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Application of the modified simple equation method for solving two nonlinear time-fractional long water wave equations

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Recently, nonlinear fractional partial differential equations have been used to model many phenomena in applied sciences and engineering. In this study, the modified simple equation scheme is implemented to obtain some new traveling wave solutions of the nonlinear conformable time-fractional approximate long water wave equation and the nonlinear conformable coupled time-fractional Boussinesq-Burger equation, which are used in the expression of shallow-water waves. The time-fractional derivatives are described in terms of conformable fractional derivative sense. Consequently, new exact traveling wave solutions of both equations are achieved.

Keywords: Fractional partial differential equations; modified simple equation method; conformable fractional derivative; approximate long water wave equation; coupled Boussinesq-Burger equation.

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1. Introduction

Nonlinear problems with fractional-order derivatives are used in several fields of engineering, and natural sciences such as mathematical physics, fluid mechanics, nonlinear optics, signal processing, plasma physics, mathematical biology, chemical physics [1–3]. In many research in the literature, mathematical models of real-world problems are expressed more realistically and effectively with fractional-order differential equations. For this reason, many systems and processes have intensified studies on creating more realistic models with fractional-order derivatives instead of integer-order derivatives. Obtaining the exact or approximate solutions of nonlinear physical problems involving fractional derivatives is not always possible by using the classical methods. Therefore, researchers have been doing great efforts to introduce new powerful and effective methods to solve these equations. Among which the modified simple equation (MSE) method [4], the generalized Kudryashov method [5], the (G'/G)-expansion method [6], the improved F-expansion method [7], the generalized exponential rational function method [8], the first integral method [9], the generalized bifurcation method [10], the modified trial equation method [11], the extended auxiliary equation method [12], the conformable double Sumudu transform [13], the conformable sub-equation method [14], the new extended direct algebraic method [15], the generalized (G'/G)-expansion method [16], the homotopy perturbation method [17] and collocation methods [18, 19] are just a few to name. In the literature, there are several types of fractional derivatives. The most widely used are Riemann-Liouville, Caputo, Grünwald-Letnikov, and Jumarie's modified Riemann-Liouville derivatives [2, 20–24]. Unfortunately, all these fractional derivatives do not have some classical properties, like the chain, product, and quotient rules. Khalil *et al.* [25] proposed a new interesting type of fractional derivative called conformable fractional derivative to overcome these difficulties. Furthermore, Abdeljawad [26] has contributed to the conformable fractional derivative definition, such as he proposed the definitions of right and left conformable fractional derivatives and chain rule. In this work, we propose the MSE method, which is an analytical method that has gained notable popularity in recent times. This method's robustness arises from the general solution form that is defined by the sum of finite series and includes the unknown function. In addition, thanks to this characteristic of the technique, the new and more general solitary wave solutions are derived by selecting special values of arbitrary coefficients in the exact solutions. On the other hand, when comparing the MSE method with the existing methods in the literature, such as the (G'/G)-expansion method [6], the modified extended tanh function method [27], the sine-cosine method [28], the generalized Kudryashov method and the improved F-expansion method [7] etc., these methods have more complex solution steps. The basic idea behind these methods has to do with some special pre-defined functions or a solution of the auxiliary equation. These techniques require symbolic computational software programs to solve the system of algebraic equations. We observe that the proposed method is highly practical, systematic, potential, and straightforward. The MSE method has been used to get exact solutions of various fractional partial differential equations such as the space-time fractional modified regularized long-wave equation, the space-time fractional modified Korteweg-de Vries equation, the space-time fractional coupled Burgers' equations [29], the nonlinear time-fractional Sharma-Tasso-Oliver equation [30], the fractional generalized reaction Duffing equation is and the fractional nonlinear Cahn-Allen equation [31]. We apply this efficient method to the nonlinear conformable time-fractional approximate long water wave equation (ALW) [32] and the nonlinear coupled conformable time-fractional Boussinesq-Burger equation [33] to obtain analytical solutions of physical shallow water equations. There are many effective studies investigating the exact solutions of these equations using various methods. The generalized $\exp\left(-\varphi\left(\xi\right)\right)$-expansion method [34] and the improved Bernoulli sub-equation function method [35] have been applied to obtain wave solutions of the the nonlinear time-fractional ALW equation. Also, the generalized Kudryashov method [32] has been implemented to the nonlinear time-fractional ALW and the nonlinear coupled time-fractional Boussinesq-Burger equations to attain different solutions schemes. Later, the first integral method [36], the two-variable $(G'/G, 1/G)$-expansion method [37] and the (G'/G)-expansion method [38] have been performed to find traveling wave solutions of the nonlinear coupled time-fractional Boussinesq-Burger equation. To the best of our knowledge, the exact solutions we achieved are new and more general than the previous solutions of these equations. We obtain eight distinct traveling wave solutions for each two equations. According to the values of wave speed and order of fractional derivative, these solutions are convertible to different wave shapes. Thus, we demonstrate 3-dimensional, 2-dimensional, contour, and density graphs of wave solutions with proper parameters to explain the complex nonlinear phenomena. This study is constructed as follows: In Sec. 2, we present the definition of the conformable fractional derivative and some basic properties. In Sec. 3, we summarize the steps of the MSE method. In Sec. 4, the proposed method is implemented to the physical equations. Section 5 presents the graphs of solutions and physical explanations. Conclusions are outlined in Sec. 6.

2. Preliminaries

In this segment we present the definition and several properties of the conformable fractional derivative [25, 26].

Definition 1. *The conformable fractional derivative of a function $f = f\left(t\right)$ of order $\alpha \in (0, 1]$ is given as*

$$D_t^{\alpha} f = \lim_{\tau \to 0} \frac{f\left(t + \tau t^{1-\alpha}\right) - f\left(t\right)}{\tau} \quad \textit{for each} \ \ t > 0,$$

where f is a real-valued function defined on $(0, \infty)$.

Theorem 1. *Assume $\alpha \in (0, 1]$ and $f = f\left(t\right),\ h = h\left(t\right)$ are α-conformable differentiable functions at $t > 0$, then*

(i) $D_t^{\alpha}\left(kf + mh\right) = kD_t^{\alpha} f + mD_t^{\alpha} h$ *for each* $k, m \in \mathbb{R}$,

(ii) $D_t^{\alpha}\left(t^r\right) = rt^{r-\alpha}$ *for each* $r \in \mathbb{R}$,

(iii) $D_t^{\alpha}(fh) = hD_t^{\alpha} f + fD_t^{\alpha} h$,

(iv) $D_t^{\alpha}\left(\dfrac{f}{h}\right) = \dfrac{hD_t^{\alpha} f - fD_t^{\alpha} h}{h^2}$.

Moreover, if f is a differentiable function, then $D_t^{\alpha} f = t^{1-\alpha} f'$.

Theorem 2. *Suppose $f = f\left(t\right)$ is a real-valued function defined on $(0, \infty)$ such that f is differentiable and α-conformable differentiable. Also, suppose $h = h\left(t\right)$ is a differentiable function defined in the range of f. Then, the chain rule is obtained as*

$$D_t^{\alpha}\left(f \circ h\right)\left(t\right) = t^{1-\alpha} h\left(t\right)^{\alpha-1} h'\left(t\right) D_t^{\alpha}\left(f\left(t\right)\right)_{t=h(t)}.$$

3. The modified simple equation method

This section introduces the fundamental steps of the MSE method [30]:

Assume the nonlinear time-fractional evolution equation as follows:

$$F\left(u, D_t^{\alpha} u, u_x, u_{xx}, \ldots\right) = 0, \tag{1}$$

where $\alpha \in (0, 1]$ and F is a polynomial of $u\left(x, t\right)$.

Step 1. We perform the wave transformation

$$u\left(x, t\right) = u\left(\varepsilon\right),\ \varepsilon = kx \pm c\frac{t^{\alpha}}{\alpha}, \tag{2}$$

to reduce Eq. (1) into the nonlinear ordinary differential equation (ODE):

$$Q\left(u, u', u'', u''', \ldots\right) = 0, \tag{3}$$

where Q is a polynomial in $u\left(\varepsilon\right)$ and its all derivatives, wherein $u'\left(\varepsilon\right) = du/d\varepsilon$ and so on.

Step 2. We suppose the formal solution of Eq. (3) in the form

$$u\left(\varepsilon\right) = \sum_{k=0}^{N} A_k \left[\frac{\varphi'\left(\varepsilon\right)}{\varphi\left(\varepsilon\right)}\right]^k, \tag{4}$$

where A_k are real constants such as $A_N \neq 0$ and $\varphi\left(\varepsilon\right)$ is an indetermined function to be calculated.

Step 3. We compute the integer $N > 0$ in Eq. (4) by consulting the homogeneous balance between the highest order derivatives and the highest order nonlinear terms consisted in Eq. (3).

Step 4. We replace Eq. (4) and all its essential derivatives into Eq. (3). As a conclusion of this replacement, we have a polynomial of $\varphi^{-i}\left(\varepsilon\right)$ $(i = 0, 1, 2, \ldots)$. Furthermore, we arrange all terms of the same power of $\varphi^{-i}\left(\varepsilon\right)$ and we equate all the coefficients to zero. This process ensures a system of equations that can be solved to calculate both A_k and $\varphi\left(\varepsilon\right)$.

4. Applications

In this part, the MSE method is applied to nonlinear time fractional systems of physical equations.

4.1. The nonlinear conformable time-fractional approximate long water wave equation

The ALW equation is a special form of the Whitham-Broer-Kaup (WBK) equation, so we first take the WBK equation into account. The equation refers to the dispersion of shallow-water waves with various distributions. The nonlinear time-fractional WBK equation, which is frequently used in fluid mechanics, is defined as follows:

$$\begin{aligned} D_t^\alpha u + uu_x + v_x + bu_{xx} = 0,\\ D_t^\alpha v + (uv)_x + au_{xxx} - bv_{xx} = 0, \end{aligned} \tag{5}$$

wherein $t > 0$, $\alpha \in (0, 1]$, $u(x, t)$ is the velocity at the horizontal, $v(x, t)$ is the height at which the fluid deviates from the equilibrium, a and b are constants at different diffusions. In addition, if $\alpha = 1$ in Eq. (5), the system becomes the original integer-order WBK equation. Also, when it takes the values $a = 1$ and $b = 0$, it converts to the fractional modified Boussinesq equation [39–41]. In conclusion, when we give the values $a = 0$ and $b = 1/2$ at Eq. (5), the equation turns into the nonlinear time-fractional ALW equation as follows [32]:

$$\begin{aligned} D_t^\alpha u + uu_x + v_x + \frac{1}{2}u_{xx} = 0,\\ D_t^\alpha v + (uv)_x - \frac{1}{2}v_{xx} = 0, \end{aligned} \tag{6}$$

where $\alpha \in (0, 1]$, x represents the position of the wave and t represents the time $t > 0$. Using the wave transformation $\varepsilon = x - c(t^\alpha/\alpha)$ for $k = 1$, equality $v = cu - (1/2)u^2 - (1/2)u'$ and integrating with respect to ε, Eq. (6) may be reduced to integer order nonlinear ODE:

$$-\frac{1}{4}u'' + \frac{1}{2}u^3 + c^2u - \frac{3}{2}cu^2 = 0. \tag{7}$$

Additionally, we obtain $N = 1$ from balancing principle in Eq. (7). Therefore, Eq. (4) turns into the following form:

$$u(\varepsilon) = A_0 + A_1\frac{\varphi'(\varepsilon)}{\varphi(\varepsilon)}. \tag{8}$$

Substituting Eq. (8) and its derivatives into Eq. (7) then editing each terms with the same power of $\varphi^{-i}(\varepsilon)$, we get a system as follows:

$$(\varphi)^0 : \frac{1}{2}A_0^3 + c^2A_0 - \frac{3}{2}cA_0^2 = 0, \tag{9}$$

$$\begin{aligned} (\varphi)^{-1} : &-\frac{1}{4}A_1\varphi'''(\varepsilon) + \frac{3}{2}A_0^2A_1\varphi'(\varepsilon)\\ &+ c^2A_1\varphi'(\varepsilon) - 3cA_0A_1\varphi'(\varepsilon) = 0, \end{aligned} \tag{10}$$

$$\begin{aligned} (\varphi)^{-2} : &\frac{3}{4}A_1\varphi'(\varepsilon)\varphi''(\varepsilon)\\ &+ \frac{3}{2}A_0A_1^2(\varphi'(\varepsilon))^2 - \frac{3}{2}cA_1^2(\varphi'(\varepsilon))^2 = 0, \end{aligned} \tag{11}$$

$$(\varphi)^{-3} : -\frac{1}{2}A_1(\varphi'(\varepsilon))^3 + \frac{1}{2}A_1^3(\varphi'(\varepsilon))^3 = 0. \tag{12}$$

We deduce that $A_0 = 0$, $A_0 = c$, $A_0 = 2c$ and $A_1 = \pm 1$ from Eq. (9) and Eq. (12).

Case 1: If $A_0 = c$, there is a trivial solution. So, this case is discarded.

Case 2: If $A_0 = 0$ and $A_1 = \pm 1$. Solving Eq. (10) and Eq. (11), we obtain $\varphi'(\varepsilon) = \pm(e^{\pm 2c\varepsilon}c_1/2c)$ and $\varphi(\varepsilon) = (c_1/4c^2)e^{\pm 2c\varepsilon} + c_2$. Here and in the sequel, c_1 and c_2 are arbitrary constants of integration. Now, substituting $\varphi(\varepsilon)$, $\varphi'(\varepsilon)$ into Eq. (8), we get the exact wave solutions:

$$u(\varepsilon) = \frac{2ce^{\pm 2c\varepsilon}c_1}{c_1e^{\pm 2c\varepsilon} + 4c^2c_2}. \tag{13}$$

So, we use hyperbolic function properties, and we attain wave solutions when $c_1 = 1$, $c_2 = 1/4c^2$;

$$u_{1,2}(x, t) = c\left(1 \pm \tanh\left[c\left\{x - c\frac{t^\alpha}{\alpha}\right\}\right]\right), \tag{14}$$

and when $c_1 = 1$, $c_2 = -1/4c^2$;

$$u_{3,4}(x, t) = c\left(1 \pm \coth\left[c\left\{x - c\frac{t^\alpha}{\alpha}\right\}\right]\right). \tag{15}$$

Case 3: If $A_0 = 2c$ and $A_1 = \pm 1$. Using Eq. (10) and Eq. (11), we obtain $\varphi'(\varepsilon) = \pm(e^{\pm 2c\varepsilon}c_1/2c)$ and $\varphi(\varepsilon) = (c_1/4c^2)e^{\pm 2c\varepsilon} + c_2$. We substitute, $\varphi(\varepsilon)$, $\varphi'(\varepsilon)$ in Eq. (8) and we attain the exact wave solutions:

$$u(\varepsilon) = 2c - c\frac{2ce^{\pm 2c\varepsilon}c_1}{c_1e^{\pm 2c\varepsilon} + 4c^2c_2}. \tag{16}$$

So, we use hyperbolic function properties and we obtain the solitary wave solutions when $c_1 = c$, $c_2 = 1/4c^2$;

$$u_{5,6}(x, t) = 2c - c\left(1 \mp \tanh\left[c\left\{x - c\frac{t^\alpha}{\alpha}\right\}\right]\right), \tag{17}$$

and when $c_1 = c$, $c_2 = -(1/4c^2)$;

$$u_{7,8}(x, t) = 2c - c\left(1 \mp \coth\left[c\left\{x - c\frac{t^\alpha}{\alpha}\right\}\right]\right). \tag{18}$$

4.2. The nonlinear conformable coupled time-fractional Boussinesq-Burger equation

The nonlinear coupled time-fractional Boussinesq-Burger equation is used in several fields of science such as mathematical physics and fluid mechanics in the investigation of fluid flow in physical systems. This system refers to the expansion of shallow-water waves. Furthermore, this equation regulates partial differential equations to identify the flow under a compression surface in a fluid, the movement of water bodies, and the motion is vertically well-mixed water bodies. Investigating the solutions of the equation is very significant for civil and coastal engineers to enforce the nonlinear water wave equations to side designs and port construction [42, 43]. The nonlinear coupled time-fractional Boussinesq-Burger equation is defined as in the following [33]:

$$D_t^\alpha u - \frac{1}{2}v_x + 2uu_x = 0,$$
$$D_t^\alpha v - \frac{1}{2}u_{xxx} + 2\left(uv\right)_x = 0, \tag{19}$$

where $t > 0$, $\alpha \in (0,1]$, $v\left(x,t\right)$ is the height of the water surface above a horizontal level at the bottom and $u\left(x,t\right)$ is the horizontal velocity field. Then, using the wave transformation $\varepsilon = x - c(t^\alpha/\alpha)$ for $k=1$, the relation $v = 2\left(u^2 - cu\right)$ and integrating for ε, Eq. (19) may be reduced to integer order nonlinear ODE:

$$-\frac{1}{2}u'' + 4u^3 + 2c^2u - 6cu^2 = 0. \tag{20}$$

Moreover, we acquire $N=1$ from balancing principle in Eq. (20). For this reason, Eq. (4) turns into the following form:

$$u(\varepsilon) = A_0 + A_1\frac{\varphi'\left(\varepsilon\right)}{\varphi\left(\varepsilon\right)}. \tag{21}$$

Inserting Eq. (21) and its derivatives into Eq. (20) and compiling each terms with the same power of $\varphi^{-i}\left(\varepsilon\right)$, we have a system as follows:

$$\left(\varphi\right)^0 : 4A_0^3 + 2c^2A_0 - 6cA_0^2 = 0, \tag{22}$$

$$\left(\varphi\right)^{-1} : -\frac{1}{2}A_1\varphi'''(\varepsilon) + 12A_0^2A_1\varphi'(\varepsilon) + 2c^2A_1\varphi'(\varepsilon) - 12cA_0A_1\varphi'(\varepsilon) = 0, \tag{23}$$

$$\left(\varphi\right)^{-2} : \frac{3}{2}A_1\varphi'(\varepsilon)\varphi''(\varepsilon) + 12A_0A_1^2(\varphi'(\varepsilon))^2 - 6cA_1^2(\varphi'(\varepsilon))^2 = 0, \tag{24}$$

$$\left(\varphi\right)^{-3} : -A_1(\varphi'(\varepsilon))^3 + 4A_1^3(\varphi'(\varepsilon))^3 = 0. \tag{25}$$

We conclude that $A_0 = 0$, $A_0 = c$, $A_0 = c/2$ and $A_1 = \pm 1/2$ from Eq. (22) and Eq. (25).

Case 1: If $A_0 = c/2$, there is a trivial solution. Thus, this case is rejected.

Case 2: If $A_0 = 0$ and $A_1 = \pm 1/2$. Using Eq. (23) and Eq. (24), we achieve $\varphi'\left(\varepsilon\right) = \pm(e^{\pm 2c\varepsilon}c_1/2c)$ and $\varphi\left(\varepsilon\right) = (c_1e^{\pm 2c\varepsilon}/4c^2) + c_2$. Substituting $\varphi(\varepsilon)$, $\varphi'(\varepsilon)$ into Eq. (21), we have the exact wave solutions:

$$u\left(\varepsilon\right) = \frac{c_1ce^{\pm 2c\varepsilon}}{c_1e^{\pm 2c\varepsilon} + 4c^2c_2}. \tag{26}$$

Now, using hyperbolic function properties, we get wave solutions when $c_1 = 1$, $c_2 = 1/4c^2$;

$$u_{1,2}\left(x,t\right) = \frac{c}{2}\left(1 \pm \tanh\left[c\left\{x - c\frac{t^\alpha}{\alpha}\right\}\right]\right), \tag{27}$$

and when $c_1 = 1$, $c_2 = -1/4c^2$;

$$u_{3,4}\left(x,t\right) = \frac{c}{2}\left(1 \pm \coth\left[c\left\{x - c\frac{t^\alpha}{\alpha}\right\}\right]\right). \tag{28}$$

Case 3: If $A_0 = c$ and $A_1 = \pm 1/2$. Solving Eq. (23) and Eq. (24), we take $\varphi'\left(\varepsilon\right) = \pm(e^{\pm 2c\varepsilon}c_1/2c)$ and $\varphi(\varepsilon) = (c_1e^{\pm 2c\varepsilon}/4c^2) + c_2$. Substituting, $\varphi\left(\varepsilon\right)$, $\varphi'\left(\varepsilon\right)$ into Eq. (21), we get the exact wave solutions:

$$u\left(\varepsilon\right) = c - \frac{cc_1e^{\pm 2c\varepsilon}}{c_1e^{\pm 2c\varepsilon} + 4c^2c_2}. \tag{29}$$

Hence, we use hyperbolic function properties and we have wave solutions when $c_1 = 1$, $c_2 = (1/4c^2)$;

$$u_{5,6}\left(x,t\right) = c - \frac{c}{2}\left(1 \mp \tanh\left[c\left\{x - c\frac{t^\alpha}{\alpha}\right\}\right]\right), \tag{30}$$

and when $c_1 = 1$, $c_2 = -(1/4c^2)$;

$$u_{7,8}\left(x,t\right) = c - \frac{c}{2}\left(1 \mp \coth\left[c\left\{x - c\frac{t^\alpha}{\alpha}\right\}\right]\right). \tag{31}$$

Consequently, the analytical solutions for the nonlinear time-fractional ALW equation and the nonlinear coupled time-fractional Boussinesq-Burger equation can be expanded by selecting various arbitrary constants c_1 and c_2. Moreover, $v\left(x,t\right)$ values can be calculated according to equalities in equations.

5. Physical interpretation and graphs

This section presents the attained exact wave solutions of both the nonlinear conformable time-fractional ALW equation and the nonlinear coupled conformable time-fractional Boussinesq-Burger equation. We have obtained the traveling wave solutions of long water wave equations. These solutions are plotted with convenient values in several types like 3D, 2D, contour, and density graphs. The graphics have particular intervals such as 3D graphs on $-8 \leq x, t \leq 8$, 2D graphs on $-8 \leq x \leq 8$, contour and density graphs on $0 \leq x, t \leq 10$.

Figure 1a) show Eq. (14) $u_1\left(x,t\right)$ for $\alpha = 0.56$, $k=1$ and $c = 0.45$. Figure 1b) represents $u_1(x,t)$ fixed at point $t=1$ with the same values in Fig. 1a) for $\alpha = 0.56$ and $\alpha = 1$. Figure 1c) denotes contour and density graphs of $u_1(x,t)$ with the same cases in 3D-graph. Figure 2a) gives Eq. (15) $u_3\left(x,t\right)$ for $\alpha = 0.70$, $k=1$ and $c = 0.61$. Figure 2b) demonstrates $u_3(x,t)$ fixed at point $t=1$ with the same cases in Fig. 2a) for $\alpha = 0.70$ and $\alpha = 1$. Figure 2c) shows contour and density graphs of $u_3(x,t)$ with the same values in 3D-graph. Figure 3a) expresses Eq. (27) $u_1\left(x,t\right)$ for $\alpha = 0.74$, $k=1$ and $c = 0.49$. Figure 3b) represents $u_1(x,t)$ fixed at point $t=1$ with the same values in Fig. 3a) for $\alpha = 0.74$ and $\alpha = 1$. Figure 3c) indicates contour and density graphs of $u_1(x,t)$ with the same cases in 3D-graph. Figure 4a) demonstrates Eq. (28) $u_3\left(x,t\right)$ for $\alpha = 0.82$, $k=1$ and $c = 0.18$. Figure 4b) expresses $u_3(x,t)$ fixed at point $t=1$ with the same cases in Fig. 4a) for $\alpha = 0.82$ and $\alpha = 1$. Figure 4c) indicates contour and density graphs of $u_3(x,t)$ with the same values in 3D-graph. As a result, we have obtained solitary wave solutions which have various shapes according to the values of wave speed c and fractional order α. Equations (14), (17), (27) and (30) represent kink shape soliton solutions when parameter $c > 0$ and $c < 0$. In addition, Eqs. (15), (18), (28) and (31) denote singular soliton solutions when $c > 0$ also when the parameter $c < 0$ these equations express dark singular shape soliton solutions. The rest of the solutions have similar graphs like as in drawn results.

5.1. Graphs of solutions for the ALW equation

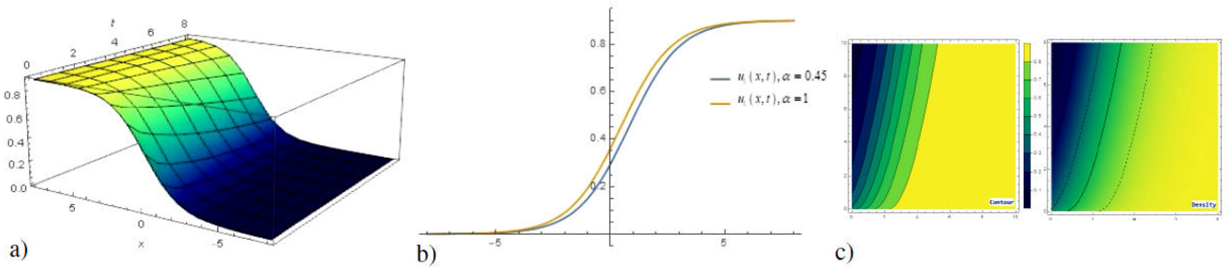

FIGURE 1. a) 3D-graph. b) 2D-graph. c) Contour and Density graphs.

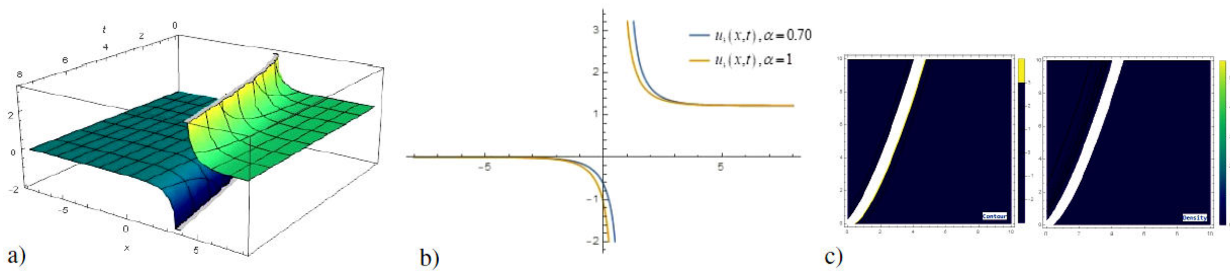

FIGURE 2. a) 3D-graph. b) 2D-graph. c) Contour and Density graphs.

5.2. Graphs of solutions for the Boussinesq-Burger equation

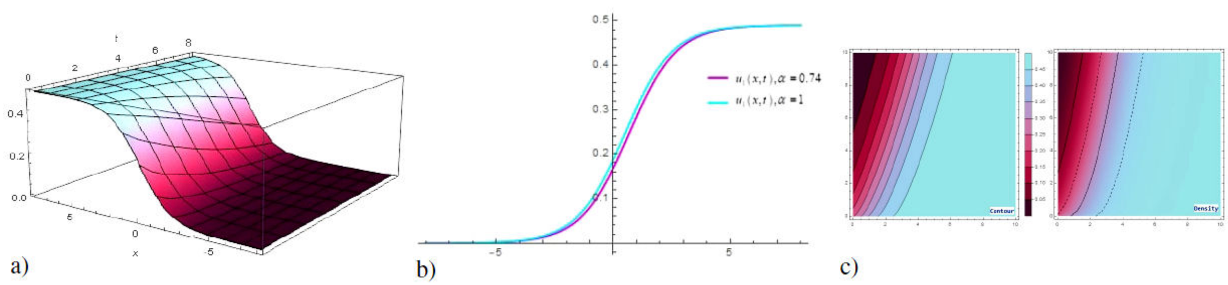

FIGURE 3. a) 3D-graph. b) 2D-graph. c) Contour and Density graphs.

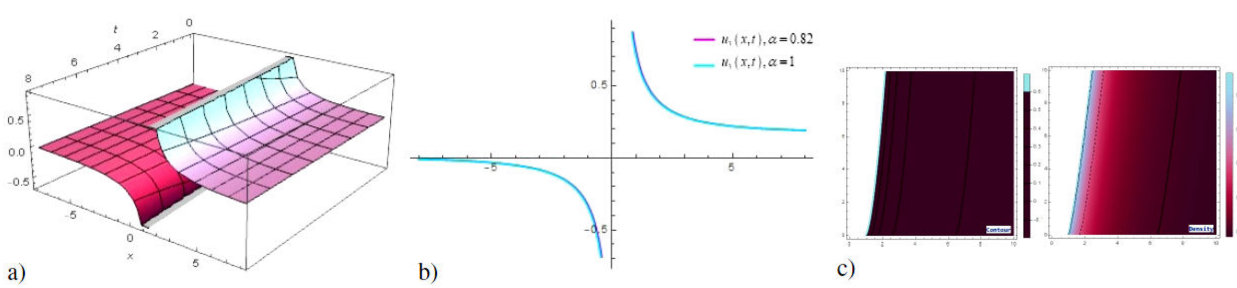

FIGURE 4. a) 3D-graph. b) 2D-graph. c) Contour and Density graphs.

6. Conclusion

We have applied the MSE method to acquire some traveling wave solutions to the nonlinear conformable time-fractional ALW equation and the nonlinear conformable coupled time-fractional Boussinesq -Burger equation. We have checked the correctness of the solutions by using the `Mathematica` program. The graphics of the solution functions have been constructed concerning the convenient values. The MSE method successfully attains analytical traveling wave solutions of some nonlinear partial differential equations with fractional order derivatives. This method also explains new physical solutions in complex structures by obtaining new types of solutions thanks to its general solution form, unknown function, and independent parameters. The results show that the MSE method is useful, effective, and innovative to solve such equations in physics, applied mathematics, and engineering.

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