}{\delta\sqrt{\delta^2 - 1}}, \qquad \delta = \delta, \qquad \gamma = \gamma, \qquad A_0 = 0, \qquad A_1 = A_1, \qquad B_1 = A_1.$$

We have replaced the above values with Eqs. (3.7) and (3.8) together with (3.5)

$$\mathscr{U}_2(\xi) = \frac{\left(2\,\left[\cosh\left(\xi\right)\right]^2 - 1\right)A_1}{\cosh\left(\xi\right)\sinh\left(\xi\right)},$$
$$\mathscr{V}_2(\xi) = \frac{2\,i\lambda\,{A_1}^2\left(4\,\xi\,\cosh\left[\xi\right]\sinh\left[\xi\right] - 2\,\left[\cosh\left\{\xi\right\}\right]^2 + 1\right)}{\cosh\left(\xi\right)\sinh\left(\xi\right)\mu\,\left(\delta^2 - 1\right)}. \tag{3.11}$$

Hence, the following exact solution has been reached for Eq. (1.1):

$$q_2\left(x, y, t\right) = \left(\frac{\left[2\,\left\{\cosh\left(\xi\right)\right\}^2 - 1\right]A_1}{\cosh\left[\xi\right]\sinh\left[\xi\right]}\right)e^{i(\alpha x+\beta y+\gamma t)},$$
$$\phi_2\left(x, y, t\right) = \left(\frac{2\,i\lambda\,{A_1}^2\left[4\,\xi\,\cosh\left(\xi\right)\sinh\left(\xi\right) - 2\,\left\{\cosh\left(\xi\right)\right\}^2 + 1\right]}{\cosh\left[\xi\right]\sinh\left[\xi\right]\mu\,\left[\delta^2 - 1\right]}\right)e^{i(\alpha x+\beta y+\gamma t)}. \tag{3.12}$$

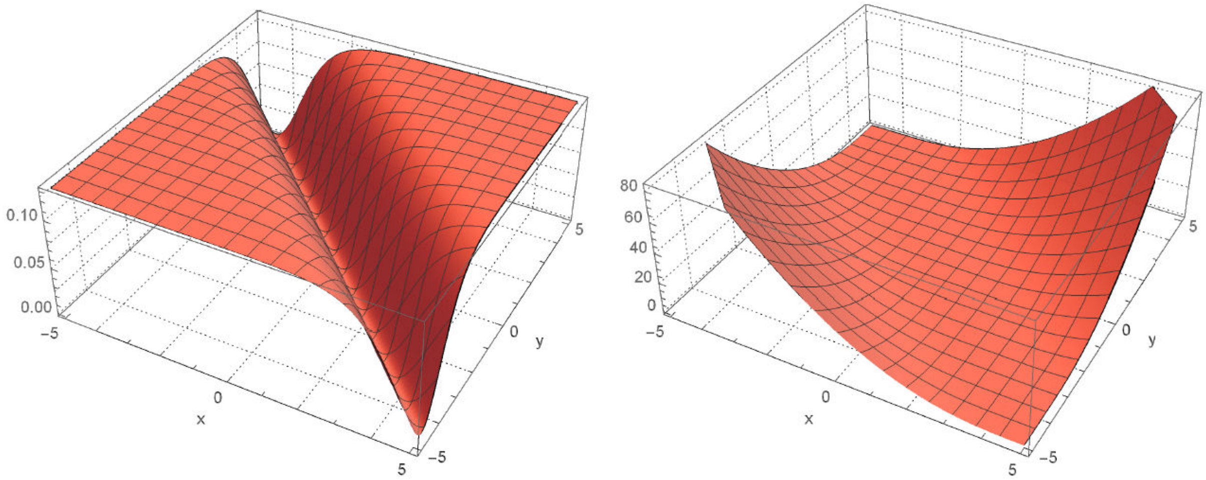

FIGURE 1. Dynamic behaviours modulus of solutions $q_3\,(x,y,t)$ (left) and $\phi_3\,(x,y,t)$ (right) for $\delta=1.05, \gamma=1.3, \alpha=\beta=0.5, \lambda=1.5$, and $t=1$.

Case 3:

$$\alpha=\alpha, \qquad \beta=\beta, \qquad \mu=\frac{i/2\sqrt{2}\sqrt{\beta^2\delta^4+\alpha^2\delta^2+2\,\gamma}}{\delta\sqrt{\delta^2+1}}, \qquad \delta=\delta, \qquad \gamma=\gamma, \qquad A_0=0,$$
$$A_1=0, \qquad B_1=\frac{-i\sqrt{2}\sqrt{\delta^4-1}\sqrt{\beta^2\delta^4+\alpha^2\delta^2+2\,\gamma}}{\sqrt{\lambda}\,(2\,\delta^2+2)}.$$

We have replaced the above values with Eqs. (3.7) and (3.8) together with (3.5)

$$\mathscr{U}_3(\xi)=\frac{-i/2\sqrt{2}\sqrt{\delta^4-1}\sqrt{\beta^2\delta^4+\alpha^2\delta^2+2\,\gamma}\,\sinh(\xi)}{\sqrt{\lambda}\,(\delta^2+1)\cosh(\xi)},$$
$$\mathscr{V}_3(\xi)=\frac{-i\left(\delta^4-1\right)\left(\beta^2\delta^4+\alpha^2\delta^2+2\,\gamma\right)(\xi-\tanh(\xi))}{\mu\,(\delta^2-1)\,(\delta^2+1)^2}. \tag{3.13}$$

Hence, the following exact solution has been reached for Eq. (1.1):

$$q_3\,(x,y,t)=\left(\frac{-i/2\sqrt{2}\sqrt{\delta^4-1}\sqrt{\beta^2\delta^4+\alpha^2\delta^2+2\,\gamma}\,\sinh(\xi)}{\sqrt{\lambda}\,[\delta^2+1]\cosh[\xi]}\right)e^{i(\alpha x+\beta y+\gamma t)},$$
$$\phi_3\,(x,y,t)=\left(\frac{-i\left[\delta^4-1\right]\left[\beta^2\delta^4+\alpha^2\delta^2+2\,\gamma\right][\xi-\tanh(\xi)]}{\mu\,[\delta^2-1]\,[\delta^2+1]^2}\right)e^{i(\alpha x+\beta y+\gamma t)}. \tag{3.14}$$

Figure 1 shows the dynamic behavior of modulus of solutions $q_3\,(x,y,t)$ (left) and $\phi_3\,(x,y,t)$ for $\delta=1.05, \gamma=1.3$, $\alpha=\beta=0.5, \lambda=1.5$, and $t=1$.

Family 2: We attain the results for $p=[i,-i,1,1]$ and $q=[i,-i,i,-i]$, and thus one gets

$$\Xi\,(\xi)=-\frac{\sin(\xi)}{\cos(\xi)}. \tag{3.15}$$

Case 1:

$$\alpha=\alpha, \qquad \beta=\frac{i\sqrt{\alpha^2\delta^4+\left(-8\,{A_1}^2\lambda-\alpha^2+2\,\gamma\right)\delta^2-8\,{A_1}^2\lambda-2\,\gamma}}{\delta^2\sqrt{\delta^2-1}},$$
$$\mu=\frac{\sqrt{\lambda}A_1}{\delta\sqrt{\delta^2-1}}, \qquad \delta=\delta, \qquad \gamma=\gamma, \qquad A_0=0, \qquad A_1=A_1, \qquad B_1=-A_1.$$

We have replaced the above values with Eqs. (3.7) and (3.8) together with (3.5)

$$\mathscr{U}_4(\xi) = \frac{\left(2\,[\cos(\xi)]^2 - 1\right) A_1}{\sin(\xi)\cos(\xi)},$$
$$\mathscr{V}_4(\xi) = \frac{2\,i\lambda\,{A_1}^2\left(-2\,[\cos\{\xi\}]^2 - 4\,\xi\,\cos[\xi]\sin[\xi] + 1\right)}{\mu\,(\delta^2-1)\sin(\xi)\cos(\xi)}. \tag{3.16}$$

Hence, the following exact solution has been reached for Eq. (1.1):

$$q_4(x,y,t) = \left(\frac{\left[2\,\{\cos(\xi)\}^2 - 1\right] A_1}{\sin(\xi)\cos(\xi)}\right) e^{i(\alpha x+\beta y+\gamma t)},$$
$$\phi_4(x,y,t) = \left(\frac{2\,i\lambda\,{A_1}^2\left[-2\,\{\cos(\xi)\}^2 - 4\,\xi\,\cos(\xi)\sin(\xi) + 1\right]}{\mu\,[\delta^2-1]\sin(\xi)\cos(\xi)}\right) e^{i(\alpha x+\beta y+\gamma t)}. \tag{3.17}$$

Case 2:

$$\alpha = \alpha, \qquad \beta = \frac{i\sqrt{\alpha^2\delta^4 + \left(4\,{B_1}^2\lambda - \alpha^2 + 2\,\gamma\right)\delta^2 + 4\,{B_1}^2\lambda - 2\,\gamma}}{\delta^2\sqrt{\delta^2-1}},$$
$$\mu = \frac{\sqrt{\lambda}B_1}{\delta\sqrt{\delta^2-1}}, \qquad \delta=\delta, \qquad \gamma=\gamma, \qquad A_0 = 0, \qquad A_1 = B_1, \qquad B_1 = B_1.$$

We have replaced the above values with Eqs. (3.7) and (3.8) together with (3.5)

$$\mathscr{U}_5(\xi) = -\frac{B_1}{\sin(\xi)\cos(\xi)},$$
$$\mathscr{V}_5(\xi) = \frac{2\,i\lambda\,{B_1}^2\left(-2\,[\cos(\xi)]^2 + 1\right)}{\mu\,(\delta^2-1)\sin(\xi)\cos(\xi)}. \tag{3.18}$$

Hence, the following exact solution has been reached for Eq. (1.1):

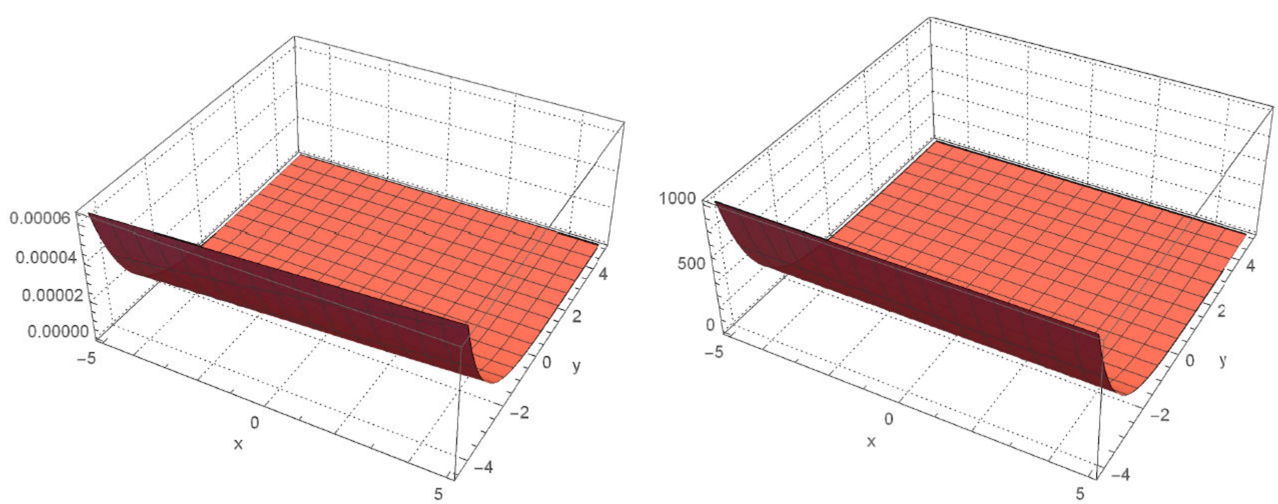

FIGURE 2. Dynamic behaviours modulus of solutions $q_5(x,y,t)$ (left) and $\phi_5(x,y,t)$ (right) for $B_1 = 1, \delta = 8.5, \gamma = 2.01$, $\alpha = 4.25, \lambda = 0.92$, and $t = 1$.

$$q_5(x,y,t) = \left(-\frac{B_1}{\sin(\xi)\cos(\xi)}\right) e^{i(\alpha x+\beta y+\gamma t)},$$

$$\phi_5(x,y,t) = \left(\frac{2\,i\lambda\,{B_1}^2\left[-2\,\{\cos(\xi)\}^2+1\right]}{\mu\,[\delta^2-1]\sin(\xi)\cos(\xi)}\right) e^{i(\alpha x+\beta y+\gamma t)}. \tag{3.19}$$

Figure 2 shows the dynamic behavior of modulus of solutions $q_5(x,y,t)$ (left) and $\phi_5(x,y,t)$ for $B_1 = 1, \delta = 8.5, \gamma = 2.01, \alpha = 4.25, \lambda = 0.92$, and $t = 1$.

Case 3:

$$\alpha = \alpha, \qquad \beta = \frac{i\sqrt{\alpha^2\delta^4 + \left(-2\,{B_1}^2\lambda - \alpha^2 + 2\,\gamma\right)\delta^2 - 2\,{B_1}^2\lambda - 2\,\gamma}}{\sqrt{\delta^2-1}\delta^2},$$

$$\mu = \frac{\sqrt{\lambda}B_1}{\delta}\sqrt{\delta^2-1}, \qquad \delta=\delta, \qquad \gamma=\gamma, \qquad A_0=0, \qquad A_1=0, \qquad B_1=B_1.$$

We have replaced the above values with Eqs. (3.7) and (3.8) together with (3.5)

$$\mathscr{U}_6(\xi) = -\frac{B_1\cos(\xi)}{\sin(\xi)},$$

$$\mathscr{V}_6(\xi) = \frac{2\,i\lambda\,{B_1}^2\left(-\cot(\xi)-\xi\right)}{\mu\,(\delta^2-1)}. \tag{3.20}$$

Hence, the following exact solution has been reached for equation (1.1):

$$q_6(x,y,t) = \left(-\frac{B_1\cos(\xi)}{\sin(\xi)}\right) e^{i(\alpha x+\beta y+\gamma t)},$$

$$\phi_6(x,y,t) = \left(\frac{2\,i\lambda\,{B_1}^2\left[-\cot(\xi)-\xi\right]}{\mu\,[\delta^2-1]}\right) e^{i(\alpha x+\beta y+\gamma t)}. \tag{3.21}$$

Family 3: We attain $p = [1,1,-1,1]$ and $q = [2,0,2,0]$, which gives

$$\Xi(\xi) = \frac{\mathrm{e}^{2\,\xi}+1}{\mathrm{e}^{2\,\xi}-1}. \tag{3.22}$$

Case 1:

$$\alpha = \alpha, \qquad \beta = \frac{i\sqrt{\alpha^2\delta^4 + \left(8\,{A_1}^2\lambda - \alpha^2 + 2\,\gamma\right)\delta^2 + 8\,{A_1}^2\lambda - 2\,\gamma}}{\sqrt{\delta^2-1}\delta^2},$$

$$\mu = \frac{\sqrt{\lambda}A_1}{\delta}\sqrt{\delta^2-1}, \qquad \delta=\delta, \qquad \gamma=\gamma, \qquad A_0=0, \qquad A_1=A_1, \qquad B_1=A_1.$$

We have replaced the above values with Eqs. (3.7) and (3.8) together with (3.5)

$$\mathscr{U}_7(\xi) = \frac{2A_1\left(\mathrm{e}^{4\,\xi}+1\right)}{\mathrm{e}^{4\,\xi}-1},$$

$$\mathscr{V}_7(\xi) = \frac{2\,i\lambda\,\left(-4+4\,\xi\left[\mathrm{e}^{4\,\xi}-1\right]\right){A_1}^2}{\mu\,(\delta^2-1)\left(\mathrm{e}^{4\,\xi}-1\right)}. \tag{3.23}$$

Hence, the following exact solution has been reached for Eq. (1.1):

$$q_7(x,y,t) = \left(\frac{2A_1\left[\mathrm{e}^{4\,\xi}+1\right]}{\mathrm{e}^{4\,\xi}-1}\right) e^{i(\alpha x+\beta y+\gamma t)},$$

$$\phi_7(x,y,t) = \left(\frac{2\,i\lambda\,\left[-4+4\,\xi\left\{\mathrm{e}^{4\,\xi}-1\right\}\right]A_1^2}{\mu\,[\delta^2-1]\left[\mathrm{e}^{4\,\xi}-1\right]}\right) e^{i(\alpha x+\beta y+\gamma t)}. \tag{3.24}$$

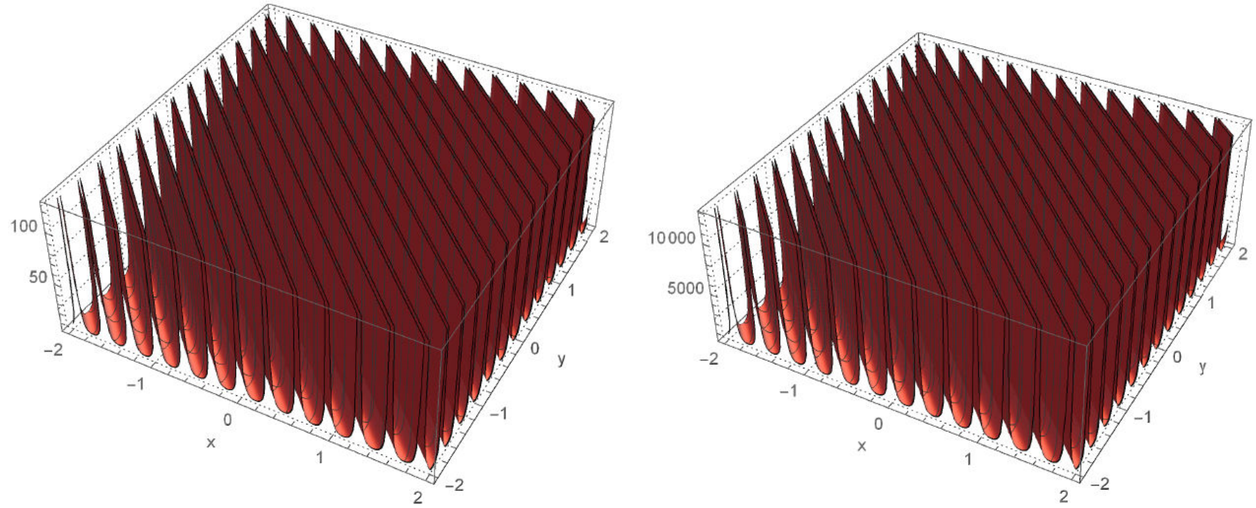

FIGURE 3.Dynamic behaviours modulus of solutions $q_8\,(x,y,t)$ (left) and $\phi_8\,(x,y,t)$ (right) for $A_1 = 1, \delta = 1.01, \gamma = 2.3, \alpha = 0.2$, $\lambda = 0.5$, and $t = 1$.

Case 2:

$$\alpha = \alpha, \qquad \beta = \frac{i\sqrt{\alpha^2\delta^4 + \left(-4\,{A_1}^2\lambda - \alpha^2 + 2\,\gamma\right)\delta^2 - 4\,{A_1}^2\lambda - 2\,\gamma}}{\sqrt{\delta^2-1}\delta^2},$$

$$\mu = \frac{\sqrt{\lambda}A_1}{\sqrt{\delta^2-1}\delta}, \qquad \delta = \delta, \qquad \gamma = \gamma, \qquad A_0 = 0, \qquad A_1 = A_1, \qquad B_1 = -A_1.$$

We have replaced the above values with Eqs. (3.7) and (3.8) together with (3.5)

$$\mathscr{U}_8(\xi) = \frac{4\mathrm{e}^{2\,\xi}A_1}{\mathrm{e}^{4\,\xi}-1},$$
$$\mathscr{V}_8(\xi) = \frac{-8\,i\lambda\,{A_1}^2}{\mu\,\left(\delta^2-1\right)\left(\mathrm{e}^{4\,\xi}-1\right)}. \tag{3.25}$$

Hence, the following exact solution has been reached for Eq. (1.1):

$$q_8\,(x,y,t) = \left(\frac{4\mathrm{e}^{2\,\xi}A_1}{\mathrm{e}^{4\,\xi}-1}\right)e^{i(\alpha x+\beta y+\gamma t)},$$
$$\phi_8\,(x,y,t) = \left(\frac{-8\,i\lambda\,{A_1}^2}{\mu\,\left[\delta^2-1\right]\left[\mathrm{e}^{4\,\xi}-1\right]}\right)e^{i(\alpha x+\beta y+\gamma t)}. \tag{3.26}$$

Figure 3 shows the dynamic behavior of modulus of solutions $q_8\,(x,y,t)$ (left) and $\phi_8\,(x,y,t)$ for $A_1 = 1, \delta = 1.01, \gamma = 2.3, \alpha = 0.2, \lambda = 0.5$, and $t = 1$.

Family 4: We attain the results for $p = [-1,3,1,-1]$ and $q = [2,0,2,0]$. So, it reads

$$\Xi\,(\xi) = \frac{-\mathrm{e}^{2\,\xi}+3}{\mathrm{e}^{2\,\xi}-1}. \tag{3.27}$$

Case 1:

$$\alpha = \alpha, \beta = \frac{i\sqrt{\alpha^2\delta^4 + \left(2\,{A_1}^2\lambda - \alpha^2 + 2\,\gamma\right)\delta^2 + 2\,{A_1}^2\lambda - 2\,\gamma}}{\sqrt{\delta^2-1}\delta^2},$$

$$\mu = \frac{\sqrt{\lambda}A_1}{\sqrt{\delta^2-1}\delta}, \qquad \delta = \delta, \qquad \gamma = \gamma, \qquad A_0 = 2\,A_1, \qquad A_1 = A_1, \qquad B_1 = 0.$$

We have replaced the above values with Eqs. (3.7) and (3.8) together with (3.5)

$$\mathscr{U}_9(\xi) = \frac{A_1\left(\mathrm{e}^{2\,\xi}+1\right)}{\mathrm{e}^{2\,\xi}-1},$$
$$\mathscr{V}_9(\xi) = \frac{2\,i\lambda\left(-4+2\,\xi\left[\mathrm{e}^{2\,\xi}-1\right]\right)A_1{}^2}{\mu\,(\delta^2-1)\,(2\,\mathrm{e}^{2\,\xi}-2)}. \tag{3.28}$$

Hence, the following exact solution has been reached for equation (1.1):

$$q_9\,(x,y,t) = \left(\frac{A_1\left[\mathrm{e}^{2\,\xi}+1\right]}{\mathrm{e}^{2\,\xi}-1}\right)e^{i(\alpha x+\beta y+\gamma t)},$$
$$\phi_9\,(x,y,t) = \left(\frac{2\,i\lambda\left[-4+2\,\xi\left\{\mathrm{e}^{2\,\xi}-1\right\}\right]A_1{}^2}{\mu\,[\delta^2-1]\,[2\,\mathrm{e}^{2\,\xi}-2]}\right)e^{i(\alpha x+\beta y+\gamma t)}. \tag{3.29}$$

Case 2:

$$\alpha=\alpha, \qquad \beta = \frac{i\sqrt{2}\sqrt{2\,\alpha^2\delta^4+\left(A_0{}^2\lambda-2\,\alpha^2+4\,\gamma\right)\delta^2+A_0{}^2\lambda-4\,\gamma}}{2\sqrt{\delta^2-1}\delta^2},$$
$$\mu=\frac{A_0\sqrt{\lambda}}{2\delta\sqrt{\delta^2-1}}, \qquad \delta=\delta, \qquad \gamma=\gamma, \qquad A_0=A_0, \qquad A_1=0, \qquad B_1=3/2\,A_0.$$

We have replaced the above values with Eqs. (3.7) and (3.8) together with (3.5)

$$\mathscr{U}_{10}(\xi) = -\frac{A_0\left(\mathrm{e}^{2\,\xi}+3\right)}{2\,\mathrm{e}^{2\,\xi}-6},$$
$$\mathscr{V}_{10}(\xi) = \frac{2\,i\lambda\,A_0{}^2\left(-12+2\,\xi\left[\mathrm{e}^{2\,\xi}-3\right]\right)}{\mu\,(\delta^2-1)\,(8\,\mathrm{e}^{2\,\xi}-24)}. \tag{3.30}$$

Hence, the following exact solution has been reached for equation (1.1):

$$q_{10}\,(x,y,t) = \left(-\frac{A_0\left[\mathrm{e}^{2\,\xi}+3\right]}{2\,\mathrm{e}^{2\,\xi}-6}\right)e^{i(\alpha x+\beta y+\gamma t)},$$
$$\phi_{10}\,(x,y,t) = \left(\frac{2\,i\lambda\,A_0{}^2\left[-12+2\,\xi\left\{\mathrm{e}^{2\,\xi}-3\right\}\right]}{\mu\,[\delta^2-1]\,[8\,\mathrm{e}^{2\,\xi}-24]}\right)e^{i(\alpha x+\beta y+\gamma t)}. \tag{3.31}$$

Family 5: We attain the results for $p=[-1,1,1,1]$ and $q=[1,-1,1,-1]$, which gives

$$\Xi\,(\xi) = -\frac{\sinh(\xi)}{\cosh(\xi)}. \tag{3.32}$$

Case 1:

$$\alpha=\alpha, \qquad \beta=\frac{i\sqrt{\alpha^2\delta^4+\left(-4\,A_1{}^2\lambda-\alpha^2+2\,\gamma\right)\delta^2-4\,A_1{}^2\lambda-2\,\gamma}}{\sqrt{\delta^2-1}\delta^2},$$
$$\mu=\frac{\sqrt{\lambda}A_1}{\sqrt{\delta^2-1}\delta}, \qquad \delta=\delta, \qquad \gamma=\gamma, \qquad A_0=0, \qquad A_1=A_1, \qquad B_1=-A_1.$$

We have replaced the above values with Eqs. (3.7) and (3.8) together with (3.5)

$$\mathscr{U}_{11}(\xi) = \frac{A_1}{\cosh(\xi)\sinh(\xi)},$$
$$\mathscr{V}_{11}(\xi) = \frac{2\,i\lambda\,A_1{}^2\left(-2\,\left[\cosh(\xi)\right]^2+1\right)}{\mu\,(\delta^2-1)\cosh(\xi)\sinh(\xi)}. \tag{3.33}$$

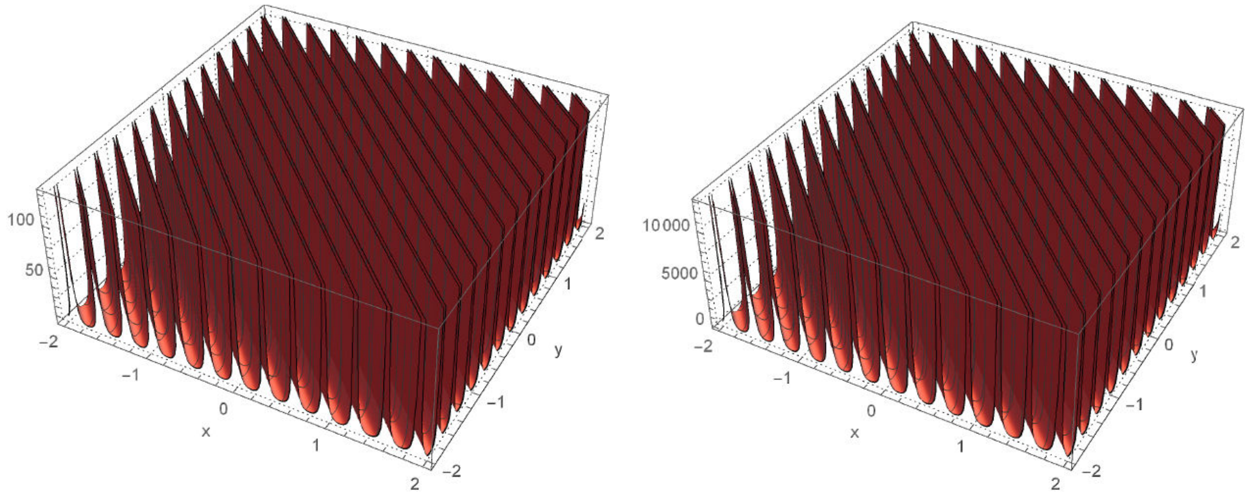

FIGURE 4. Dynamic behaviours modulus of solutions $q_{11}(x,y,t)$ (left) and $\phi_{11}(x,y,t)$ (right) for $A_1 = 1, \delta = 1.01, \gamma = 2.3, \alpha = 0.2, \lambda = 0.5$, and $t = 1$.

Hence, the following exact solution has been reached for Eq. (1.1):

$$q_{11}(x,y,t) = \left(\frac{A_1}{\cosh(\xi)\sinh(\xi)}\right) e^{i(\alpha x+\beta y+\gamma t)},$$

$$\phi_{11}(x,y,t) = \left(\frac{2\,i\lambda\,{A_1}^2\left[-2\,\{\cosh(\xi)\}^2+1\right]}{\mu\,[\delta^2-1]\cosh(\xi)\sinh(\xi)}\right) e^{i(\alpha x+\beta y+\gamma t)}. \tag{3.34}$$

Figure 4 shows the dynamic behavior of modulus of solutions $q_{11}(x,y,t)$ (left) and $\phi_{11}(x,y,t)$ for $A_1 = 1, \delta = 1.01, \gamma = 2.3, \alpha = 0.2, \lambda = 0.5$, and $t = 1$.

Family 6: We attain the results for $p = [-1-i, 1-i, -1, 1]$ and $q = [i, -i, i, -i]$, and one obtains

$$\Xi(\xi) = \frac{\cos(\xi) + \sin(\xi)}{\sin(\xi)}. \tag{3.35}$$

Case 1:

$$\alpha = \alpha, \qquad \beta = \frac{i\sqrt{\alpha^2\delta^4 + \left(-2\,{A_0}^2\lambda - \alpha^2 + 2\,\gamma\right)\delta^2 - 2\,{A_0}^2\lambda - 2\,\gamma}}{\sqrt{\delta^2-1}\,\delta^2},$$

$$\mu = \frac{A_0\sqrt{\lambda}}{\sqrt{\delta^2-1}\,\delta}, \qquad \delta = \delta, \qquad \gamma = \gamma, \qquad A_0 = A_0, \qquad A_1 = 0, \qquad B_1 = -2\,A_0.$$

We have replaced the above values with Eqs. (3.7) and (3.8) together with (3.5)

$$\mathscr{U}_{12}(\xi) = \frac{A_0\left(-\sin(\xi) + \cos(\xi)\right)}{\cos(\xi) + \sin(\xi)},$$

$$\mathscr{V}_{12}(\xi) = \frac{-2\,i\lambda\,{A_0}^2\left(\xi\,\tan(\xi) + \xi + 2\right)}{\mu\,(\delta^2-1)\,(\tan(\xi)+1)}. \tag{3.36}$$

Hence, the following exact solution has been reached for Eq. (1.1):

$$q_{12}(x,y,t) = \left(\frac{A_0\left[-\sin(\xi) + \cos(\xi)\right]}{\cos(\xi) + \sin(\xi)}\right) e^{i(\alpha x+\beta y+\gamma t)},$$

$$\phi_{12}(x,y,t) = \left(\frac{-2\,i\lambda\,{A_0}^2\left[\xi\,\tan(\xi) + \xi + 2\right]}{\mu\,[\delta^2-1]\,[\tan(\xi)+1]}\right) e^{i(\alpha x+\beta y+\gamma t)}. \tag{3.37}$$

Family 7: We obtain $p = [-2-i, 2-i, -1, 1]$ and $q = [i, -i, i, -i]$, and thus one attains

$$\Xi(\xi) = \frac{\cos(\xi) + 2\,\sin(\xi)}{\sin(\xi)}. \tag{3.38}$$

Case 1:

$$\alpha=\alpha,\qquad \beta=\frac{i/2\sqrt{2}\sqrt{2\,\alpha^2\delta^4+\left(-{A_0}^2\lambda-2\,\alpha^2+4\,\gamma\right)\delta^2-{A_0}^2\lambda-4\,\gamma}}{\sqrt{\delta^2-1}\delta^2},$$

$$\mu=1/2\,\frac{A_0\sqrt{\lambda}}{\sqrt{\delta^2-1}\delta},\qquad \delta=\delta,\qquad \gamma=\gamma,\qquad A_0=A_0,\qquad A_1=0,\qquad B_1=-5/2\,A_0.$$

We have replaced the above values with Eqs. (3.7) and (3.8) together with (3.5)

$$\mathscr{U}_{13}(\xi)=\frac{A_0\left(-\sin\left(\xi\right)+2\,\cos\left(\xi\right)\right)}{2\,\cos\left(\xi\right)+4\,\sin\left(\xi\right)},$$

$$\mathscr{V}_{13}(\xi)=\frac{-2\,i\lambda\,{A_0}^2\left(4\,\xi\,\tan\left(\xi\right)+2\,\xi+5\right)}{\mu\,\left(\delta^2-1\right)\left(8+16\,\tan\left(\xi\right)\right)}. \tag{3.39}$$

Hence, the following exact solution has been reached for equation (1.1):

$$q_{13}\left(x,y,t\right)=\left(\frac{A_0\left[-\sin\left(\xi\right)+2\,\cos\left(\xi\right)\right]}{2\,\cos\left(\xi\right)+4\,\sin\left(\xi\right)}\right)e^{i(\alpha x+\beta y+\gamma t)},$$

$$\phi_{13}\left(x,y,t\right)=\left(\frac{-2\,i\lambda\,{A_0}^2\left[4\,\xi\,\tan\left(\xi\right)+2\,\xi+5\right]}{\mu\,\left[\delta^2-1\right]\left[8+16\,\tan\left(\xi\right)\right]}\right)e^{i(\alpha x+\beta y+\gamma t)}. \tag{3.40}$$

Family 8: We attain the results for $p=[1-i,-1-i,-1,1]$ and $q=[i,-i,i,-i]$, and thus we obtain

$$\Xi\left(\xi\right)=\frac{-\sin\left(\xi\right)+\cos\left(\xi\right)}{\sin\left(\xi\right)}. \tag{3.41}$$

Case 1:

$$\alpha=\alpha,\qquad \beta=\frac{i\sqrt{\alpha^2\delta^4+\left(-2\,{A_0}^2\lambda-\alpha^2+2\,\gamma\right)\delta^2-2\,{A_0}^2\lambda-2\,\gamma}}{\sqrt{\delta^2-1}\delta^2},$$

$$\mu=\frac{A_0\sqrt{\lambda}}{\sqrt{\delta^2-1}\delta},\qquad \delta=\delta,\qquad \gamma=\gamma,\qquad A_0=A_0,\qquad A_1=0,\qquad B_1=2\,A_0.$$

We have replaced the above values with Eqs. (3.7) and (3.8) together with (3.5)

$$\mathscr{U}_{14}(\xi)=\frac{2\,A_0\sin\left(\xi\right)\cos\left(\xi\right)+A_0}{2\,\left(\cos\left(\xi\right)\right)^2-1},$$

$$\mathscr{V}_{14}(\xi)=\frac{-2\,i\lambda\,{A_0}^2\left(\xi\,\tan\left(\xi\right)-\xi+2\right)}{\mu\,\left(\delta^2-1\right)\left(\tan\left(\xi\right)-1\right)}. \tag{3.42}$$

Hence, the following exact solution has been reached for Eq. (1.1):

$$q_{14}\left(x,y,t\right)=\left(\frac{2\,A_0\sin\left(\xi\right)\cos\left(\xi\right)+A_0}{2\,\left[\cos\left(\xi\right)\right]^2-1}\right)e^{i(\alpha x+\beta y+\gamma t)},$$

$$\phi_{14}\left(x,y,t\right)=\left(\frac{-2\,i\lambda\,{A_0}^2\left[\xi\,\tan\left(\xi\right)-\xi+2\right]}{\mu\,\left[\delta^2-1\right]\left[\tan\left(\xi\right)-1\right]}\right)e^{i(\alpha x+\beta y+\gamma t)}. \tag{3.43}$$

Family 9: We attain the results for$p=[-3,-1,1,1]$ and $q=[1,-1,1,-1]$, and thus we have

$$\Xi\left(\xi\right)=\frac{-\sinh\left(\xi\right)-2\,\cosh\left(\xi\right)}{\cosh\left(\xi\right)}. \tag{3.44}$$

Case 1:

$$\alpha=\alpha,\qquad \beta=\frac{i\sqrt{2}\sqrt{2\,\alpha^2\delta^4+\left({A_0}^2\lambda-2\,\alpha^2+4\,\gamma\right)\delta^2+{A_0}^2\lambda-4\,\gamma}}{2\sqrt{\delta^2-1}\delta^2},$$

$$\mu=\frac{A_0\sqrt{\lambda}}{2\sqrt{\delta^2-1}\delta},\qquad \delta=\delta,\qquad \gamma=\gamma,\qquad A_0=A_0,\qquad A_1=0,\qquad B_1=3/2\,A_0.$$

We have replaced the above values with Eqs. (3.7) and (3.8) together with (3.5)

$$\mathscr{U}_{15}(\xi) = \frac{A_0\,(2\,\sinh(\xi) + \cosh(\xi))}{4\,\cosh(\xi) + 2\,\sinh(\xi)},$$
$$\mathscr{V}_{15}(\xi) = \frac{2i\lambda\,(4\cosh(\xi)\ln(\cosh(\xi) - 1 + \sinh(\xi)) + 2\sinh(\xi)\ln(\cosh(\xi) - 1 + \sinh(\xi)))\,{A_0}^2}{\mu\,(\delta^2 - 1)\,(16\cosh(\xi) + 8\sinh(\xi))}$$
$$+ \frac{2i\lambda\,(-4\cosh(\xi)\ln(\cosh(\xi) - 1 - \sinh(\xi)) - 2\sinh(\xi)\ln(\cosh(\xi) - 1 - \sinh(\xi)) - 3\sinh(\xi))\,{A_0}^2}{\mu\,(\delta^2 - 1)\,(16\cosh(\xi) + 8\sinh(\xi))}. \quad (3.45)$$

Hence, the following exact solution has been reached for Eq. (1.1):

$$q_{15}(x, y, t) = \left(\frac{A_0\,[2\,\sinh(\xi) + \cosh(\xi)]}{4\,\cosh(\xi) + 2\,\sinh(\xi)}\right) e^{i(\alpha x + \beta y + \gamma t)},$$
$$\phi_{15}(x, y, t) = \mathscr{V}_{15}(\xi) e^{i(\alpha x + \beta y + \gamma t)}. \quad (3.46)$$

Family 10: We attain the results for $p = [-2 - i, -2 + i, 1, 1]$ and $q = [i, -i, i, -i]$, and thus one has

$$\Xi(\xi) = \frac{\sin(\xi) - 2\,\cos(\xi)}{\cos(\xi)}. \quad (3.47)$$

Case 1:

$$\alpha = \alpha, \qquad \beta = \frac{i\sqrt{2}\sqrt{2\,\alpha^2\delta^4 + \left(-{A_0}^2\lambda - 2\,\alpha^2 + 4\,\gamma\right)\delta^2 - {A_0}^2\lambda - 4\,\gamma}}{2\sqrt{\delta^2 - 1}\,\delta^2},$$
$$\mu = \frac{A_0\sqrt{\lambda}}{2\sqrt{\delta^2 - 1}\,\delta}, \qquad \delta = \delta, \qquad \gamma = \gamma, \qquad A_0 = A_0, \qquad A_1 = 0, \qquad B_1 = 5/2\,A_0.$$

We have replaced the above values with Eqs. (3.7) and (3.8) together with (3.5)

$$\mathscr{U}_{16}(\xi) = \frac{A_0\,(\cos(\xi) + 2\,\sin(\xi))}{2\,\sin(\xi) - 4\,\cos(\xi)},$$
$$\mathscr{V}_{16}(\xi) = \frac{-2\,i\lambda\,{A_0}^2\,(\xi\,\tan(\xi) - 2\,\xi + 5)}{\mu\,(\delta^2 - 1)\,(4\,\tan(\xi) - 8)}. \quad (3.48)$$

Hence, the following exact solution has been reached for Eq. (1.1):

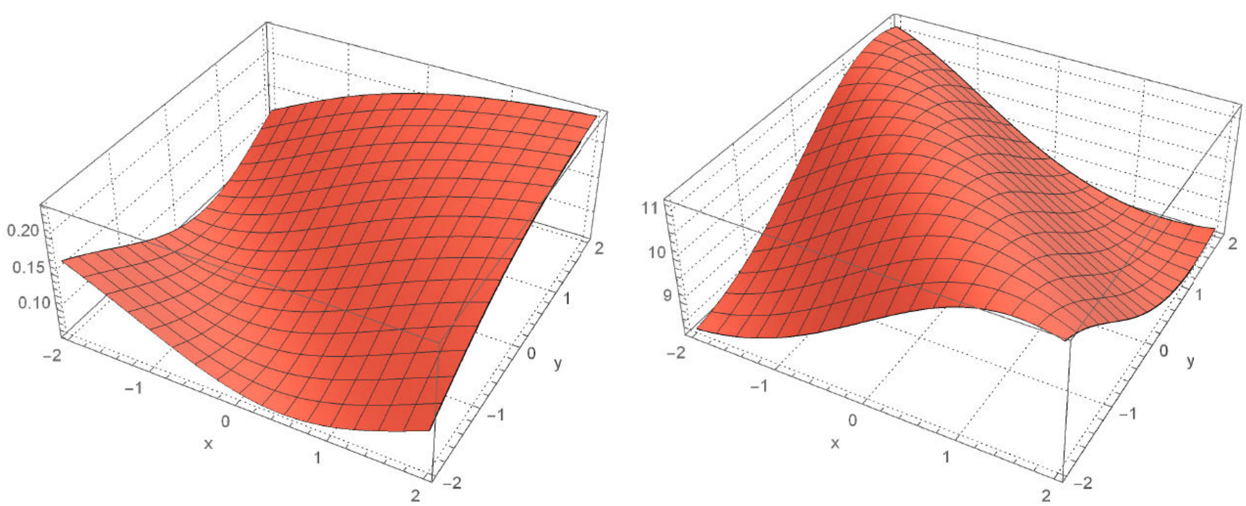

FIGURE 5. Dynamic behaviours modulus of solutions $q_{16}(x, y, t)$ (left) and $\phi_{16}(x, y, t)$ (right) for $A_0 = 1, \delta = 1.1, \gamma = 0.8, \alpha = 0.9$, $\lambda = 0.2$, and $t = 1$.

$$q_{16}(x,y,t) = \left(\frac{A_0 \left[\cos(\xi) + 2\,\sin(\xi)\right]}{2\,\sin(\xi) - 4\,\cos(\xi)}\right) e^{i(\alpha x+\beta y+\gamma t)},$$

$$\phi_{16}(x,y,t) = \left(\frac{-2\,i\lambda\,{A_0}^2\left[\xi\,\tan(\xi) - 2\,\xi + 5\right]}{\mu\,\left[\delta^2-1\right]\left[4\,\tan(\xi) - 8\right]}\right) e^{i(\alpha x+\beta y+\gamma t)}. \tag{3.49}$$

Figure 5 shows the dynamic behavior of modulus of solutions $q_{16}(x,y,t)$ (left) and $\phi_{16}(x,y,t)$ for $A_0 = 1, \delta = 1.1$, $\gamma = 0.8$, $\alpha = 0.9$, $\lambda = 0.2$, and $t = 1$.

Family 11: We attain the results for $p = [1-i, 1+i, 1, 1]$ and $q = [i, -i, i, -i]$, and then one results

$$\Xi(\xi) = \frac{\cos(\xi) + \sin(\xi)}{\cos(\xi)}. \tag{3.50}$$

Case 1:

$$\alpha = \alpha, \qquad \beta = \frac{i\sqrt{\alpha^2\delta^4 + \left(-2\,{A_1}^2\lambda - \alpha^2 + 2\,\gamma\right)\delta^2 - 2\,{A_1}^2\lambda - 2\,\gamma}}{\sqrt{\delta^2-1}\,\delta^2},$$

$$\mu = \frac{\sqrt{\lambda}A_1}{\sqrt{\delta^2-1}\,\delta}, \qquad \delta = \delta, \qquad \gamma = \gamma, \qquad A_0 = -A_1, \qquad A_1 = A_1, \qquad B_1 = 0,$$

We have replaced the above values with Eqs. (3.7) and (3.8) together with (3.5)

$$\mathscr{U}_{17}(\xi) = \frac{A_1\sin(\xi)}{\cos(\xi)},$$

$$\mathscr{V}_{17}(\xi) = \frac{2\,i\lambda\,A_1^2\left(\tan(\xi) - \xi\right)}{\mu\,\left(\delta^2-1\right)}. \tag{3.51}$$

Hence, the following exact solution was reached for Eq. (1.1):

$$q_{17}(x,y,t) = \left(\frac{A_1\sin(\xi)}{\cos(\xi)}\right) e^{i(\alpha x+\beta y+\gamma t)},$$

$$\phi_{17}(x,y,t) = \left(\frac{2\,i\lambda\,A_1^2\left[\tan(\xi) - \xi\right]}{\mu\,\left[\delta^2-1\right]}\right) e^{i(\alpha x+\beta y+\gamma t)}. \tag{3.52}$$

Figure 6 shows the dynamic behavior of modulus of solutions $q_{17}(x,y,t)$ (left) and $\phi_{17}(x,y,t)$ for $A_1 = 1, \delta = 1.5$, $\gamma = 0.5$, $\alpha = 0.1$, $\lambda = 0.2$, and $t = 1$.

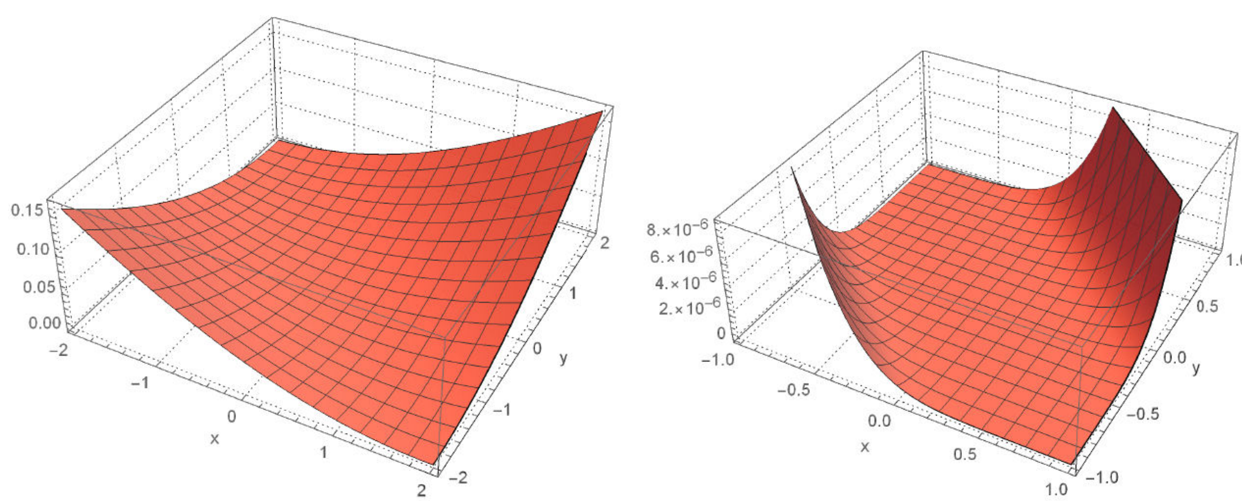

FIGURE 6. Dynamic behaviours modulus of solutions $q_{17}(x,y,t)$ (left) and $\phi_{17}(x,y,t)$ (right) for $A_1 = 1, \delta = 1.5$, $\gamma = 0.5$, $\alpha = 0.1$, $\lambda = 0.2$, and $t = 1$.

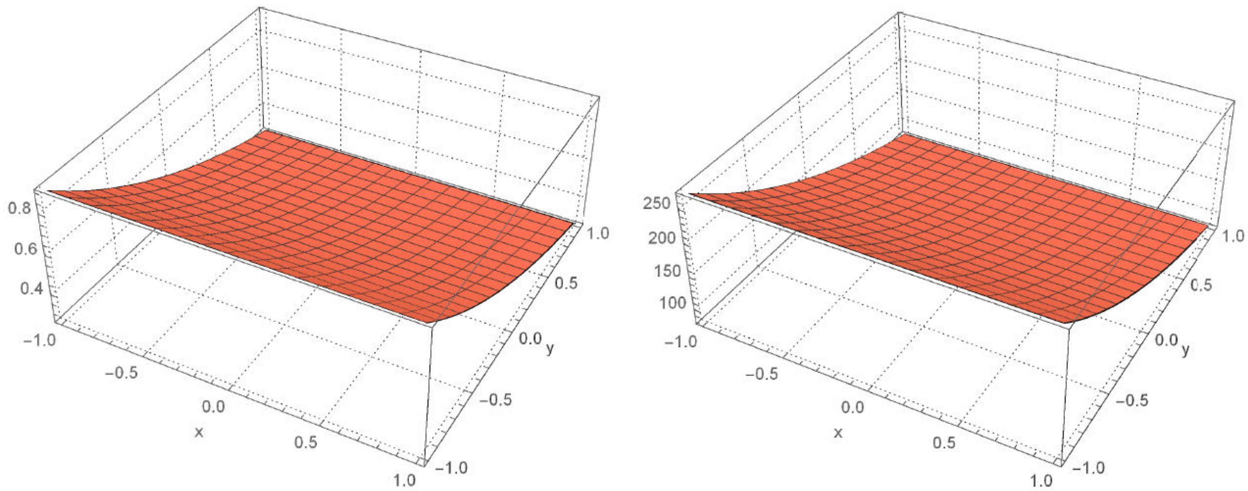

FIGURE 7. Dynamic behaviours modulus of solutions $q_{18}\left(x,y,t\right)$ (left) and $\phi_{18}\left(x,y,t\right)$ (right) for $A_0=1, \delta=5, \gamma=0.1, \alpha=1.5$, $\lambda=0.3$, and $t=1$.

Family 12:

We attain the results for $p=[-3,-2,1,1]$ and $q=[0,1,0,1]$, and we get

$$\Xi\left(\xi\right)=\frac{-3-2\,\mathrm{e}^{\xi}}{1+\mathrm{e}^{\xi}}. \tag{3.53}$$

Case 1:

$$\alpha=\alpha, \qquad \beta=\frac{i/5\sqrt{25\,\alpha^2\delta^4+\left(2\,{A_0}^2\lambda-25\,\alpha^2+50\,\gamma\right)\delta^2+2\,{A_0}^2\lambda-50\,\gamma}}{\sqrt{\delta^2-1}\delta^2},$$

$$\mu=\frac{2A_0\sqrt{\lambda}}{5\sqrt{\delta^2-1}\delta}, \qquad \delta=\delta, \qquad \gamma=\gamma, \qquad A_0=A_0, \qquad A_1=0, \qquad B_1=\frac{12\,A_0}{5},$$

We have replaced the above values with Eqs. (3.7) and (3.8) together with (3.5)

$$\begin{aligned}\mathscr{U}_{18}(\xi)&=-\frac{A_0\left(2\,\mathrm{e}^{\xi}-3\right)}{15+10\,\mathrm{e}^{\xi}},\\ \mathscr{V}_{18}(\xi)&=\frac{2\,i\lambda\,{A_0}^2\left(2\,\xi\,\mathrm{e}^{\xi}+3\,\xi+12\right)}{\mu\,\left(\delta^2-1\right)\left(75+50\,\mathrm{e}^{\xi}\right)}.\end{aligned} \tag{3.54}$$

Hence, the following exact solution has been reached for equation (1.1):

$$\begin{aligned}q_{18}\left(x,y,t\right)&=\left(-\frac{A_0\left[2\,\mathrm{e}^{\xi}-3\right]}{15+10\,\mathrm{e}^{\xi}}\right)e^{i(\alpha x+\beta y+\gamma t)},\\ \phi_{18}\left(x,y,t\right)&=\left(\frac{2\,i\lambda\,{A_0}^2\left[2\,\xi\,\mathrm{e}^{\xi}+3\,\xi+12\right]}{\mu\,\left[\delta^2-1\right]\left[75+50\,\mathrm{e}^{\xi}\right]}\right)e^{i(\alpha x+\beta y+\gamma t)}.\end{aligned} \tag{3.55}$$

Figure 7 shows the dynamic behavior of modulus of solutions $q_{18}\left(x,y,t\right)$ (left) and $\phi_{18}\left(x,y,t\right)$ for $A_0=1, \delta=5, \gamma=0.1$, $\alpha=1.5$, $\lambda=0.3$, and $t=1$.

Family 13:

We attain the results for $p=[-1,-2,1,1]$ and $q=[1,0,1,0]$, and one finds

$$\Xi\left(\xi\right)=\frac{-\mathrm{e}^{\xi}-2}{\mathrm{e}^{\xi}+1}. \tag{3.56}$$

Case 1:

$$\alpha=\alpha,\qquad \beta=\frac{i\sqrt{9\,\alpha^2\delta^4+\left(2\,{A_0}^2\lambda-9\,\alpha^2+18\,\gamma\right)\delta^2+2\,{A_0}^2\lambda-18\,\gamma}}{3\sqrt{\delta^2-1}\delta^2},$$
$$\mu=\frac{2A_0\sqrt{\lambda}}{3\sqrt{\delta^2-1}\delta},\qquad \delta=\delta,\qquad \gamma=\gamma,\qquad A_0=A_0,\qquad A_1=0,\qquad B_1=4/3\,A_0.$$

We have replaced the above values with Eqs. (3.7) and (3.8) together with (3.5)

$$\begin{aligned}\mathscr{U}_{19}(\xi)&=-\frac{A_0\left(\mathrm{e}^{\xi}-2\right)}{3\,\mathrm{e}^{\xi}+6},\\ \mathscr{V}_{19}(\xi)&=\frac{2\,i\lambda\,{A_0}^2\left(\xi\,\mathrm{e}^{\xi}+2\,\xi+8\right)}{\mu\,\left(\delta^2-1\right)\left(9\,\mathrm{e}^{\xi}+18\right)}.\end{aligned}\tag{3.57}$$

Hence, the following exact solution has been reached for equation (1.1):

$$\begin{aligned}q_{19}\left(x,y,t\right)&=\left(-\frac{A_0\left[\mathrm{e}^{\xi}-2\right]}{3\,\mathrm{e}^{\xi}+6}\right)e^{i(\alpha x+\beta y+\gamma t)},\\ \phi_{19}\left(x,y,t\right)&=\left(\frac{2\,i\lambda\,{A_0}^2\left[\xi\,\mathrm{e}^{\xi}+2\,\xi+8\right]}{\mu\,\left[\delta^2-1\right]\left[9\,\mathrm{e}^{\xi}+18\right]}\right)e^{i(\alpha x+\beta y+\gamma t)}.\end{aligned}\tag{3.58}$$

Family 14: We attain the results for $p=[2,1,1,1]$ and $q=[1,0,1,0]$, and then we obtain

$$\Xi\left(\xi\right)=\frac{2\,\mathrm{e}^{\xi}+1}{\mathrm{e}^{\xi}+1}.\tag{3.59}$$

Case 1:

$$\alpha=\alpha,\qquad \beta=\frac{i\sqrt{9\,\alpha^2\delta^4+\left(2\,{A_0}^2\lambda-9\,\alpha^2+18\,\gamma\right)\delta^2+2\,{A_0}^2\lambda-18\,\gamma}}{3\sqrt{\delta^2-1}\delta^2},$$
$$\mu=\frac{2A_0\sqrt{\lambda}}{3\sqrt{\delta^2-1}\delta},\qquad \delta=\delta,\qquad \gamma=\gamma,\qquad A_0=A_0,\qquad A_1=0,\qquad B_1=-4/3\,A_0.$$

We have replaced the above values with Eqs. (3.7) and (3.8) together with (3.5)

$$\begin{aligned}\mathscr{U}_{20}(\xi)&=\frac{A_0\left(2\,\mathrm{e}^{\xi}-1\right)}{6\,\mathrm{e}^{\xi}+3},\\ \mathscr{V}_{20}(\xi)&=\frac{2\,i\lambda\,{A_0}^2\left(2\,\xi\,\mathrm{e}^{\xi}+\xi+4\right)}{\mu\,\left(\delta^2-1\right)\left(18\,\mathrm{e}^{\xi}+9\right)}.\end{aligned}\tag{3.60}$$

Hence, the following exact solution has been reached for equation (1.1):

$$\begin{aligned}q_{20}\left(x,y,t\right)&=\left(\frac{A_0\left[2\,\mathrm{e}^{\xi}-1\right]}{6\,\mathrm{e}^{\xi}+3}\right)e^{i(\alpha x+\beta y+\gamma t)},\\ \phi_{20}\left(x,y,t\right)&=\left(\frac{2\,i\lambda\,{A_0}^2\left[2\,\xi\,\mathrm{e}^{\xi}+\xi+4\right]}{\mu\,\left[\delta^2-1\right]\left[18\,\mathrm{e}^{\xi}+9\right]}\right)e^{i(\alpha x+\beta y+\gamma t)}.\end{aligned}\tag{3.61}$$

Family 15: We attain the results for $p=[-1,0,1,1]$ and $q=[0,0,1,0]$, and then we find

$$\Xi\left(\xi\right)=-\frac{1}{1+e^{\xi}}.\tag{3.62}$$

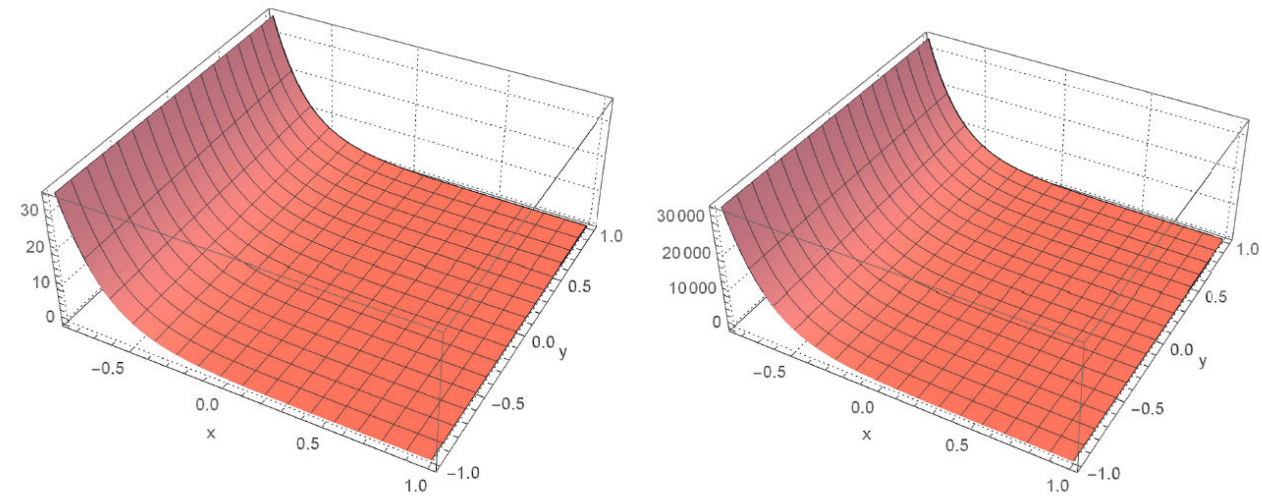

FIGURE 8. Dynamic behaviours modulus of solutions $q_{21}\left(x,y,t\right)$ (left) and $\phi_{21}\left(x,y,t\right)$ (right) for $A_0=1, \delta=2, \gamma=0.9, \beta=0.5, \lambda=0.7$, and $t=1$.

Case 1:

$$\alpha=\frac{i\sqrt{\beta^2\delta^6-\beta^2\delta^4+\left(2\,{A_0}^2\lambda+2\,\gamma\right)\delta^2+2\,{A_0}^2\lambda-2\,\gamma}}{\sqrt{\delta^2-1}\delta},\qquad \beta=\beta,$$

$$\mu=\frac{2A_0\sqrt{\lambda}}{\delta\sqrt{\delta^2-1}},\qquad \delta=\delta,\qquad \gamma=\gamma,\qquad A_0=A_0,\qquad A_1=2\,A_0,\qquad B_1=0.$$

We have replaced the above values with Eqs. (3.7) and (3.8) together with (3.5)

$$\begin{aligned}\mathscr{U}_{21}(\xi)&=\frac{A_0\left(\mathrm{e}^{\xi}-1\right)}{1+\mathrm{e}^{\xi}},\\ \mathscr{V}_{21}(\xi)&=\frac{2\,i\lambda\,{A_0}^2\left(\xi\,\mathrm{e}^{\xi}+\xi+4\right)}{\mu\,\left(\delta^2-1\right)\left(1+\mathrm{e}^{\xi}\right)}\end{aligned}\tag{3.63}$$

Hence, the following exact solution has been reached for equation (1.1):

$$\begin{aligned}q_{21}\left(x,y,t\right)&=\left(\frac{A_0\left[\mathrm{e}^{\xi}-1\right]}{1+\mathrm{e}^{\xi}}\right)e^{i(\alpha x+\beta y+\gamma t)},\\ \phi_{21}\left(x,y,t\right)&=\left(\frac{2\,i\lambda\,{A_0}^2\left[\xi\,\mathrm{e}^{\xi}+\xi+4\right]}{\mu\,\left[\delta^2-1\right]\left[1+\mathrm{e}^{\xi}\right]}\right)e^{i(\alpha x+\beta y+\gamma t)}.\end{aligned}\tag{3.64}$$

Figure 8 shows the dynamic behavior of modulus of solutions $q_{21}\left(x,y,t\right)$ (left) and $\phi_{21}\left(x,y,t\right)$ for $A_0=1, \delta=2, \gamma=0.9, \beta=0.5, \lambda=0.7$, and $t=1$.

Remark 1 *In each of the above cases, we take* $\xi=i\mu\left(x+y-\left(\alpha\delta^2+\beta\delta^4\right)t\right)$.

4. Conclusion

Partial differential equations have many applications in modeling practical problems in our lives. This importance has created additional motivation for researchers to develop new and efficient methods. Some of these techniques enable us to achieve exact solutions to such problems. However, determining such solutions is impossible or very difficult for some categories of equations. The method used in this paper, called the GERFM, is a powerful technique to determine the exact solutions to different types of PDEs. In this survey, the method has been utilized to solve the Davey-Stewartson equation. It was shown that the method is a suitable technique to solve the Davey-Stewartson equation with this study. The results are quite reliable for solving this problem. Further, we believe that the presented methods and results in this paper are valuable to all researchers in the field of mathematical physics. Therefore, GERFM offers an excellent opportunity for future research studies on related topics of the research. This emphasizes the power of the method used in providing exact solutions to various real-world applied models.

1. A. Davey, K. Stewartson, On three-dimensional packets of surface waves, *Proc. R. Soc. London Ser. A*, **338** (1974) 101. https://doi.org/10.1098/rspa.1974.0076.
2. C. Babaoglu, Long-wave short-wave resonance case for a generalized Davey-Stewartson system, *Chaos Solitons Fractal* **38** (2008) 48. https://doi.org/10.1016/j.chaos.2008.02.007.
3. K.W. Chow, S.Y. Lou, Propagating wave patterns and peakons of the Davey-Stewartson system, *Chaos Solitons Fractal*, **27** (2006) 561. https://doi.org/10.1016/j.chaos.2005.04.036.
4. M. D. Groves, S-M Sun, E. Wahln, Periodic solitons for the elliptic-elliptic focussing Davey-Stewartson equations, *C. R. Math.* **354** (2016) 486. https://doi.org/10.1016/j.crma.2016.02.005.
5. R. F. Zinati and J. Manafian, Applications of He's semi-inverse method, ITEM and GGM to the Davey-Stewartson equation, *Eur. Phys. J. Plus.* **132** (2017) 155. https://doi.org/10.1140/epjp/i2017-11463-3.
6. G. Ebadi and A. Biswas, The G'/G method and 1-soliton solution of the Davey-Stewartson equation, *Math. Comput. Model.* **53** (2011) 694. https://doi.org/10.1016/j.mcm.2010.10.005.
7. Y. Gao, L. Mei, R. Li, Galerkin methods for the Davey-Stewartson equations, *Appl. Math. Comput.* **328** (2018) 144. https://doi.org/10.1016/j.amc.2018.01.044.
8. L. Chang, Y. Pan, X. Ma, New exact travelling wave solutions of Davey-Stewartson equation, *J. Comput. Inf. Syst.* **9** (2013) 1687.
9. S. Tuluce Demiray, H. Bulut, New soliton solutions of Davey-Stewartson equation with power-law nonlinearity, *Opt. Quant. Electron.* **49** (2017) 117. https://doi.org/10.1007/s11082-017-0950-6.
10. M. Song, A. Biswas, Topological defects and bifurcation analysis of the DS equation with power law nonlinearity, *Appl. Math. Inf. Sci.* **9** (2015) 1719.
11. H. Jafari, A. Sooraki, Y. Talebi, A. Biswas, The first integral method and traveling wave solutions to Davey-Stewartson equation, *Nonlinear Anal. Model. Control*, **17** (2012) 182. https://doi.org/10.15388/NA.17.2.14067.
12. O.H. El-Kalaawy, R.S. Ibrahim, Solitary wave solution of the two-dimensional regularized long-wave and Davey-Stewartson equations in fluids and plasmas, *Appl. Math.* **3** (2012) 833. https://doi.org/10.4236/am.2012.38124.
13. G. Ebadi, E.V. Krishnan, M. Labidi, E.Zerrad, A. Biswas-Analytical and numerical solutions to the Davey-Stewartson equation with power-law nonlinearity, *Wave Random Complex Media* **21** (2011) 559. https://doi.org/10.1080/17455030.2011.606853.
14. J. Shi, J. Li, S. Li, Analytical travelling wave solutions and parameter analysis for the $(2+1)$-dimensional Davey-Stewartson-type equations, *Pramana J. Phys.* **81** (2013) 747. https://doi.org/10.1007/s12043-013-0612-6.
15. V. A. Arkadiev, A. K. Pogrebkov, and M.C. Polivanov, Inverse scattering transform method and soliton solutions for the Davey-Stewartson II equation, *Phys. D* **36** (1989) 189. https://doi.org/10.1016/0167-2789(89)90258-3.
16. X. Zhao, Self-similar solutions to a generalized Davey-Stewartson system, *Math. Comput. Model.* **50** (2009) 1394, https://doi.org/10.1016/j.mcm.2009.04.023.
17. Y. Ohta and J. Yang, Dynamics of rogue waves in the Davey-Stewartson II equation, *J. Phys. A* **46** (2013) 105202. https://doi.org/10.1088/1751-8113/46/10/105202.
18. Y. Ohta and J. Yang, Rogue waves in the Davey-Stewartson I equation, *Phys. Rev. A Math. Theor.* **86** (2012) 036604. https://doi.org/10.1103/PhysRevE.86.036604.
19. D. Anker and N.C. Freeman, On the Solition Solutions of the Davey-Stewartson Equation for Long Waves, *Proc. R. Soc. Lond. A.* **360** (1978) 529. https://doi.org/10.1098/rspa.1978.0083.
20. B. Zhang, M.N. Xiong, L. Chen, Many New Exact Solutions for Generalized Davey-Stewartson Equation with Arbitrary Power Nonlinearities Using Novel (G'/G)-Expansion Method, *J. Adv. Appl. Math.* **4** (2019) 10. https://doi.org/10.22606/jaam.2019.41002.
21. M. Fazli Aghdaei, H. Adibi, New methods to solve the resonant nonlinear schrödinger equation with time-dependent coefficients, *Opt. Quantum Electronics* **49** (2017) 316. https://doi.org/10.1007/s11082-017-1152-y.
22. M. Song, Z.R. Liu, "Qualitative analysis and explicit traveling wave solutions for the Davey-Stewartson equation, *Math. Methods Appl. Sci.* **37** (2014) 393. https://doi.org/10.1002/mma.2798.
23. J. Cao, H.Y. Lu, Exact traveling wave solutions of the generalized Davey-Stewartson equation, *J. Shanghai Norm. Univ.* **44** (2015) 330.
24. K. Munusamy, C. Ravichandran, K. S. Nisar and B. Ghanbari, Existence of solutions for some functional integrodifferential equations with nonlocal conditions, *Math. Methods Appl. Sci.* **43** (2020) 10319. https://doi.org/10.1002/mma.6698.
25. G. Rahman, K. S. Nisar, B. Ghanbari, and T. Abdeljawad, On generalized fractional integral inequalities for the monotone weighted Chebyshev functionals, *Adv. Differ. Equ.* **2020** (2020) 368. https://doi.org/10.1186/s13662-020-02830-7.
26. M. Eslami and H. Rezazadeh, The first integral method for Wu-Zhang system with conformable time-fractional derivative, *Calcolo* **53** (2016) 475. https://doi.org/10.1007/s10092-015-0158-8.
27. R. M. Jena Rajarama, S. Chakraverty, H. Rezazadeh, and D. Domiri Ganji, On the solution of time-fractional dynamical model of Brusselator reaction-diffusion system arising in chemical reactions, *Mathematical Methods in the Applied Sciences* **43** (2020) 3903. https://doi.org/10.1002/mma.6141.

28. M. Inc *et al.*, New solitary wave solutions for the conformable Klein-Gordon equation with quantic nonlinearity, *AIMS Math.* **5** (2020) 6972. https://doi.org/10.3934/math.2020447.

29. M. Inc, M. Miah, A. Akher Chowdhury Shahadat, H. Rezazadeh, M.A. Akinlar, Y.M. Chu, New exact solutions for the Kaup-Kupershmidt equation, *AIMS Math.*, **5** (2020) 6726. https://doi.org/10.3934/math.2020432.

30. A. C. Cevikel, A. Bekir, S. San, M. B. Gucen, Construction of periodic and solitary wave solutions for the complex nonlinear evolution equations, *J. Franklin Inst.* **351** (2014) 694. https://doi.org/10.1016/j.jfranklin.2013.04.017.

31. A. C. Çevikel, A. Bekir, M. Akar, S. San, A procedure to construct exact solutions of nonlinear evolution equations, *Pramana* **79** (2012) 337. https://doi.org/10.1007/s12043-012-0326-1.

32. A. Bekir, A. C. Cevikel, New solitons and periodic solutions for nonlinear physical models in mathematical physics, *Nonlinear Anal. Real World Appl.* **11** (2010) 3275. https://doi.org/10.1016/j.nonrwa.2009.10.015.

33. A. Bekir and A. C. Cevikel, Solitary wave solutions of two nonlinear physical models by tanh–coth method, *Commun. Nonlinear Sci. Numer. Simul.* **14** (2009) 1804. https://doi.org/10.1016/j.cnsns.2008.07.004.

34. A. C. Çevikel and A. Bekir, New solitons and periodic solutions for (2+1)-dimensional Davey-Stewartson equations, *Chin. J. Phys.*, **51** (2013) 1. https://doi.org/10.6122/CJP.51.1.

35. A. Bekir and A. C. Çevikel, New exact travelling wave solutions of non-linear physical models, *Chaos Solitons Fractals* **41** (2009) 1733. https://doi.org/10.1016/j.chaos.2008.07.017.

36. A. Bekir, A. C. Çevikel, O. Guner, S. San, Bright and Dark Soliton Solutions of the (2+1)-Dimensional Evolution Equations, *Math. Model. Anal.* **19** (2014) 118. https://doi.org/10.3846/13926292.2014.893456.

37. Ö. Güner, A. Bekir, and A. C. Çevikel, Dark soliton and periodic wave solutions of nonlinear evolution equations, *Adv. Differ. Equ.* **2013** (2013) 68. https://doi.org/10.1186/1687-1847-2013-68.

38. A. C. Çevikel, New Exact Solutions of The Space-Time Fractional KdV-Burgers and Non-Linear Fractional Foam Drainage Equation, *Therm. Sci.* **22** (2018) 15. https://doi.org/10.2298/TSCI170615267C.

39. A. Bekir,O. Guner, A. C. Çevikel, The Exp-function Method for Some Time-fractional Differential Equations, *IEEE/CAA J. Autom. Sinica* **4** (2017) 315. https://doi.org/10.1109/JAS.2016.7510172.

40. A. Bekir, O. Guner, A. Burcu, A. C. Çevikel, Exact Solutions for Fractional Differential-Difference Equations by (G/G)-Expansion Method with Modified Riemann-Liouville Derivative, *Adv. Appl. Math. Mech.* **8** (2016) 293. https://doi.org/10.4208/aamm.2014.m798.

41. E. Aksoy, A. Bekir, A. C. Çevikel, Study on Fractional Differential Equations with Modified Riemann-Liouville Derivative via Kudryashov Method, *Int. J. Nonlinear Sci. Numer. Simul.* **20** (2019) 511. https://doi.org/10.1515/ijnsns-2015-0151.

42. H. Günerhan, Exact traveling wave solutions of the Gardner equation by the improved $\tan(\Theta(\vartheta))$-expansion method and the wave ansatz method, *Math. Probl. Eng.* **2020** (2020) 5926836, https://doi.org/10.1155/2020/5926836.

43. H. Dutta, H. Günerhan, K.K. Ali, R. Yilmazer, Exact Soliton Solutions to the Cubic-Quartic Non-linear SchrÃ¶dinger Equation With Conformable Derivative, *Front. Phys.*, **8** (2020) 62. https://doi.org/10.3389/fphy.2020.00062.

44. B. Ghanbari and M. Inc, A new generalized exponential rational function method to find exact special solutions for the resonance nonlinear Schrödinger equation, *Eur. Phys. J. Plus* **133** (2018) 142. https://doi.org/10.1140/epjp/i2018-11984-1.

45. B. Ghanbari and J.F. Gómez-Aguilar, The generalized exponential rational function method for Radhakrishnan-Kundu-Lakshmanan equation with β-conformable time derivative, *Rev. Mex. Fis.* **65** (2019) 503. https://doi.org/10.31349/RevMexFis.65.503.

46. H.M. Srivastava, H. Günerhan, B. Ghanbari, Exact traveling-wave solutions for resonance nonlinear Schrödinger equation with intermodal dispersions and the Kerr law nonlinearity, *Math Meth Appl Sci.*, **42** (2019) 7210. https://doi.org/10.1002/mma.5827.

47. B. Ghanbari, H. Günerhan, O.A. Íhan, and H.M. Baskonus, Some new families of exact solutions to a new extension of nonlinear Schrödinger equation, *Phys. Scr.* **95** (2020) 075208. https://doi.org/10.1088/1402-4896/ab8f42.

48. B. Ghanbari and C-Ku. Kuo, A variety of solitary wave solutions to the (2+1)-dimensional bidirectional SK and variable-coefficient SK equations, *Results in Physics* **18** (2020) 103266. https://doi.org/10.1016/j.rinp.2020.103266.

49. B. Ghanbari, M. Inc and L. Rada, Solitary wave solutions to the Tzitzeica type equations obtained by a new efficient approach, *J. Appl. Anal. Comput.* **9** (2018) 568. https://doi.org/10.11948/2156-907X.20180103.

50. B. Ghanbari, H. Günerhan, S. Momani, Exact optical solutions for the regularized long-wave Kadomtsev-Petviashvili equation, *Phys. Scr.* **95** (2020) 105208. https://doi.org/10.1088/1402-4896/abb5c8.

51. B. Ghanbari, K. S. Nisar, and M. Aldhaifallah, Abundant solitary wave solutions to an extended nonlinear Schrödinger's equation with conformable derivative using an efficient integration method, *Adv. Differ. Eq.* **2020** (2020) 328. https://doi.org/10.1186/s13662-020-02787-7.

52. M. S. Osman, B. Ghanbari, and J. A. T. Machado, New complex waves in nonlinear optics based on the complex Ginzburg-Landau equation with Kerr law nonlinearity, *Eur. Phys. J. Plus.* **134** (2019) 20. https://doi.org/10.1140/epjp/i2019-12442-4.

53. B. Ghanbari, A. Yusuf, and D. Baleanu, The new exact solitary wave solutions and stability analysis for the(2+1)-dimensional Zakharov-Kuznetsov equation, *Adv. Differ. Eq.*, **2019** (2019) 49. https://doi.org/10.1186/s13662-019-1964-0.

54. S. Kumar, A. Kumar, A-M Wazwaz, New exact solitary wave solutions of the strain wave equation in microstructured solids via the generalized exponential rational function method, *Eur. Phys. J. Plus* **135** (2020) 1.

55. B. Ghanbari, On novel non-differentiable exact solutions to local fractional Gardner's equation using an effective technique, *Math. Methods Appl. Sci.* (2020) 4673. `https://doi.org/10.1002/mma.7060`.

56. B. Ghanbari, On the non-differentiable exact solutions to Schamel's equation with local fractional derivative on Cantor sets, *Numer. Methods Partial Differ. Equ.* (to be published), `https://doi.org/10.1002/num.22740`.