A theoretical study of the modified equal scalar and vector Manning-Rosen potential within the deformed Klein-Gordon and Schrödinger in relativistic noncommutative quantum mechanics and nonrelativistic noncommutative quantum mechanics symmetries

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In this research work, within the framework of relativistic and nonrelativistic noncommutative quantum mechanics, the deformed Klein-Gordon and Schrödinger equations were solved with the modified equal vector scalar Manning-Rosen potential that has been of significance interest in recent years using Bopp's shift method and standard perturbation theory in the first-order in the noncommutativity parameters (Θ, σ, χ) in 3-dimensions noncommutative quantum mechanics. By employing the improved approximation of the centrifugal term, the relativistic and nonrelativistic bound state energies were obtained for some diatomic molecules such as (HCl, CH, LiH, CO, NO, O_2, I_2, N_2, H_2, and Ar_2). The obtained energy eigenvalues appear as a function of the generalized Gamma function, the parameters of noncommutativity, and the parameters (β, A, α) of studied potential, in addition to the atomic quantum numbers (n, j, l, s, m). In both relativistic and nonrelativistic problems, we show that the corrections on the spectrum energy are smaller than the main energy in the ordinary cases of RQM and NRQM. A straightforward limit of our results to ordinary quantum mechanics shows that the present result is consistent with what is obtained in the literature. We have seen that the improved approximation of the centrifugal term is better than the other approximations in finding the approximate analytical solutions of the Klein-Gordon and Schrödinger equations for the modified Manning-Rosen potential in relativistic noncommutative quantum mechanics and nonrelativistic noncommutative quantum mechanics.

Keywords: Klein-Gordon equation; Schrödinger equation; Manning-Rosen potential; the diatomic molecules; noncommutative geometry; Bopp's shift method and star products.

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1. Introduction

During the past two decades, considerable efforts have been made by many researchers in various fields of physics and chemistry to reach relativistic and nonrelativistic solutions using many potentials by adopting different methods such as the Nikiforov-Uvarov method [1], the Wentzel-Kramers-Brillouin method [2], the proper quantization rule [3], and the exact quantization rule [4], in addition to many other methods. The exact solutions of the fundamental equations are only possible in some exceptional cases like the harmonic oscillator and the Hydrogen atom as two typical models. As for most of the cases treated by researchers, it is done by using approximations and numerical methods such as the Pekeris approximation [5], the Greene and Aldrich approximation [6], the good approximation proposed by B.H. Yazarloo *et al*. in the study of the oscillator strengths based on the Möbius square potential under Schrödinger equation [7] and the new approximation of the centrifugal term proposed C. S. Jia *et al*. [8]. The Manning-Rosen potential has been of relevant interest in recent years as it can be applied to various fields such as atomic, condensed matter, particle, and nuclear physics in both relativistic and non-relativistic regimes [9-12]. Furthermore, it is used to describe the vibrations of diatomic molecules such as HCl, CH, LiH, CO, NO, O_2, I_2, N_2, H_2, and Ar_2 [13,14]. Many authors have studied Manning-Rosen potential in the nonrelativistic case, in both the s-wave and l-wave cases (see for example [15-18]). Furthermore, this potential was also studied in the relativistic Klein-Gordon, and Dirac equations [19-23].

As a result of several considerations and many physical problems apparat at the level of the non-renormalizable of the electroweak interaction, quantum gravity, string theory, where the idea of non-commutativity resulting from properties of deformation of space-space (W. Heisenberg in 1930 is the first to suggest the idea and then it was developed by H. Snyder in 1947) was one of the major solutions to these problems. In the past two decades, in particular, it has attracted a great attention [24-35].

The main objective of this work is to develop the study of B.J. Falaye *et al*., A.I. Ahmadov *et al*., and Z. H. Chen *et al.* [13,18,19] within the framework of the Klein Gordon and Schrödinger equations. But in the context of symmetries of noncommutative quantum mechanics for the purpose to get more investigation in the microscopic scales and from achieving more scientific knowledge of elementary particles in the field of nano-scales. The relativisti energy levels under the modified Manning-Rosen potential have not been obtained yet in the context of the RNCQM and NRNCQM. Furthermore, we hope to find new applications and profound physical interpretations using a new, updated model of the modified Manning-Rosen potential, which takes the form:

$$V_{mp}(r) = \frac{1}{2M\beta^2}\left(\frac{\alpha\left(\alpha - 1e^{-2r/\beta}\right)}{\left(1-e^{-r/\beta}\right)^2} - \frac{Ae^{-r/\beta}}{1-e^{-r/\beta}}\right) \rightarrow$$
$$V_{mp}(\hat{r}) \equiv V_{mp}(r) - \frac{\partial V_{mp}(r)}{\partial r}\frac{\vec{\mathbf{L}}\vec{\Theta}}{2r} + O(\Theta^2), \tag{1}$$

and

$$S_{mp}(r) = \frac{1}{2M\beta^2}\left(\frac{\lambda\left(\lambda - 1\right)e^{-2r/\beta}}{\left(1-e^{-r/\beta}\right)^2} - \frac{Be^{-r/\beta}}{1-e^{-r/\beta}}\right) \rightarrow$$
$$S_{mp}(\hat{r}) \equiv S_{mp}(r) - \frac{\partial S_{mp}(r)}{\partial r}\frac{\vec{\mathbf{L}}\vec{\Theta}}{2r} + O(\Theta^2), \tag{2}$$

where the parameter β relates to the potential range while A and α are two dimensionless parameters, and $1/\alpha$ is its range and r is the distance between the two particles. The coupling $\vec{\mathbf{L}}\vec{\Theta}$ equals $L_x\Theta_{12} + L_y\Theta_{23} + L_z\Theta_{13}$, where L_x, L_y, and L_z are the usual components of angular momentum operator $\vec{\mathbf{L}}$ in RQM while the new noncommutativity parameter $\Theta_{\mu\nu}$ equals $\theta_{\mu\nu}/2$. The new structure of (RNCQM) based on new covariant noncommutative canonical commutations relations (CNCCRs) in Schrödinger, Heisenberg and interactions pictures (SP, HP, and IP), respectively, as follows [36-46]:

$$\left[\hat{x}_\mu \overset{*}{,} \hat{P}_\nu\right] = \left[\hat{x}_\mu(t) \overset{*}{,} \hat{P}_\nu(t)\right] = \left[\hat{x}^I_\mu(t) \overset{*}{,} \hat{P}^I_\nu(t)\right] = i\hbar_{\text{eff}}\delta_{\mu\nu}, \tag{3.1}$$
$$\left[\hat{x}_\mu \overset{*}{,} \hat{x}_\nu\right] = \left[\hat{x}_\mu(t) \overset{*}{,} \hat{x}_\nu(t)\right] = \left[\hat{x}^I_\mu(t) \overset{*}{,} \hat{x}^I_\nu(t)\right] = i\theta_{\mu\nu}. \tag{3.2}$$

We generalize the CNCCRs to include HP and IP. It should be noted that, in our calculation, we have used the natural units $c = \hbar = 1$. Here $\hbar_{\text{eff}} = \hbar(1 + Tr[\theta\bar{\theta}/4]) \approx \hbar$ is the effective Planck constant and here $\bar{\theta}$ observed in the NC commutator

$$\left[\hat{x}_\mu \overset{*}{,} \hat{P}_\nu\right] = \left[\hat{x}_\mu(t) \overset{*}{,} \hat{P}_\nu(t)\right] = \left[\hat{x}^I_\mu(t) \overset{*}{,} \hat{P}^I_\nu(t)\right] = i\hbar\bar{\theta}_{\mu\nu},$$

and $\theta^{\mu\nu} = \varepsilon^{\mu\nu}\theta$ (θ is the non-commutative parameter), which is an infinitesimals parameter if compared to the energy values and elements of antisymmetric 3×3 real matrix and $\delta_{\mu\nu}$ is the identity matrix. The symbol $(*)$ denotes the Weyl Moyal star product, which is generalized between two ordinary functions $f(x)g(x)$ to the new modified form $\hat{f}(\hat{x})\hat{g}(\hat{x}) \equiv f(x) * g(x)$ in the symmetries of RNCQM as follows [47-55]:

$$f(x)g(x) \rightarrow (f*g)(x) = \exp(i\theta\varepsilon^{\mu\nu}\partial_{x_\mu}\partial_{x_\nu})f(x_\mu)g(x_\nu) \cong fg(x) - \frac{i\varepsilon^{\mu\nu}}{2}\theta\partial^x_\mu f\partial^x_\nu g\Big|_{x_\mu = x_\nu} + O(\theta^2). \tag{4}$$

The indices $\mu,\nu \equiv \overline{1,3}$ and $O(\theta^2)$ stand for the second and higher-order terms of the NC parameter. Physically, the second term in Eq. (4) presents the effects of space-space noncommutativity properties. Furthermore, the new unified two operators $\hat{\xi}^H_\mu(t) = (\hat{x}_\mu \text{ or } \hat{p}_\mu)(t)$ and $\hat{\xi}^I_\mu(t) = (\hat{x}^I_\mu \text{ or } \hat{p}^I_\mu)(t)$ in HP and IP are depending on the corresponding new operators $\hat{\xi}^H_\mu(t) \equiv \hat{x}_\mu$ or $\hat{p}_\mu$ in SP from the following projection relations, respectively:

$$\xi^H_\mu(t) = \exp(i\hat{H}^{mp}_r T)\xi^S_\mu\exp(-i\hat{H}^{mp}_r T) \Rightarrow \hat{\xi}^H_\mu(t) = \exp(i\hat{H}^{mp}_{nc-r}T) * \hat{\xi}^S_\mu * \exp(-i\hat{H}^{mp}_{nc-r}T), \tag{5.1}$$

and

$$\xi^H_\mu(t) = \exp(i\hat{H}^{mp}_{or}T)\xi^S_\mu\exp(-i\hat{H}^{mp}_{or}T) \Rightarrow \hat{\xi}^H_\mu(t) = \exp(i\hat{H}^{mp}_{nc-or}T) * \hat{\xi}^S_\mu * \exp(-i\hat{H}^{mp}_{nc-or}T), \tag{5.2}$$

where $T = t - t_0$ and the three unified coordinates $\xi^S_\mu \equiv (x_\mu \text{ or } p_\mu)$, $\xi^H_\mu(t) \equiv (x_\mu \text{ or } p_\mu)(t)$ and $\xi^I_\mu(t) \equiv (x^I_\mu \text{ or } p^I_\mu)(t)$ are represented in three relativistic quantum mechanics pictures, while the dynamics of new systems $d\hat{\xi}_H(t)/dt$ are described from the following motion equations in the modified Heisenberg picture as follows:

$$\frac{d\xi^H_\mu(t)}{dt} = \left[\xi^H_\mu(t), \hat{H}^{mp}_r\right] + \frac{\partial\xi^H_\mu(t)}{\partial t} \Rightarrow \frac{d\hat{\xi}_H(t)}{dt} = \left[\hat{\xi}^H_\mu(t) \overset{*}{,} \hat{H}^{mp}_{nc-r}\right] + \frac{\partial\hat{\xi}^H_\mu(t)}{\partial t}. \tag{6}$$

The operators ($\hat{H}_{or}^{mp}$ and $\hat{H}_{r}^{mp}$) are the free and global Hamiltonian for equal vector scalar Manning-Rosen potential while ($\hat{H}_{nc-or}^{mp}$ and $\hat{H}_{nc-r}^{mp}$) the corresponding Hamiltonians for the modified Manning-Rosen potentials. The present investigation aims at constructing a relativistic noncommutative effective scheme for the modified Manning-Rosen potential.

The paper is sketched in six sections. The rest of the five sections is organized as follows: We briefly review the usual relativistic Klein-Gordon equation with equal vector scalar Manning-Rosen potentials in the next section. Section 3 is devoted to the solutions of the deformed Klein-Gordon equation with the modified equal vector and scalar equal vector scalar Manning-Rosen potentials using Bopp's shift method and improved approximation of the centrifugal term to obtain the corresponding effective potential in addition to the standard perturbation theory in the first-order in the noncommutativity parameters (Θ, σ, χ) we find the expectation values of some radial terms. Section 4 is reserved to present the new main global energy shift and the global energy spectra of the molecular physics such as (HCl, CH, LiH, CO, NO, O_2, I_2, N_2, H_2, and Ar_2, and HCl) produced studied potential in the RNCQM symmetries. In Sec. 5, we apply our study for determining the energy spectra under this potential in the nonrelativistic noncommutative quantum mechanics (NRNCQM). Finally, we conclude this paper in Sec. 6.

2. Revised of the eigenfunctions and the energy eigenvalues for equal vector scalar Manning-Rosen in RQM

To achieve the main objective of the current study of finding solutions of deformed Klein-Gordon equation (KGE) in the RNCQM symmetries under the modified equal vector and scalar modified Manning-Rosen potential, it is helpful for the reader to see solutions in RQM. The equal vector and scalar modified Manning-Rosen potential [19] is given as:

$$V_{mp}(r)=\frac{\hbar^2}{2M\beta^2}\left(\frac{\alpha(\alpha-1)e^{-2r/\beta}}{(1-e^{-r/\beta})^2}-\frac{Ae^{-r/\beta}}{1-e^{-r/\beta}}\right), \tag{7.1}$$

and

$$S_{mp}(r)=\frac{\hbar^2}{2M\beta^2}\left(\frac{\lambda(\lambda-1)e^{-2r/\beta}}{(1-e^{-r/\beta})^2}-\frac{Be^{-r/\beta}}{1-e^{-r/\beta}}\right). \tag{7.2}$$

To achieve this goal of our current research it is useful to make a summary for the Klein?Gordon equation KGE three-dimensional relativistic quantum mechanics:

$$\Big\{-\nabla^2+[M(r)+S_{mp}(r)]^2-[E_{nl}-V_{mp}(r)]^2\Big\}\Psi(r,\theta,\varphi)=0. \tag{8}$$

The vector potential $V_{mp}(r)$ due to the four-vector linear momentum operator $A^{\mu}(V_{mp}(r),\ \vec{A}=0)$ and the space-time scalar potential $S_{my}(r)$ whereas the interaction of scalar and vector bosons are considering by usual substitutions ($M\rightarrow M+S_{mp}$ and $p^{\mu}\rightarrow p^{\mu}-A^{\mu}$), E_{nl} is the relativistic energy eigenvalues, $\vec{\nabla}$ is the ordinary 3-dimensional Nabla operator while ($n=0,1,2,\ldots$ and l) are represents the principal and orbital quantum numbers, respectively. Since equal vector scalar, Manning-Rosen potential has spherical symmetry, allowing the solutions of the time-independent KGE of the known form $\Psi(r,\theta,\varphi)=(U_{nl}(r)/r)Y_l^m(\theta,\varphi)$ to separate the radial $U_{nl}(r)$ and angular $Y_l^m(\theta,\varphi)$ parts of the wave function, thus Eq. (8) becomes:

$$\left(\frac{d^2}{dr^2}-\left[M^2-E_{nl}^2\right]-2\left[E_{nl}V_{mp}(r)+MS_{mp}(r)\right]+V_{mp}^2(r)-S_{mp}^2(r)-\frac{l[l+1]}{r^2}\right)U_{nl}(r)=0. \tag{9}$$

Using the shorthand notation

$$V_{\text{eff}}^{my}(r)\equiv 2(E_{nl}V_{mp}(r)+MS_{mp}(r))-V_{mp}^2(r)+S_{mp}^2(r)+\frac{l(l+1)}{r^2}$$

and $E_{\text{eff}}^{mp}\equiv M^2-E_{nl}^2$, we obtain the following second-order Schrödinger-like equation:

$$\left(\frac{d^2}{dr^2}-[E_{\text{eff}}^{my}+V_{\text{eff}}^{my}(r)]\right)U_{nl}(r)=0. \tag{10}$$

When the vector potential is equal to the scalar potential $V_{mp}(r)=S_{mp}(r)$ the effective potential leads to the following simple form:

$$V_{\text{eff}}^{mp}(r)\equiv 2(E_{nl}+M)V_{mp}(r)+\frac{l(l+1)}{r^2}. \tag{11}$$

The Ref. [19] gives the total wave function and the corresponding energy eigenvalues E_{nl} of the KGE with equal scalar and vector scalar and vector modified Manning-Rosen potential as follows:

$$\Psi(r,\theta,\varphi)=\frac{N_{nl}}{r}e^{(-r/\beta)\lambda}(1-e^{(-r/\beta)})^{(1/2)+(\vartheta_l/2)}{}_2F_1(-n,2\lambda_{nl}+\vartheta_l+n+1;1+2\lambda_{nl},e^{-r/\beta})Y_l^m(\theta,\varphi), \tag{12}$$

and

$$\lambda_{nl}\equiv\beta\sqrt{M^2-E_{nl}^2}=\frac{-n(n+\frac{1}{2})(1+\vartheta_{nl})-n^2+A\eta-\omega l(l+1)}{2n+1+\vartheta_{nl}}, \tag{13}$$

where $s=e^{-r/\beta}$, $\lambda_{nl}\equiv\beta\sqrt{M^2-E_{nl}^2}$, $\vartheta_{nl}=\sqrt{4\eta_{nl}\alpha(\alpha-1)+(2l+1)^2}$, $\eta_{nl}=(E_{nl}+M/M)$, N_{nl} is a normalization constant and ${}_2F_1(-n,2\lambda_{nl}+\vartheta_l+n+1;1+2\varepsilon_{nl},s)$ are the hypergeometric polynomials. From the definition of Jacobi polynomials [56]:

$$ {}_2F_1(-n,2\lambda_{nl}+\vartheta_{nl}+n+1;1+2\varepsilon_{nl},s)=\frac{n!\Gamma(2\lambda_{nl}+1)}{\Gamma(n+2\lambda_{nl}+1)}P_n^{(2\lambda_{nl},\vartheta_{nl})}(1-2s). \tag{14}$$

In terms of the definition of Jacobi polynomials, Eq. (12) can be written as:

$$\Psi(r,\theta,\varphi)=\frac{n!\Gamma(2\lambda_{nl}+1)N_{nl}}{\Gamma(n+2\lambda_{nl}+1)}\frac{s^{\lambda_{nl}}}{r}(1-s)^{([1/2]+[\vartheta_{nl}/2])}P_n^{(2\lambda_{nl},\vartheta_{nl})}(1-2s)Y_l^m(\theta,\varphi). \tag{15}$$

3. The solution of DKGE under modified Manning-Rosen potential in RNCQM

The beginning of this section is devoted to reformulate the Manning-Rosen potential in the relativistic noncommutative quantum mechanics symmetries (RNCQM). We achieve this goal by rewriting the KGE by applying the notion of the Weyl-Moyal star product introduced previously (see Eq. (3)) on the differential equation that satisfied by the radial wave function $U_{nl}(r)$ in the second section (see Eq. Eq. (9)); thus, the radial wave function $U_{nl}(r)$ in the RNCQM symmetries becomes as follows [56-67]:

$$\left\{\frac{d^2}{dr^2}-(M^2-E_{nl}^2)-2(E_{nl}+M)V_{mp}(r)-\frac{l(l+1)}{r^2}\right\}*U_{nl}(r)=0. \tag{16}$$

It is known to the specialized physicists that F. Bopp was the first to propose pseudo-differential operators obtained from a symbol by the quantization rules $(x\rightarrow x_{nc}=x-(i/2)\partial/\partial p)$ and $(p\rightarrow p_{nc}=p-(i/2)\partial/\partial x)$ instead of the ordinary correspondence $x\rightarrow x$ and $(p\rightarrow(i/2)\partial/\partial x)$; the latter are known as Bopp's shifts and the quantization procedure is the so-called Bopp quantization [55,68-70]. This method has attracted the attention of many researchers and is used as an alternative to the complicated star product calculations. As a consequence, we can rewrite the deformed Schrödinger equation, deformed Klien-Gordon equation, and deformed Dirac equation with the notion of star product to the Schrödinger equation, Klien-Gordon equation, and Dirac equation with the notion of ordinary product, respectively. This useful simplification can be achieved through reformulating the new algebraic relations which are known as noncommutative canonical commutation relations in the symmetries of relativistic noncommutative quantum mechanics with star product in Eqs. (2) and (3.1) and (3.2) without the notion of star product as follows (see, *e.g.*, [56,59,61,62]):

$$\left[\hat{x}_\mu^S,\hat{x}_\nu^S\right]=\left[\hat{x}_\mu^H(t),\hat{x}_\nu^H(t)\right]=\left[\hat{x}_\mu^I(t),\hat{x}_\nu^I(t)\right]=i\theta_{\mu\nu}. \tag{17}$$

The generalized positions and momentum coordinates: $(\hat{x}_\mu^S,\hat{p}_\mu^S)$, $(\hat{x}_\mu^H,\hat{p}_\mu^H)(t)$ and $(\hat{x}_\mu^I,\hat{p}_\mu^I)(t)$ in the symmetries of RNCQM are defined in terms of the corresponding coordinates (x_μ^S,p_μ^S), $(x_\mu^H,p_\mu^H)(t)$ and $(x_\mu^I,p_\mu^I)(t)$ in the symmetries of RQM via, respectively [39-49]:

$$(x_\mu^S,p_\mu^S)\Rightarrow\left(\hat{x}_\mu^S=x_\mu^S-\frac{\theta_{\mu\nu}}{2}P_\nu^S,\hat{p}_\mu^S=p_\mu^S\right), \tag{18.1}$$

$$(x_\mu^H,p_\mu^H)(t)\Rightarrow\left(\hat{x}_\mu^H(t)=x_\mu^H(t)-\frac{\theta_{\mu\nu}}{2}P_\nu^H(t),\hat{p}_\mu^H=p_\mu^H(t)\right), \tag{18.2}$$

$$(x_\mu^I,p_\mu^I)(t)\Rightarrow\left(\hat{x}_\mu^I(t)=x_\mu^I(t)-\frac{\theta_{\mu\nu}}{2}P_\nu^I(t),\hat{p}_\mu^I=p_\mu^I(t)\right), \tag{18.3}$$

This allows us to find the operator $r_{nc}^2 = r^2 - \vec{\mathbf{L}}\vec{\Theta}$ in the symmetries of RNCQM [59-61]. It is convenient to introduce a shorthand notation which will save us a lot of writing $r_{nc} \rightarrow \hat{r}$ the previously relation reduced to the $\hat{r}^2 = r^2 - \vec{\mathbf{L}}\vec{\Theta}$. According to the Bopp shift method, Eq. (17) becomes similar to the following like the Schrödinger equation (without the notions of star product):

$$\left\{\frac{d^2}{dr^2} - (M^2 - E_{nl}^2) - 2(E_{nl} + M)V_{mp}(\hat{r}) - \frac{l(l+1)}{\hat{r}^2}\right\}U_{nl}(r) = 0. \tag{19}$$

The new operators $V_{mp}(\hat{r})$ and $1/\hat{r}^2$ are expressed as in RNCQM symmetries as follows:

$$V_{mp}(\hat{r}) = V_{mp}(r) - \frac{\vec{\mathbf{L}}\vec{\Theta}}{2r}\frac{\partial V_{mp}(r)}{\partial r} + O(\Theta^2), \tag{20.1}$$

and

$$\frac{1}{\hat{r}^2} = \frac{1}{\hat{r}^2} + \frac{\vec{\mathbf{L}}\vec{\Theta}}{r^4} + O(\Theta^2). \tag{20.2}$$

Consequently, we can rewrite:

$$(E_{nl} + M)V_{mp}(\hat{r}) = (E_{nl} + M)V_{mp}(r) - (E_{nl} + M)\frac{\vec{\mathbf{L}}\vec{\Theta}}{2r}\frac{\partial V_{mp}(r)}{\partial r} + O(\Theta^2). \tag{21}$$

Moreover, to illustrate the above equation in a simple mathematical way and attractive form, it is useful to enter the following symbol $V_{nc-\text{eff}}^{mp}(r)$, thus the radial Eq. (20) becomes:

$$\left(\frac{d^2}{dr^2} - \left[E_{\text{eff}}^{mp} + V_{nc-\text{eff}}^{mp}(r)\right]\right)U_{nl}(r) = 0, \tag{22}$$

with

$$V_{nc-eff}^{mp}(r) = V_{eff}^{mp}(r) + V_{pert}^{my}(r), \tag{23}$$

where $V_{\text{pert}}^{mp}(r)$ is given by the following relation:

$$V_{\text{pert}}^{mp}(r) = \left(\frac{l(l+1)}{r^4} - \frac{E_{nl} + M}{r}\frac{\partial V_{mp}(r)}{\partial r}\right)\vec{L}(\vec{\Theta}). \tag{24}$$

It becomes obvious that the radial modified differential equation obtained in Eq. (22) cannot be solved analytically for any state $l \neq 0$ because of the centrifugal term. The effective perturbative potential, given in Eq. (24), has a strong singularity $r \rightarrow 0$; we need to use the new approximation of the centrifugal term proposed by C. S. Jia *et al*. [8] for a short-range potential, an excellent approximation to the centrifugal term. Unlike the following new approximation used in the previous work in ordinary quantum mechanics [8,20,21,23]:

$$\frac{1}{r^2} \approx \frac{1}{\beta^2}\left(\frac{\omega\exp[-r/\beta]}{1-\exp[-r/\beta]} + \frac{\exp[-2r/\beta]}{[1-\exp\{-r/\beta\}]^2}\right) = \frac{1}{\beta^2}\left(\frac{\omega s}{1-s} + \frac{s^2}{[1-s]^2}\right), \tag{25.1}$$

where ω is an adjustable dimensionless parameter. This allows us to obtain:

$$\frac{1}{r^4} \approx \frac{1}{\beta^4}\left(\frac{\omega\exp[-r/\beta]}{1-\exp[-r/\beta]} + \frac{\exp[-2r/\beta]}{[1-\exp\{-r/\beta\}]^2}\right)^2 = \frac{1}{\beta^4}\left(\frac{\omega^2 s^2}{[1-s]^2} + \frac{2\omega s^3}{[1-s]^3} + \frac{s^4}{[1-s]^4}\right), \tag{25.2}$$

which after straightforward calculations we obtain $\partial V_{my}(r)/\partial r$ as follows:

$$\frac{\partial V_{my}(r)}{\partial r} = \frac{\alpha(\alpha-1)\delta}{M\beta^2}\left(\frac{e^{-3\delta r}}{\left[1-e^{-\delta r}\right]^3} - \frac{e^{-2\delta r}}{\left[1-e^{-\delta r}\right]^2}\right) + \frac{A}{2M\beta^2}\frac{\delta e^{-\delta r}}{1-e^{-\delta r}} - \frac{A}{2M\beta^2}\frac{\delta e^{-2\delta r}}{(1-e^{-\delta r})^2}, \tag{26}$$

with $(1/\beta) = \delta$. The above equation can be simplified to the following form :

$$\frac{\partial V_{mp}(r)}{\partial r} = \lambda_1\frac{s^2}{(1-s)^2} + \lambda_2\frac{s^3}{(1-s)^3} + \lambda_3\frac{s}{(1-s)} \tag{27}$$

with

$$\lambda_1 \equiv -\frac{\alpha(\alpha-1)\delta}{M\beta^2} - \frac{A\delta}{2M\beta^2}, \quad \lambda_2 \equiv -\frac{\alpha(\alpha-1)\delta}{M\beta^2}$$

and

$$\lambda_3 \equiv \frac{A\delta}{2M\beta^2}.$$

Obviously, Eq. (25.1) cannot be determined from $1/r$. Therefore, we must use the improved approximation of the centrifugal term proposed by Badawi *et al.* [71]; this method proved its power and efficiency when compared with the Greene and Aldrich approximation for a short-range potential [6]. Unlike the following approximation used in the previous work in QM and NCQM [17,18,59,60,62,72] :

$$\frac{1}{r^2} \approx \frac{\exp(-r/\beta)}{\beta^2(1-\exp(-r/\beta))^2} = \frac{s}{\beta^2(1-s)^2}. \tag{28.1}$$

This allows us to obtain:

$$\frac{1}{r} \approx \frac{\exp(-r/2\beta)}{\beta(1-\exp(-r/\beta))} = \frac{s^{1/2}}{\beta(1-s)}. \tag{28.2}$$

The approximation (25.1) reduces to Eq. (28.1) when the adjustable dimensionless parameter $\omega = 1$. Inserting Eqs. (25.2), (27), (28.2) into Eq. (24) allows us to obtain the perturbed effective potential in the symmetries of RNCQM as follows:

$$V_{\text{pert}}^{mp}(r) = \begin{pmatrix} \frac{l(l+1)}{\beta^4}\frac{s^4}{(1-s)^4} - \frac{E_{nl}+M}{\beta} \\ \lambda_1\frac{s^{5/2}}{(1-s)^3} + \lambda_2\frac{s^{7/2}}{(1-s)^4} + \lambda_3\frac{s^{3/2}}{(1-s)^2} \end{pmatrix} \vec{\mathbf{L}}\vec{\Theta}. \tag{29}$$

The Manning-Rosen potential is extended by including new terms proportional to the radial terms

$$\frac{s^4}{(1-s)^4}, \quad \frac{s^{5/2}}{(1-s)^3}, \quad \frac{s^{7/2}}{(1-s)^4}$$

and

$$\frac{s^{3/2}}{(1-s)^2}.$$

to become the modified Manning-Rosen potential in RNCQM symmetries. Obviously, the newly generated effective potential $V_{\text{pert}}^{mp}(r)$ for the modified Manning-Rosen potential is also proportional to the infinitesimal vector $\vec{\Theta}$, allowing us to consider it as a perturbation potential compared with the parent potential operator $V_{\text{eff}}^{mp}(r)$ in the symmetries RNCQM, that is, the inequality $V_{\text{pert}}^{mp}(r) \ll V_{\text{eff}}^{mp}(r)$ has become achieved. In other words, all the physical justifications for applying the time-independent perturbation theory become satisfied. Now, we apply the perturbative theory, in the case of RNCQM, we find the expectation values of the radial terms

$$\frac{s^4}{(1-s)^4}, \quad \frac{s^{5/2}}{(1-s)^3}, \quad \frac{s^{7/2}}{(1-s)^4}$$

and

$$\frac{s^{3/2}}{(1-s)^2},$$

taking into account the wave function of Manning-Rosen potential which we have seen previously in the second section. After straightforward calculations, we obtain the following expectations values:

$$\left\langle \frac{s^4}{(1-s)^4} \right\rangle_{(n,l,m)} = \left[\frac{n!\Gamma(2\lambda_{nl}+1)N_{nl}}{\Gamma(n+2\lambda_{nl}+1)}\right]^2 \int_0^{+\infty} s^{2\lambda_{nl}}(1-s)^{1+\vartheta_{nl}}(P_n^{(2\lambda_{nl},\vartheta_{nl})}[1-2s])^2 \frac{s^4 dr}{(1-s)^4}, \tag{30.1}$$

$$\left\langle \frac{s^{5/2}}{(1-s)^3} \right\rangle_{(n,l,m)} = \left[\frac{n!\Gamma(2\lambda_{nl}+1)N_{nl}}{\Gamma(n+2\lambda_{nl}+1)}\right]^2 \int_0^{+\infty} s^{2\lambda_{nl}}(1-s)^{1+\vartheta_{nl}}(P_n^{(2\lambda_{nl},\vartheta_{nl})}[1-2s])^2 \frac{s^{5/2} dr}{(1-s)^3}, \tag{30.2}$$

$$\left\langle \frac{s^{7/2}}{(1-s)^4} \right\rangle_{(n,l,m)} = \left[\frac{n!\Gamma(2\lambda_{nl}+1)N_{nl}}{\Gamma(n+2\lambda_{nl}+1)}\right]^2 \int_0^{+\infty} s^{2\lambda_{nl}}(1-s)^{1+\vartheta_{nl}}(P_n^{(2\lambda_{nl},\vartheta_{nl})}[1-2s])^2 \frac{s^{7/2} dr}{(1-s)^4}. \tag{30.3}$$

and

$$\left\langle \frac{s^{3/2}}{(1-s)^2} \right\rangle_{(n,l,m)} = \left[\frac{n!\Gamma(2\lambda_{nl}+1)N_{nl}}{\Gamma(n+2\lambda_{nl}+1)}\right]^2 \int_0^{+\infty} s^{2\lambda_{nl}}(1-s)^{1+\vartheta_{nl}}(P_n^{(2\lambda_{nl},\vartheta_{nl})}[1-2s])^2 \frac{s^{3/2}dr}{(1-s)^2}. \tag{30.4}$$

We have used useful abbreviations $\langle n,l,m|\hat{B}|n,l,m\rangle \equiv \langle \hat{B}\rangle_{(n,l,m)}$ to avoid the extra burden of writing equations. Furthermore, we have applied the property of the spherical harmonics, which has the form

$$\int Y_l^m(\theta,\varphi)Y_{l'}^{m'}(\theta,\varphi)\sin(\theta)d\theta d\varphi = \delta_{ll'}\delta_{mm'}.$$

We have $s=\exp(-\delta r)$ (with $\delta = 1/\beta$), implying $dr = -(1/\delta)(ds/s)$. After introducing a new variable $z = 1-2s$, we have $s=(1-z)/2$, $dr=(1/\delta)(dz/1-z)$ and $1-s=(z+1)/2$. From the asymptotic behavior of $s=\exp(-\delta r)$ and $z=1-2s$, when $r\to 0$ $(z\to -1)$ and $r\to +\infty$ $(z\to 1)$ this allows the reformulation of Eqs. (30, $i=\overline{1,4}$) as follows:

$$2^{2\lambda_{nl}+\vartheta_{nl}+1}\delta\left\langle \frac{s^4}{(1-s)^4}\right\rangle_{(n,l,m)} = \left[\frac{n!\Gamma(2\lambda_{nl}+1)N_{nl}}{\Gamma(n+2\lambda_{nl}+1)}\right]^2 \int_{-1}^{+1}(1-z)^{2\lambda_{nl}+3}(1+z)^{\vartheta_{nl}-3}(P_n^{(2\lambda_{nl},\vartheta_{nl})}(z))^2 dz, \tag{31.1}$$

$$2^{2\lambda_{nl}+\vartheta_{nl}+1/2}\delta\left\langle \frac{s^{5/2}}{(1-s)^3}\right\rangle_{(n,l,m)} = \left[\frac{n!\Gamma(2\lambda_{nl}+1)N_{nl}}{\Gamma(n+2\lambda_{nl}+1)}\right]^2 \int_{-1}^{+1}(1-z)^{2\lambda_{nl}-3/2}(1+z)^{\vartheta_{nl}-2}(P_n^{(2\lambda_{nl},\vartheta_{nl})}(z))^2 dz, \tag{31.2}$$

$$2^{2\lambda_{nl}+\vartheta_{nl}+1/2}\delta\left\langle \frac{s^{7/2}}{(1-s)^4}\right\rangle_{(n,l,m)} = \left[\frac{n!\Gamma(2\lambda_{nl}+1)N_{nl}}{\Gamma(n+2\lambda_{nl}+1)}\right]^2 \int_{-1}^{+1}(1-z)^{2\lambda_{nl}+5/2}(1+z)^{\vartheta_{nl}-3}(P_n^{(2\lambda_{nl},\vartheta_{nl})}(z))^2 dz, \tag{31.3}$$

and

$$2^{2\lambda_{nl}+\vartheta_{nl}+1/2}\delta\left\langle \frac{s^{3/2}}{(1-s)^2}\right\rangle_{(n,l,m)} = \left[\frac{n!\Gamma(2\lambda_{nl}+1)N_{nl}}{\Gamma(n+2\lambda_{nl}+1)}\right]^2 \int_{-1}^{+1}(1-z)^{2\lambda_{nl}+1/2}(1+z)^{\vartheta_{nl}-1}(P_n^{(2\lambda_{nl},\vartheta_{nl})}(z))^2 dz. \tag{31.4}$$

For the ground state $n=0$, we have $P_{n=0}^{(2\lambda_{0l},\vartheta_{0l})}(z)=1$, thus the above expectation values in Eqs. (31, $i=\overline{1,4}$) are reduced to the following simple form:

$$2^{2\lambda_{0l}+\vartheta_{0l}+1}\delta\left\langle \frac{s^4}{(1-s)^4}\right\rangle_{(0,l,m)} = N_{0l}^2\int_{-1}^{+1}(1-z)^{2\lambda_{0l}+3}(1+z)^{\vartheta_{0l}-3}dz, \tag{32.1}$$

$$2^{2\lambda_{0l}+\vartheta_{0l}+1/2}\delta\left\langle \frac{s^{5/2}}{(1-s)^3}\right\rangle_{(0,l,m)} = N_{0l}^2\int_{-1}^{+1}(1-z)^{2\lambda_{0l}-3/2}(1+z)^{\vartheta_{0l}-2}dz, \tag{32.2}$$

$$2^{2\lambda_{0l}+\vartheta_{0l}+1/2}\delta\left\langle \frac{s^{7/2}}{(1-s)^4}\right\rangle_{(0,l,m)} = N_{0l}^2\int_{-1}^{+1}(1-z)^{2\lambda_{0l}+5/2}(1+z)^{\vartheta_{0l}-3}dz, \tag{32.3}$$

and

$$2^{2\lambda_{0l}+\vartheta_{0l}+1/2}\delta\left\langle \frac{s^{3/2}}{(1-s)^2}\right\rangle_{(0,l,m)} = N_{0l}^2\int_{-1}^{+1}(1-z)^{2\lambda_{0l}+1/2}(1+z)^{\vartheta_{0l}-1}dz. \tag{32.4}$$

where $\lambda_{0l}\equiv\beta\sqrt{M^2-E_{0l}^2}=(A\eta_{0l}-\omega l[l+1])/(1+\vartheta_{0l})$, $\vartheta_{0l}=\sqrt{4\eta_{0l}\alpha(\alpha-1)+(2l+1)^2}$ and $\eta_{0l}=(E_{0l}+M)/M$ Comparing Eqs. (31, $i=\overline{1,3}$) with the integral of the form [73]:

$$\int_{-1}^{+1}(1-x)^{\alpha}(1+x)^{\beta}P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)dx = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)n!}\delta_{mn} \Rightarrow$$

$$\int_{-1}^{+1}(1-x)^{n+\alpha}(1+x)^{n+\beta}dx = \frac{2^{2n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(2n+\alpha+\beta+1)}. \tag{33}$$

A direct calculation gives the expectation values in Eqs. (34. $i=\overline{1,3}$). Namely,

$$\left\langle \frac{s^4}{(1-s)^4}\right\rangle_{(0,l,m)} = \frac{N_{0l}^2\Gamma(2\lambda_{0l}+4)\Gamma(\vartheta_{0l}-2)}{(\rho_{0l}+1)\delta\Gamma(\rho_{0l}+1)}, \tag{34.1}$$

$$\left\langle \frac{s^{5/2}}{(1-s)^3}\right\rangle_{(0,l,m)} = \frac{N_{0l}^2\Gamma(2\lambda_{0l}-1/2)\Gamma(\vartheta_{0l}-1)}{4(\rho_{0l}-5/2)\delta\Gamma(\rho_{0l}-5/2)}, \tag{34.2}$$

$$\left\langle \frac{s^{7/2}}{(1-s)^4}\right\rangle_{(0,l,m)} = \frac{N_{0l}^2\Gamma(2\lambda_{0l}+7/2)\Gamma(\vartheta_{0l}-2)}{(\rho_{0l}+1/2)\delta\Gamma(\rho_{0l}+1/2)}, \tag{34.3}$$

$$\left\langle \frac{s^{3/2}}{(1-s)^2}\right\rangle_{(0,l,m)} = \frac{N_{0l}^2\Gamma(2\lambda_{0l}+3/2)\Gamma(\vartheta_{0l})}{(\rho_{0l}+1/2)\delta\Gamma(\rho_{0l}+1/2)}, \tag{34.4}$$

where ρ_{0l} equal $2\lambda_{0l}+\vartheta_{0l}$. For the first excited state $n=1$, the Jacobi polynomial reduced to $P_{n=1}^{(2\lambda_{1l},\vartheta_{1l})}(z)=\Omega_{1l}+\Xi_{1l}(1-z)$, here $\Omega_{1l}=\vartheta_{1l}+1$, $\Xi_{1l}=-(2\lambda_{1l}+\vartheta_{1l}+2)$, with $\lambda_{1l}\equiv\beta\sqrt{M^2-E_{1l}^2}=(-[3/2]\vartheta_{1l}-2+A\eta_{1l}-\omega l[l+1])/(3+\vartheta_{1l})$, $\vartheta_{1l}=\sqrt{4\eta_{1l}\alpha(\alpha-1)+(2l+1)^2}$, $\eta_{1l}=(E_{1l}+M)/M$. Thus, the expectation values in Eqs. (33 $i=\overline{1,6}$). Are reduces to the following simple form:

$$\left\langle \frac{s^4}{(1-s)^4}\right\rangle_{(1,l,m)} = T_1^{(1)}+T_1^{(2)}+T_1^{(3)}, \qquad \left\langle \frac{s^{5/2}}{(1-s)^3}\right\rangle_{(1,l,m)} = T_2^{(1)}+T_2^{(2)}+T_2^{(3)}, \tag{35.1}$$

and

$$\left\langle \frac{s^{7/2}}{(1-s)^4}\right\rangle_{(1,l,m)} = T_3^{(1)}+T_3^{(2)}+T_3^{(3)}, \qquad \left\langle \frac{s^{3/2}}{(1-s)^2}\right\rangle_{(1,l,m)} = T_4^{(1)}+T_4^{(2)}+T_4^{(3)}. \tag{35.2}$$

where the 12-factors $T_i^{(j)}(i=\overline{1,4},j=1,3)$ are given by:

$$\begin{pmatrix} T_1^{(1)} \\ T_1^{(2)} \\ T_1^{(3)} \end{pmatrix} = \frac{1}{2^{2\lambda_{1l}+\vartheta_{1l}+1}\delta}\left[\frac{N_{1l}}{2\lambda_{1l}+1}\right]^2 \begin{pmatrix} \Omega_{1l}^2\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+3}(1+z)^{\vartheta_{1l}-3}dz \\ 2\Omega_{1l}\Xi_{1l}\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+4}(1+z)^{\vartheta_{1l}-3}dz \\ \Xi_{1l}^2\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+5}(1+z)^{\vartheta_{1l}-3}dz \end{pmatrix}, \tag{36.1}$$

$$\begin{pmatrix} T_2^{(1)} \\ T_2^{(2)} \\ T_2^{(3)} \end{pmatrix} = \frac{1}{2^{2\lambda_{1l}+\vartheta_{1l}+1/2}\delta}\left[\frac{N_{1l}}{2\lambda_{1l}+1}\right]^2 \begin{pmatrix} \Omega_{1l}^2\int_{-1}^{+1}(1-z)^{2\lambda_{1l}-3/2}(1+z)^{\vartheta_{1l}-2}dz \\ 2\Omega_{1l}\Xi_{1l}\int_{-1}^{+1}(1-z)^{2\lambda_{1l}-1/2}(1+z)^{\vartheta_{1l}-2}dz \\ \Xi_{1l}^2\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+1/2}(1+z)^{\vartheta_{1l}-2}dz \end{pmatrix}, \tag{36.2}$$

$$\begin{pmatrix} T_3^{(1)} \\ T_3^{(2)} \\ T_3^{(3)} \end{pmatrix} = \frac{1}{2^{2\lambda_{1l}+\vartheta_{1l}+1/2}\delta}\left[\frac{N_{1l}}{2\lambda_{1l}+1}\right]^2 \begin{pmatrix} \Omega_{1l}^2\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+5/2}(1+z)^{\vartheta_{1l}-3}dz \\ 2\Omega_{1l}\Xi_{1l}\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+7/2}(1+z)^{\vartheta_{1l}-3}dz \\ \Xi_{1l}^2\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+9/2}(1+z)^{\vartheta_{1l}-3}dz \end{pmatrix}, \tag{36.3}$$

and

$$\begin{pmatrix} T_4^{(1)} \\ T_4^{(2)} \\ T_4^{(3)} \end{pmatrix} = \frac{1}{2^{2\lambda_{1l}+\vartheta_{1l}+1/2}\delta}\left[\frac{N_{1l}}{2\lambda_{1l}+1}\right]^2 \begin{pmatrix} \Omega_{1l}^2\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+1/2}(1+z)^{\vartheta_{1l}-1}dz \\ 2\Omega_{1l}\Xi_{1l}\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+3/2}(1+z)^{\vartheta_{1l}-1}dz \\ \Xi_{1l}^2\int_{-1}^{+1}(1-z)^{2\lambda_{1l}+5/2}(1+z)^{\vartheta_{1l}-1}dz \end{pmatrix}. \tag{36.4}$$

By using the integral formula in Eq. (33) we obtain the analytical expressions of the 12-factors $T_i^{(j)}(i=\overline{1,4}, j=1,3)$ as follows:

$$\begin{pmatrix} T_1^{(1)} \\ T_1^{(2)} \\ T_1^{(3)} \end{pmatrix} = \frac{1}{\delta}\left[\frac{N_{1l}}{2\lambda_{1l}+1}\right]^2 \begin{pmatrix} \dfrac{\Gamma(2\lambda_{1l}+4)\Gamma(\vartheta_{1l}-2)\Omega_{1l}^2}{(\omega_{1l}+1)\Gamma(\omega_{1l}+1)} \\ \dfrac{4\Omega_{1l}\Xi_{1l}\Gamma(2\lambda_{1l}+5)\Gamma(\vartheta_{1l}-2)}{(\omega_{1l}+2)\Gamma(\omega_{1l}+2)} \\ \dfrac{4\Xi_{1l}^2\Gamma(2\lambda_{1l}+6)\Gamma(\vartheta_{1l}-2)}{(\omega_{1l}+3)\Gamma(\omega_{1l}+3)} \end{pmatrix}, \tag{37.1}$$

$$\begin{pmatrix} T_2^{(1)} \\ T_2^{(2)} \\ T_2^{(3)} \end{pmatrix} = \frac{1}{\delta}\left[\frac{N_{1l}}{2\lambda_{1l}+1}\right]^2 \begin{pmatrix} \dfrac{\Omega_{1l}^2\Gamma(2\lambda_{1l}-1/2)\Gamma(\vartheta_{1l}-1)}{8(\omega_{1l}-5/2)\Gamma(\omega_{1l}-5/2)} \\ \dfrac{4\Omega_{1l}\Xi_{1l}\Gamma(2\lambda_{1l}+1/2)\Gamma(\vartheta_{1l}-1)}{4(\omega_{1l}-3/2)\Gamma(\omega_{1l}-3/2)} \\ \dfrac{4\Xi_{1l}^2\Gamma(2\lambda_{1l}+3/2)\Gamma(\vartheta_{1l}-1)}{2(\omega_{1l}-1/2)\Gamma(\omega_{1l}-1/2)} \end{pmatrix}, \tag{37.2}$$

$$\begin{pmatrix} T_3^{(1)} \\ T_3^{(2)} \\ T_3^{(3)} \end{pmatrix} = \frac{1}{\delta}\left[\frac{N_{1l}}{2\lambda_{1l}+1}\right]^2 \begin{pmatrix} \dfrac{\Omega_{1l}^2\Gamma(2\lambda_{1l}+7/2)\Gamma(\vartheta_{1l}-2)}{(\omega_{1l}+1/2)\Gamma(\omega_{1l}+1/2)} \\ \dfrac{4\Omega_{1l}\Xi_{1l}\Gamma(2\lambda_{1l}+9/2)\Gamma(\vartheta_{1l}-2)}{(\omega_{1l}+3/2)\Gamma(\omega_{1l}+3/2)} \\ \dfrac{4\Xi_{1l}^2\Gamma(2\lambda_{1l}+11/2)\Gamma(\vartheta_{1l}-2)}{(\omega_{1l}+5/2)\Gamma(\omega_{1l}+5/2)} \end{pmatrix}, \tag{37.3}$$

and

$$\begin{pmatrix} T_4^{(1)} \\ T_4^{(2)} \\ T_4^{(3)} \end{pmatrix} = \frac{1}{\delta}\left[\frac{N_{1l}}{2\lambda_{1l}+1}\right]^2 \begin{pmatrix} \dfrac{\Omega_{1l}^2\Gamma(2\lambda_{1l}+3/2)\Gamma(\vartheta_{1l})}{2(\omega_{1l}+1/2)\Gamma(\omega_{1l}+1/2)} \\ \dfrac{2\Omega_{1l}\Xi_{1l}\Gamma(2\lambda_{1l}+5/2)\Gamma(\vartheta_{1l})}{2(\omega_{1l}+1/2)\Gamma(\omega_{1l}+1/2)} \\ \dfrac{\Xi_{1l}^2\Gamma(2\lambda_{1l}+7/2)\Gamma(\vartheta_{1l})}{8(\omega_{1l}+5/2)\Gamma(\omega_{1l}+5/2)} \end{pmatrix}, \tag{37.4}$$

with $\omega_{1l}\equiv 2\lambda_{1l}+\vartheta_{1l}$. The substitution of Eqs. (37.1) (37.2), (37.3) and (37.4) into Eqs. (35.1), (35.2) (35.3), and (35.4) gives the expectation values in the first excited state $(1,l,m)$:

$$\begin{aligned}\left\langle\frac{s^4}{(1-s)^4}\right\rangle_{(1,l,m)} &= \frac{1}{\delta}\left[\frac{N_{1l}}{2\lambda_{1l}+1}\right]^2\Gamma(\vartheta_{1l}-2) \\ &\times\left(\frac{\Omega_{1l}^2\Gamma(2\lambda_{1l}+4)}{(\omega_{1l}+1)\Gamma(\omega_{1l}+1)}+\frac{4\Omega_{1l}\Xi_{1l}\Gamma(2\lambda_{1l}+5)}{(\omega_{1l}+2)\Gamma(\omega_{1l}+2)}+\frac{4\Xi_{1l}^2\Gamma(2\lambda_{1l}+6)}{(\omega_{1l}+3)\Gamma(\omega_{1l}+3)}\right), \qquad (38.1)\\ \left\langle\frac{S^{5/2}}{(1-s)^3}\right\rangle_{(1,l,m)} &= \left[\frac{N_{1l}}{2\lambda_{1l}+1}\right]^2\Gamma(\vartheta_{1l}-1) \\ &\times\left(\frac{\Omega_{1l}^2\Gamma(2\lambda_{1l}-1/2)}{8(\omega_{1l}-5/2)\Gamma(\omega_{1l}-5/2)}+\frac{4\Omega_{1l}\Xi_{1l}\Gamma(2\lambda_{1l}+1/2)}{4(\omega_{1l}-3/2)\Gamma(\omega_{1l}-3/2)}+\frac{4\Xi_{1l}^2\Gamma(2\lambda_{1l}+3/2)}{2(\omega_{1l}-1/2)\Gamma(\omega_{1l}-1/2)}\right), \qquad (38.2)\\ \left\langle\frac{S^{7/2}}{(1-s)^4}\right\rangle_{(1,l,m)} &= \frac{1}{\delta}\left[\frac{N_{1l}}{2\lambda_{1l}+1}\right]^2\Gamma(\vartheta_{1l}-2) \\ &\times\left(\frac{\Omega_{1l}^2\Gamma(2\lambda_{1l}+7/2)}{(\omega_{1l}+1/2)\Gamma(\omega_{1l}+1/2)}+\frac{4\Omega_{1l}\Xi_{1l}\Gamma(2\lambda_{1l}+9/2)}{(\omega_{1l}+3/2)\Gamma(\omega_{1l}+3/2)}+\frac{4\Xi_{1l}^2\Gamma(2\lambda_{1l}+11/2)}{(\omega_{1l}+5/2)\Gamma(\omega_{1l}+5/2)}\right), \qquad (38.3)\end{aligned}$$

and

$$\left\langle \frac{S^{3/2}}{(1-s)^2} \right\rangle_{(1,l,m)} = \frac{1}{\delta}\left[\frac{N_{1l}}{2\lambda_{1l}+1}\right]^2 \Gamma(\vartheta_{1l}) \times \left(\frac{\Omega_{1l}^2\Gamma(2\lambda_{1l}+3/2)}{2(\omega_{1l}+1/2)\Gamma(\omega_{1l}+1/2)} + \frac{2\Omega_{1l}\Xi_{1l}\Gamma(2\lambda_{1l}+5/2)}{2(\omega_{1l}+1/2)\Gamma(\omega_{1l}+1/2)} + \frac{\Xi_{1l}^2\Gamma(2\lambda_{1l}+7/2)}{8(\omega_{1l}+5/2)\Gamma(\omega_{1l}+5/2)} \right). \quad (38.4)$$

The relativistic study of the modified equal vector scalar Manning-Rosen potential is divided into three principal parts. The first one is devoted to studying the spin-orbit effect generated by the noncommutativity space-space. This is achieved by replacing the coupling of the angular momentum operator with noncommutativity coupling $\vec{\mathbf{L}}\vec{\Theta}$ by the new equivalent coupling $\Theta\vec{L}\vec{S}$ (with $\Theta = (\Theta_{12}^2 + \Theta_{23}^2 + \Theta_{13}^2)^{1/2}$). We have oriented the spin vector of the diatomic molecules such as HCl, CH, LiH, CO, NO, O_2, I_2, N_2, H_2, and Ar_2 to the direction of the vector $\vec{\Theta}$ under modified equal vector scalar Manning-Rosen potential. Then we replace it with the corresponding value: $(\Theta/2)(\vec{J}^2 - \vec{L}^2 - \vec{S}^2)$ Furthermore, in quantum mechanics, the operators ($\hat{H}_{nc-r}^{mp}$, J^2, L^2, S^2, and J_z) forms a complete set of conserved physics quantities, the eigenvalues of the operator $(\vec{J}^2 - \vec{L}^2 - \vec{S}^2)$ are equal the values $2k(l) \equiv j(j+1) - l(l+1) - s(s+1)$, with $|l-s| \le j \le |l+s|$. Consequently, the energy shift $\Delta\hat{E}_{myp}^{so}(n=0,\Theta,j,l,s) \equiv \Delta\hat{E}_{mp}^{so}(0,\Theta,j,l,s)$ and $\Delta\hat{E}_{mp}^{so}(n=1,\Theta,j,l,s) \equiv \Delta\hat{E}_{mp}^{so}(1,\Theta,j,l,s)$ due to the perturbed effective potential produced $V_{pert}^{mp}(r)$ for the ground state and the first excited state, respectively, in RNCQM symmetries as follows:

$$\Delta E_{mp}^{SO}(0,\Theta,j,l,s) = \frac{1}{2}(j[j+1] - l[l+1] - s(s+1)\Theta\langle X\rangle_{(0,l,m)}^{rmp}, \quad (39.1)$$

$$\Delta E_{mp}^{SO}(0,\Theta,j,l,s) = \frac{1}{2}(j[j+1] - l[l+1] - s(s+1)\Theta\langle X\rangle_{(0,l,m)}^{rmp}, \quad (39.2)$$

where the global expectation value $\langle X\rangle_{(0,l,m)}^{R}$ is determined from the following expression:

$$\langle X\rangle_{(0,l,m)}^{R} = \frac{l(l+1)}{\beta^4}\left\langle \frac{s^4}{(1-s)^4}\right\rangle_{(0,l,m)} - \frac{E_{nl}+M}{\beta} \times \left(\lambda_1\left\langle \frac{s^{5/2}}{(1-s)^3}\right\rangle_{(0,l,m)} + \lambda_2\left\langle \frac{s^{7/2}}{(1-s)^4}\right\rangle_{(0,l,m)} + \lambda_3\left\langle \frac{s^{3/2}}{(1-s)^2}\right\rangle_{(0,l,m)}\right), \quad (40)$$

while $\langle X\rangle_{(1,l,m)}^{rmp} = \langle X\rangle_{(1\to 0,l,m)}^{rmp}$. We can now generalize the above results $\Delta E_{mp}^{so}(n,\Theta,j,l,s)$ to the case of n^{th} excited states in RNCQM symmetries as follows:

$$\Delta E_{mp}^{so}(n,\Theta,j,l,s) = \frac{1}{2}(j[j+1] - l[l+1] - s(s+1)\Theta\langle X\rangle_{(n,l,m)}^{rmp}. \quad (41)$$

Thus, we can express the general expectation value as follows: $\langle X\rangle_{(n,l,m)}^{rmp}$

$$\langle X\rangle_{(n,l,m)}^{rmp} = \frac{l(l+1)}{\beta^4}\left\langle \frac{s^4}{(1-s)^4}\right\rangle_{(n,l,m)} - \frac{E_{nl}+M}{\beta} \times \left(\lambda_1\left\langle \frac{s^{5/2}}{(1-s)^3}\right\rangle_{(n,l,m)} + \lambda_2\left\langle \frac{s^{7/2}}{(1-s)^4}\right\rangle_{(n,l,m)} + \lambda_3\left\langle \frac{s^{3/2}}{(1-s)^2}\right\rangle_{(n,l,m)}\right). \quad (42)$$

The second main part in the relativistic study of the modified equal vector scalar Manning-Rosen potential is corresponding to replace both ($\vec{\mathbf{L}}\vec{\Theta}$ and Θ_{12}) by ($\sigma\aleph L_z$ and $\sigma\aleph$), respectively, here $\aleph$ and σ are, respectively symbolize the intensity of the induced magnetic field by the effect of deformation of space-space geometry and a new infinitesimal noncommutativity parameter, so that the physical unit of the original noncommutativity parameter Θ_{12} (length)^2 is the same unit of $\sigma\aleph$, we have also need to apply $\left\langle n,l,m|\hat{L}_z|n',l',m'\right\rangle = m'\delta_{nn'}\delta_{ll'}\delta_{mm'}$ (with $-(l,l') \le (m,m') \le +(l,l')$). All of this data allows for the discovery of the new energy shift $\Delta E_{mp}^{m}(n=0,\sigma,l,m) \equiv E_{my}^{m}(0,\sigma,l,m)$ and $\Delta E_{mp}^{m}(n=1,\sigma,l,m) \equiv E_{my}^{m}(1,\sigma,l,m)$ due to the perturbed Zeeman effect created by the influence of the modified equal vector scalar Manning-Rosen for the ground state and the first excited state in (RNC: 3D-RS) symmetries as follows:

$$\Delta E_{mp}^{m}(0,\sigma,l,m) = \aleph\langle X\rangle_{(0,l,m)}^{rmp}\sigma m, \quad (43.1)$$

$$\Delta E_{mp}^{m}(1,\sigma,l,m) = \aleph\langle X\rangle_{(1,\sigma,l,m)}^{rmp}\sigma m. \quad (43.2)$$

Thus, we can generalize the above particular cases to the general case $\Delta E^{m}_{mp}(n,\sigma,l,m)$ of the modified equal vector scalar Manning-Rosen potential which corresponds to the n^{th} excited states in (RNC: 3D-RS) symmetries as follows:

$$\Delta E^{m}_{mp}(n,\sigma,l,m) = \aleph\Bigg(\frac{l(l+1)}{\beta^4}\left\langle\frac{s^4}{(1-s)^4}\right\rangle_{(n,l,m)} - \frac{E_{nl}+M}{\beta} \times\left[\lambda_1\left\langle\frac{s^{5/2}}{(1-s)^3}\right\rangle_{(n0,l,m)} + \lambda_2\left\langle\frac{s^{7/2}}{(1-s)^4}\right\rangle_{(n,l,m)} + \lambda_3\left\langle\frac{s^{3/2}}{(1-s)^2}\right\rangle_{(n,l,m)}\right]\Bigg)\sigma m. \tag{44}$$

Now, for our purposes, we are interested in finding a new third automatically important symmetry for modified equal vector scalar Manning-Rosen potential at zero temperature in RNCQM symmetries. This physical phenomenon is induced automatically from the influence of a perturbed effective potential $V^{mp}_{\text{pert}}(r)$, which we have seen in Eq. (31). We discover these important physical phenomena when our studied system consists of N non-interacting is consider as Fermi gas, it is formed from all the particles in their gaseous state under rotation with angular velocity $\vec{\Omega}$ if we make the following two simultaneous transformations to ensure that previous calculations are not repeated:

$$\vec{\Theta} \rightarrow \chi\vec{\Omega}. \tag{45.1}$$

Here χ is just infinitesimal real proportional constants. We can express the effective potential $V^{mp}_{\text{pert-rot}}(r)$, which induced the rotational movements under the effect of modified equal vector scalar Manning-Rosen potential at zero temperature for the diatomic molecules as follows:

$$V^{mp}_{\text{pert-rot}}(r) = \chi\left(\frac{l(l+1)}{\beta^4}\frac{s^4}{(1-s)^4} - \frac{E_{nl}+M}{\beta}\left[\lambda_1\frac{s^{5/2}}{(1-s)^3} + \lambda_2\frac{s^{7/2}}{(1-s)^4} + \lambda_3\frac{s^{3/2}}{(1-s)^2}\right]\right)\vec{\Omega}\vec{\mathbf{L}}. \tag{45.2}$$

To simplify the calculations without compromising physical content, we choose the rotational velocity $\vec{\Omega} = \Omega e_z$. The next step is to transform the spin-orbit coupling to the new physical phenomena as follows:

$$\chi g(s)\vec{\Omega}\vec{\mathbf{L}} \rightarrow \chi g(r)\Omega L_z, \tag{46}$$

with

$$g(s) = \frac{l(l+1)}{\beta^4}\frac{s^4}{(1-s)^4} - \frac{E_{nl}+M}{\beta}\left(\lambda_1\frac{s^{5/2}}{(1-s)^3} + \lambda_2\frac{s^{7/2}}{(1-s)^4} + \lambda_3\frac{s^{3/2}}{(1-s)^2}\right). \tag{47}$$

All of this data allows for the discovery of the new energy shift $\Delta E^{f}_{mp}(n=0,\chi,l,m) \equiv \Delta E^{f}_{mp}(0,\chi,l,m)$ and $\Delta E^{f}_{mp}(n=1,\chi,l,m) \equiv \Delta E^{f}_{mp}(1,\chi,l,m)$ due to the perturbed Fermi gas effect which generated automatically by the influence of the modified equal vector scalar Manning-Rosen potential for the ground state and the first excited state in RNCQM symmetries as follows:

$$\Delta E^{f}_{mp}(0,\chi,l,m) = \langle X\rangle^{rmp}_{(0,l,m)}\chi\Omega m, \tag{48.1}$$

and

$$\Delta E^{f}_{mp}(1,\chi,l,m) = \langle X\rangle^{rmp}_{(1,l,m)}\chi\Omega m. \tag{48.1}$$

Thus, we can generalize the above particular cases to the general case which correspond to the n^{th} excited states in RNCQM symmetries as follows:

$$\Delta E^{f}_{mp}(n,\chi,l,m) = \langle X\rangle^{rmp}_{(n,l,m)}\chi\Omega m. \tag{49}$$

It is worth mentioning that K. Bencheikh *et al*. [74,75] studied rotating isotropic and anisotropic harmonically confined ultra-cold Fermi gas in a two and three-dimensional space at zero temperature but in this study, the rotational term $\chi g(s)\vec{\Omega}\,\vec{\mathbf{L}}$ was added to the Hamiltonian operator. In contrast to our case, where this rotation term automatically appears due to the large symmetries resulting from the deformation of the space-phase.

4. Results and discussion

In this section of the paper, we summarize our obtained results ($\Delta E_{mp}^{so}(0,\Theta,j,l,s)$), ($\Delta E_{mp}^{so}(1,\Theta,j,l,s)$), ($\Delta E_{mp}^{m}(0,\sigma,l,m)$), ($\Delta E_{mp}^{m}(1,\sigma,l,m)$), and ($\Delta E_{mp}^{f}(0,\chi,l,m)$), and ($\Delta E_{mp}^{f}(1,\chi,l,m)$), for the ground state and first excited state due to the spin-orbital complying, modified Zeeman effect, and perturbed Fermi gas potential which induced by $V_{eff}^{mp}(r)$ on based to the superposition principle. Accordingly, we can deduce the additive energy shift $\Delta E_{mp}^{tot}(\Theta,\sigma,\chi,0,j,l,s,m)$ and $\Delta E_{mp}^{tot}(\Theta,\sigma,\chi,1,j,l,s,m)$ under the influence of modified Manning-Rosen potential in RNCQM symmetries as follows:

$$\Delta E_{mp}^{tot}(\Theta,\sigma,\chi,0,j,l,s,m)=\langle\chi\rangle_{(0,l,m)}^{rmp}\left(k[j,l,s]\Theta+\aleph\sigma m+\chi\Omega m\right), \tag{50.1}$$

and

$$\Delta E_{mp}^{tot}(\Theta,\sigma,\chi,0,j,l,s,m)=\langle\chi\rangle_{(0,l,m)}^{rmp}\left(k[j,l,s]\Theta+\aleph\sigma m+\chi\Omega m\right). \tag{50.2}$$

It is easily to generalize the above special cases to the n^{th} excited states $\Delta E_{mp}^{tot}(\Theta,\sigma,\chi,n,j,l,s,m)$ under the influence of modified Manning-Rosen potential in RNCQM symmetries as

$$\Delta E_{mp}^{tot}(\Theta,\sigma,\chi,0,j,l,s,m)=\langle\chi\rangle_{(0,l,m)}^{rmp}\left(k[j,l,s]\Theta+\aleph\sigma m+\chi\Omega m\right). \tag{51}$$

The above results present the global energy shift, which is generated by the effect of noncommutativity properties of space-space; it depends explicitly on the noncommutativity parameters (Θ,σ,χ), the parameters of equal vector scalar Manning-Rosen (β,A,α) in addition to the atomic quantum numbers (n,j,l,s,m). We observed that the obtained global effective energy $\Delta E_{mp}^{tot}(\Theta,\sigma,\chi,n,j,l,s,m)$ under Modified Manning-Rosen potential carries units of energy because it is combined from the carrier of energy $(M^2-E_{nl}^2)$. As a direct consequence, the energy $E_{r-nc}^{mp}(\Theta,\sigma,\chi,\beta,A,\alpha,n,j,l,s,m)$ produced with modified equal vector and scalar equal vector scalar Manning-Rosen potentials, in the symmetries of RNCQM, corresponding the generalized n^{th} excited states, the sum of the square-roots $[E_{mp}^{tot}(\Theta,\sigma,\chi,n,j,l,s,m)]^{1/2}$ of the shift energy, and E_{nl} due to the effect of equal vector scalar Manning-Rosen in RQM, which determined from Eq. (12), as follows:

$$E_{r-nc}^{mp}(\Theta,\sigma,\chi,\beta,A,\alpha,n,j,l,s,m)=E_{nl}-M+\left(\langle\chi\rangle_{(n,l,m)}^{rmp}\left[k(j,l,s)\Theta+\aleph\sigma m+\chi\Omega m\right]\right)^{1/2}. \tag{52}$$

For the ground state and first excited state, the above equation can be reduced to the following form:

$$E_{r-nc}^{mp}(\Theta,\sigma,\chi,\beta,A,\alpha,n=0,j,l,s,m)=E_{0l}-M+\left(\langle\chi\rangle_{(0,l,m)}^{rmp}\left[k(j,l,s)\Theta+\aleph\sigma m+\chi\Omega m\right]\right)^{1/2}. \tag{53.1}$$

and

$$E_{r-nc}^{ym}(\Theta,\sigma,\chi,\beta,A,\alpha,n=1,j,l,s,m)=E_{1l}-M+\left(\langle\chi\rangle_{(1,l,m)}^{rmp}\left[k(j,l,s)\Theta+\aleph\sigma m+\chi\Omega m\right]\right)^{1/2}. \tag{53.2}$$

Equation (52) can de describe the relativistic energy of some diatomic molecules such as HCl, CH, LiH, CO, NO, O_2, I_2, N_2, H_2, and Ar_2 under the modified equal vector scalar Manning-Rosen potential in RNCQM symmetries.

5. Nonrelativistic spectrum under the modified Manning-Rosen potential

The radial part $U_{nl}(r)$ of the complete wave function

$$\Psi(r,\theta,\varphi)=\frac{U_{nl}(r)}{r}Y_l^m(\theta,\varphi)$$

in ordinary nonrelativistic QM satisfied the following equation for Manning-Rosen potential:

$$\frac{d^2U_{nl}(r)}{dr^2}+2M(E_{nl}^{nr-mp}-V_{eff}^{nr-mp}(r))U_{nl}(r)=0, \tag{54}$$

where

$$V_{eff}^{nr-mp}(r)=\frac{\hbar^2}{2M\beta^2}\left[\frac{\alpha(\alpha-1)e^{-2r/\beta}}{(1-e^{-r/\beta})^2}-\frac{Ae^{-r/\beta}}{1-e^{-r/\beta}}\right]+\frac{l(l+1)}{2Mr^2}$$

is the nonrelativistic effective potential in ordinary NRQM. The radial wave function $U_{nl}(r)$ in nonrelativistic noncommutative three-dimensional real space NRNCQM symmetries becomes as follows [56-67]:

$$\frac{d^2U_{nl}(r)}{dr^2}+2M(E_{nl}^{nr-mp}-V_{eff}^{nr-mp}(r))*U_{nl}(r)=0. \tag{55}$$

According to the Bopp shift method, Eq. (55) becomes similar to the Schrödinger equation (without the notions of star product):

$$\frac{d^2U_{nl}(r)}{dr^2}+2M(E_{nl}^{nr-mp}-V_{eff}^{nr-mp}(\hat{r}))U_{nl}(r)=0. \tag{56}$$

We can express the new effective potential $V_{eff}^{nr-mp}(\hat{r})$ in NRNCQM symmetries:

$$V_{eff}^{nr-mp}(\hat{r})=V_{eff}^{nr-mp}+V_{mp}^{pert}(r). \tag{57}$$

The global effective potential $V_{mp}^{pert}(r)$ is the perturbative potential produced with modified equal vector scalar Manning-Rosen potential in NRNCQM symmetries plus the additive part $(l(l+1)/2Mr^4)\vec{\mathbf{L}}\vec{\Theta}$ in Eq. (20.2):

$$V_{mp}^{pert}(r)=\frac{l(l+1)}{2Mr^4}\vec{\mathbf{L}}\vec{\Theta}-\frac{\partial V_{mp}(r)}{\partial r}\frac{\vec{\mathbf{L}}\vec{\Theta}}{2r}+O(\Theta^2). \tag{58}$$

We have applied the type of approximations suggested by Greene and Aldrich and Dong *et al.* for a short-range potential (see Eq. (28.1)) and we have calculated $\partial V_{my}(r)/\partial r$ (Eq. (27)). Now, substituting Eq. (27) into Eq. (55) and replacing $1/r$ by its corresponding approximation in Eq. (28.2), we get the perturbative potential in (NC: 3D-RS) symmetries,

$$V_{my}^{pert}(r)=\frac{l(l+1)}{2M\beta^4}\frac{s^4}{(1-s)^4}\vec{\mathbf{L}}\vec{\Theta}-\frac{1}{\beta}\left(\lambda_1\frac{s^{5/2}}{(1-s)^3}+\lambda_2\frac{s^{7/2}}{(1-s)^4}+\lambda_3\frac{s^{3/2}}{(1-s)^2}\right)\vec{\mathbf{L}}\vec{\Theta}+O(\Theta^2). \tag{59}$$

Thus, we need the expectation values of $s^4/(1-s)^4$, $s^{5/2}/(1-s)^3$, $s^{7/2}/(1-s)^4$, and $s^{3/2}/(1-s)^2$ to find the nonrelativistic energy corrections produced with the perturbative potential $V_{mp}^{pert}(r)$. By using the wave function in Eq. (15) and the expectation values in Eq. (34, $i=\overline{1,4}$), and Eq. (38, $i=\overline{1,4}$). For the ground state and first excited state, respectively, we get the corresponding global expectation values $\langle\chi\rangle_{(0,l,m)}^{nr-mp}$ and $\langle\chi\rangle_{(1,l,m)}^{nr-mp}$

$$\begin{aligned}\langle\chi\rangle_{(0,l,m)}^{nr-mp}=&\frac{l(l+1)}{2M\beta^4}\left\langle\frac{s^4}{(1-s)^4}\right\rangle_{(0,l,m)}\\&-\frac{1}{\beta}\left(\lambda_1\left\langle\frac{s^{5/2}}{(1-s)^3}\right\rangle_{(0,l,m)}+\lambda_2\left\langle\frac{s^{7/2}}{(1-s)^4}\right\rangle_{(0,l,m)}+\lambda_3\left\langle\frac{s^{3/2}}{(1-s)^2}\right\rangle_{(0,l,m)}\right),\end{aligned} \tag{60.1}$$

and

$$\begin{aligned}\langle\chi\rangle_{(1,l,m)}^{nr-mp}=&\frac{l(l+1)}{2M\beta^4}\left\langle\frac{s^4}{(1-s)^4}\right\rangle_{(1,l,m)}\\&-\frac{1}{\beta}\left(\lambda_1\left\langle\frac{s^{5/2}}{(1-s)^3}\right\rangle_{(1,l,m)}+\lambda_2\left\langle\frac{s^{7/2}}{(1-s)^4}\right\rangle_{(1,l,m)}+\lambda_3\left\langle\frac{s^{3/2}}{(1-s)^2}\right\rangle_{(1,l,m)}\right).\end{aligned} \tag{60.2}$$

By following the same physical methodology that we have developed in our previous relativistic study, the energy corrections $\Delta E_{mp}^{nr}(0,\Theta,\sigma,\chi,j,l,s,m)$ and $\Delta E_{mp}^{nr}(1,\Theta,\sigma,\chi,j,l,s,m)$ for the ground state and first excited state due to the spin-orbit complying, modified Zeeman effect and nonrelativistic perturbed Fermin gas potential which induced by $V_{mp}^{pert}(r)$ under the influence of modified Manning-Rosen potential in NRNCQM symmetries

$$\Delta E_{mp}^{nr}(\Theta,\sigma,\beta,A,\alpha,n=0,j,l,s,m)=\langle\chi\rangle_{(0,l,m)}^{nr-mp}(k(j,l,s)\Theta+\aleph\sigma m+\chi\Omega m), \tag{61.1}$$

and

$$\Delta E_{mp}^{nr}(\Theta,\sigma,\beta,A,\alpha,n=1,j,l,s,m)=\langle\chi\rangle_{(1,l,m)}^{nr-mp}(k(j,l,s)\Theta+\aleph\sigma m+\chi\Omega m). \tag{61.2}$$

It is easily to generalize the above special cases to the n^{th} excited states under the influence of modified Manning-Rosen potential in NRNCQM symmetries as follows:

$$\Delta E_{mp}^{nr}(\Theta,\sigma,\beta,A,\alpha,n,j,l,s,m)=\langle\chi\rangle_{(n,l,m)}^{nr-mp}(k(j,l,s)\Theta+\aleph\sigma m+\chi\Omega m), \tag{62}$$

with $\langle\chi\rangle_{(n,l,m)}^{nr-mp}$ is given by

$$\langle\chi\rangle_{(n,l,m)}^{nr-mp}=\frac{l(l+1)}{2M\beta^4}\left\langle\frac{s^4}{(1-s)^4}\right\rangle_{(n,l,m)} -\frac{1}{\beta}\left(\lambda_1\left\langle\frac{s^{5/2}}{(1-s)^3}\right\rangle_{(n,l,m)}+\lambda_2\left\langle\frac{s^{7/2}}{(1-s)^4}\right\rangle_{(n,l,m)}+\lambda_3\left\langle\frac{s^{3/2}}{(1-s)^2}\right\rangle_{(n,l,m)}\right). \tag{63}$$

The nonrelativistic energy $E_{nr-nc}^{mp}(\Theta,\sigma,\beta,A,\alpha,n,j,l,s,m)$ for the diatomic molecules (HCl, CH, LiH, CO, NO, O_2, I_2, N_2, H_2, and Ar_2) produced with modified equal vector scalar Manning-Rosen potential, in the symmetries of (NC: 3D-RS), corresponding to the generalized n^{th} excited states, the sum of the nonrelativistic energy E_{nl}^{nr} due to the effect of equal vector scalar Manning-Rosen in NRQM, and the corrections produced with the perturbed spin-orbit interaction and modified Zeeman effect, as follows:

$$E_{nr-nc}^{mp}(\Theta,\sigma,\beta,A,\alpha,n,j,l,s,m)=\frac{1}{2\mu b^2}l(l+1)c-\frac{\hbar^2}{2\mu b^2}\left(\frac{A+\alpha(\alpha-1)}{\tau(n,l,\alpha)}-\frac{\tau(n,l,\alpha)}{4}\right)^2 +\langle\chi\rangle_{(n,l,m)}^{nr-mp}\left(k(j,l,s)\Theta+\aleph\sigma m+\chi\Omega\right), \tag{64}$$

here $\tau(n,l,\alpha)=2n+1+\sqrt{(1-2\alpha)^2+4l(l+1)}$, $b\to\beta$ and c is a dimensionless constant and equal $1/12$ when $(r/b)\ll 1$, while the case of $c=0$ is identical to the conventional approximation given in Eq. (28.1). The first two terms are the nonrelativistic energy due to the Manning-Rosen potential in NRQM, which is determined directly from the study of Z.Y. Chen, *et al*. [18].

Now, considering composite systems such as molecules made of $N=2$ particles of masses $m_n(n=1,2)$ in the frame of noncommutative algebra, it is worth taking into account features of descriptions of the systems in the non-relativistic case, it was obtained that composite systems with different masses are described with different noncommutative parameters [76-78]:

$$\left[\hat{x}_\mu^s \overset{*}{,} \hat{x}_\mu^s\right]=\left[\hat{x}_\mu^H(t)\overset{*}{,}\hat{x}_\nu^H(t)\right]=\left[\hat{x}_\mu^I(t)\overset{*}{,}\hat{x}_\nu^I(t)\right]=i\theta_{\mu\nu}^c, \tag{65}$$

where the noncommutativity parameter $\theta_{\mu\nu}^c$ is given by:

$$\theta_{\mu\nu}^c=\sum_{n=1}^{2}\mu_n^2\theta_{\mu\nu}^{(n)}, \tag{66}$$

with $\mu_n=m_n/\sum_n m_n$ the indices $(n=1,2)$ label the particle, and $\theta_{\mu\nu}^{(n)}$ is the parameter of noncommutativity, corresponding to the particle of mass m_n. Note that in the case of a system of two particles with the same mass $m_1=m_2$ such as the diatomic (O_2, I_2, N_2, H_2, and Ar_2) molecules, the parameter $\theta_{\mu\nu}^{(n)}=\theta_{\mu\nu}$. Thus, the three parameters Θ, σ and χ which appears in Eq. (63) are changed to the new form:

$$\Theta^{c2}=\left(\sum_{n=1}^{2}\mu_n^2\Theta_{12}^{(n)}\right)^2+\left(\sum_{n=1}^{2}\mu_n^2\Theta_{23}^{(n)}\right)^2+\left(\sum_{n=1}^{2}\mu_n^2\Theta_{13}^{(n)}\right)^2, \tag{67.1}$$

$$\sigma^{c2}=\left(\sum_{n=1}^{2}\mu_n^2\sigma_{12}^{(n)}\right)^2+\left(\sum_{n=1}^{2}\mu_n^2\sigma_{23}^{(n)}\right)^2+\left(\sum_{n=1}^{2}\mu_n^2\sigma_{13}^{(n)}\right)^2, \tag{67.2}$$

$$\chi^{c2}=\left(\sum_{n=1}^{2}\mu_n^2\chi_{12}^{(n)}\right)^2+\left(\sum_{n=1}^{2}\mu_n^2\chi_{23}^{(n)}\right)^2+\left(\sum_{n=1}^{2}\mu_n^2\chi_{13}^{(n)}\right)^2. \tag{67.3}$$

As it is mentioned above, in the case of a system of two particles with the same mass $m_1=m_2$, we have $\Theta_{\mu\nu}^{(n)}=\Theta_{\mu\nu}$, $\sigma_{\mu\nu}^{(n)}=\sigma_{\mu\nu}$, and $\chi_{\mu\nu}^{(n)}=\chi_{\mu\nu}$. Finally, we can generalize the nonrelativistic global energy $E_{nr-nc}^{mp}(\Theta,\sigma,\chi,\beta,A,\alpha,n,j,l,s,m)$ under the modified Manning-Rosen potential considering that composite systems with different masses are described with different noncommutative parameters for the diatomic molecules (HCl, CH, LiH, CO, and NO) as:

$$E_{nr-nc}^{mp}(\Theta,\sigma,\chi,\beta,A,\alpha,n,j,l,s,m)=\frac{l(l+1)c}{2\mu b^2}-\frac{1}{2\mu b^2}\left(\frac{A+\alpha(\alpha-1)}{\tau(n,l,\alpha)}-\frac{\tau(n,l,\alpha)}{4}\right)^2 +\langle\chi\rangle_{(n,l,m)}^{nr-mp}\left(k(j,l,s)\Theta^c+\aleph\sigma^c m+\chi^c\Omega m\right), \tag{68}$$

The important result in this work is to consider DKGE and DSHE under modified Manning-Rosen has a physical behavior similar to the Duffin-Kemmer equation, which provides us a basis to study relativistic spin-zero and spin-one bosons [79], it can describe a dynamic state of a particle with spin one in the symmetries of relativistic noncommutative quantum mechanics. Worthwhile it is better to mention that for the two simultaneous limits $(\Theta, \sigma, \chi) \to (0,0,0)$ and $(\Theta^c, \sigma^c, \chi^c) \to (0,0,0)$, we recover the results of the in Refs. [13,18,19].

6. Summary and conclusion

This paper covers the perspective of modified equal vector scalar Manning-Rosen potential in both relativistic and nonrelativistic regimes that correspond to high and low energy physics. We have employed both simultaneous methods, the Bopp's shift and standard perturbation theory methods, to obtain the new bound state solutions the deformed Klein-Gordon and Schrödinger equations by applying the improved approximation scheme to deal with the centrifugal term. The obtained new bound state solutions $E^{mp}_{r-nc}(\Theta, \sigma, \chi, \beta, A, \alpha, n, j, l, s, m)$ and $E^{mp}_{nr-nc}(\Theta, \sigma, \chi, \beta, A, \alpha, n, j, l, s, m)$ corresponding to the generalized n^{th} excited states appear as a sum of the total shift energy $\Delta E^{tot}_{mp}(\Theta, \sigma, \chi, n, j, l, s, m)$ the nonrelativistic corrections $\Delta E^{nr}_{mp}(\Theta, \sigma, \beta, A, \alpha, n, j, l, s, m)$ relativistic energy E_{nl}, and nonrelativistic energies in RQM and NRQM, respectively, for the equal vector scalar Manning-Rosen potential. The total shift energy and nonrelativistic corrections appeared as a function of the Gamma function, the discreet atomic quantum numbers (j, l, s, m), and the potential parameters (β, A, α) in addition to noncommutativity three parameters $(\Theta, \sigma$ and $\chi)$ of noncommutativity space-space. This behavior is similar to the perturbed both modified Zeeman effect, modified perturbed spin-orbit coupling in which an external magnetic field is applied to the system and the spin-orbit couplings which are generated with the effect of the perturbed effective potential $V^{mp}_{pert}(r)$ in the symmetries of relativistic and nonrelativistic noncommutative quantum mechanics. In addition, we can conclude that the DKGE becomes similar to the Duffin-Kemmer equation under modified equal vector scalar Manning-Rosen potential, it can describe a dynamic state of a particle with spin one in the symmetries of relativistic noncommutative quantum mechanics. Furthermore, we have applied our results to composite systems such as molecules made of $N = 2$ particles of masses $m_n(n = 1, 2)$. It is worth mentioning that, for all cases, when to make the two simultaneously limits $(\Theta, \sigma, \chi) \to (0,0,0)$ and $(\Theta^c, \sigma^c, \chi^c) \to (0,0,0)$, the ordinary physical quantities are recovered. Finally, our research findings are very relevant in areas of atomic physics, vibrational and rotational spectroscopy, mass spectra, nuclear physics, and other applications [13,18,19].

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