On the London superconductors and mesoscopic RLC circuits

I. A. Pedrosa and L. Nascimento
Departamento de Física, CCEN, Universidade Federal da Paraíba,
Caixa Postal 5008, 58059-900, João Pessoa, PB, Brazil,
e-mail: iapedrosa@fisica.ufpb.br; luciano.fisicacex@gmail.com

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We investigate the connection between the London superconductor and a mesoscopic RLC circuit in both classical and quantum contexts. We show that the mathematical framework to describe the dynamics of these two different systems is identical. Based on the Lewis-Riesenfeld invariant method together with the Fock states, we solve the time-dependent Schrödinger equation for this problem and evaluate some of its important physical properties, such as coherent states, expectation values of the charge and magnetic flux, their quantum fluctuations and the corresponding uncertainty principle.

Keywords: London superconductor; RLC circuit; london equations; invariant method; coherent states.

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1. Introduction

The understanding of classical and quantum systems in different areas of physics that can be described by the same mathematical formalism is a relevant topic of study and has always attracted the attention of physicists. This study is recognized to be of great utility because solutions of problems in different branches of physics that present similar behavior and properties help us understand more about others.

It is well known that superconductors are materials that exhibit no electrical resistance below a temperature that is known as its critical temperature. Their unusual properties have made them key components in many areas of physics, such as quantum computation, quantum optical systems, nanoelectronic devices, particle accelerators and NMR magnets [1–9]. The main advantages of devices made from superconductors are low power dissipation, high-speed operation, and high sensitivity. The superconductivity was discovered by Onnes in 1911 [10], and for many years afterwards, it was thought to consist simply of a complete disappearance of its electrical resistance below the its critical temperature [9]. A major advance for the study of suprconductors was given, twenty years after its discovery, by Meissener and Ochsenfeld [11]. These researchers showed that when a magnetic field is applied to a superconductor, the applied field is excluded from its interior, except in a thin penetration region near the surface [9]. This is the so-called Meissener-Ochsenfeld. Shortly after the discovery that magnetic fields are expelled from superconductors the brothers Fritz and Heinz London in 1935 [12] proposed a phenomenological theory to describe the electromagnetic dynamics of superconductivity. Their proposed equations are consistent with the Meissner-Ochsenfeld effect and can be used with Maxwell's equations to predict how the magnetic field and surface current vary with distance from the surface of a superconductor. Further, a microscopic theory of superconductivity, the famous BCS theory, was presented in 1957 by Bardeen, Cooper and Schrieffer [9]. Since then, many physicists have contributed to the study of superconductor materials [13–18]. In this work we investigate the classical and quantum electromagnetic dynamics of a conventional superconductor based on the phenomenological London equations [12].

Over the last decades, a great deal of attention has been paid to the study of quantum effects of mesoscopic circuits [17–23]. This interest is mainly motivated by the fact that their applications in nanoscience and specially in nanoelectronics seem endless. In this context an LC (inductance L and capacitance C) represents a typical and fundamental circuit. In history, Louisell [24] was the first physicits who proposed a quantization scheme for this circuit. Another, more complicated, mesoscopic electromagnetic oscillation system is the RLC circuit. For this case, one has to consider the effect of the resistance R on the circuit, that is, the dissipation. The quantization scheme and quantum properties of the RLC circuit are certainly of great theoretical and experimental physical interest. In fact, in recent years, many works on the quantum behavior of this circuit have been published in the literature [19, 22, 23, 25, 26, 26–28].

The main purpose of this paper is to discuss the connection between the London superconductor and a mesoscopic RLC circuit. We demonstrate that the classical and quantum dynamics of these two different systems are similar and both can be described by similar Hamiltonians. Based on the quantum invariant theory introduced by Lewis and Riesenfeld [29], together with Fock states, we easily solve the time-dependent Schrödinger equation for our problem and use its solutions to construct coherent states for the quantized RLC circuit. Finally, we evaluate various quantum properties of these systems, such as the expectation values of the charge and magnetic flux, their quantum fluctuations and the uncertainty principle.

We organize this paper in the following order. In Sec. 2 we discuss the classical dynamics of the London superconductor and of the mesoscopic RLC circuit. In Sec. 3, we present the quantum dynamics of the RLC circuit. In Sec. 4, we construct coherent states for the quantized RLC circuit and calculate some physical properties of this system. We conclude the paper with a short summary in Sec. 5.

2. Classical dynamics analysis

2.1. London Superconductor

With the objective of discussing the classical electromagnetic behavior of the London superconductor, we first need to look at Maxwell's equations. The classical electromagnetic field with charge and current sources is described by the set of equations

$$\nabla \cdot \vec{D} = \rho, \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{1}$$

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}, \tag{2}$$

with the total charge and current densities, respectively, ρ and $\vec{J}$ satisfying the the continuity equation (charge conservation)

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}, \tag{3}$$

and the fields being related by

$$\vec{D} = \varepsilon \vec{E}, \quad \vec{B} = \mu \vec{H}, \tag{4}$$

where ε and μ are, respectively, the electric permittivity and the magnetic permeability.

Now, to analyze the electromagnetic behavior of a conventional superconductor we must take in account the London's equations. These equations can be written as [12]

$$\frac{\partial \vec{J}_s}{\partial t} = \frac{1}{\mu \lambda_L^2} \vec{E}, \tag{5}$$

$$\nabla \times \vec{J}_s = -\frac{n_s e_s^2}{m_s} \vec{B}, \tag{6}$$

where

$$\lambda_L = \left(\frac{m_s}{\mu n_s e_s^2} \right)^{1/2}. \tag{7}$$

The length λ_L, which is associated with the decay of the magnetic field at the surface of the superconductor, is the London penetration depth. Here, $\vec{J}_s$ is the supercurrent density and n_s, m_s and e_s are, respectively, the density of electrons in a superconducting state, the mass and the charge of the superconducting electrons.

Consider next, the total current density. It is given by

$$\vec{J} = \vec{J}_n + \vec{J}_s, \tag{8}$$

with

$$\vec{J}_n = \sigma \vec{E}, \tag{9}$$

where $\vec{J}_n$ and σ are, respectively, the current density due to the normal electrons and the electric conductivity. If we now take the time-derivative of Eq. (8) and use the Maxwell and London equations together with the continuity Eq. (3) we find that

$$\ddot{q}(t) + \frac{\sigma}{\varepsilon} \dot{q}(t) + \omega^2 q(t) = 0, \tag{10}$$

where the dots indicate differentiation with respect to time, $\omega^2 = c^2/\lambda_L^2$ is the frequency and $c = 1/\sqrt{\mu\varepsilon}$ is the velocity of the light within the superconductor. The Eq. (10) represents the equation of motion for the total charge inside a certain volume of the superconductor. As everybody knows, the solution of this equation is

$$q(t) = A e^{-\sigma t/2\varepsilon} \sin(\Omega t + \delta), \tag{11}$$

where Ω is given by

$$\Omega = \sqrt{\omega^2 - \left(\frac{\sigma}{2\varepsilon} \right)^2}, \tag{12}$$

with A and δ being arbitrary constants . Here we have only considered oscillatory solutions, that is, the $\Omega > 0$ case. Now, it is easy to verify that the classical Hamiltonian

$$H(t) = e^{-\sigma t/\varepsilon} \frac{\Phi^2}{2\varepsilon} + \frac{1}{2} e^{\sigma t/\varepsilon} \varepsilon \omega^2 q^2, \tag{13}$$

leads to the Eq. (10), where the dynamical variable Φ represents the magnetic flux. The Hamiltonian (13) is the famous Caldirola-Kanai Hamiltonian, which has been employed by many authors to study time-dependent Hamiltonian systems in many branches of physics [22, 30–38]. At this point, it is worth noticing that the Hamiltonian (13) that describes the dynamics of the London superconductor is time-dependent as long as the conductivity σ exists ($\sigma \neq 0$). It can also be helpful to remark that the Eq. (10) is formally identical to that describing the behavior of the mode amplitudes of the propagation of electromagnetic waves in conducting media [36–38]. For this case, the classical and quantum dynamics of electromagnetic waves are, of course, also described by the Caldirola-Kanai Hamiltonian [36–38].

In what follows, we calculate some properties of the London superconductor. By using Eq. (13) together with the Hamilton's equations we find the magnetic flux as

$$\Phi = \varepsilon e^{\sigma t/\varepsilon} \dot{q}. \tag{14}$$

We can rewrite this flux in a more convenient form as

$$\Phi = L'(t) i, \tag{15}$$

with

$$L'(t) = \varepsilon e^{\sigma t/\varepsilon}, \tag{16}$$

where $L'(t)$ and $i = \dot{q}$ are, respectively, the time-dependent inductance the current in the superconductor. Further, by making use of (15) we find the Faraday's law for the London superconductor as

$$\epsilon = -\frac{d\Phi}{dt} = \varepsilon e^{\sigma t/\varepsilon} \omega^2(t) q = L'(t) \omega^2 q. \tag{17}$$

Therefore, the previous results represent the classical electromagnetic dynamics of the London superconductor. In the following subsection, we discuss the classical mesoscopic or nanoscale RLC circuit.

2.2. Mesoscopic RLC Circuit

In this subsection, we consider the classical RLC circuit. It consists of a resistance R, inductance L, and capacitance C. In this mesoscopic circuit the resistance is given by Ohm's law Ri_1 where $i_1 = \dot{q_1}$ is the current in the circuit. The inductance induces a magnetic flux Li_1 and the capacitance enters in the total voltage as q_1/C. For this circuit, the equation of motion for the charge q_1 is

$$\ddot{q_1}(t) + \frac{R}{L}\dot{q_1}(t) + \omega_1^2 q_1(t) = 0, \tag{18}$$

with ω_1^2=$1/LC$ being the frequency of the circuit in the absence of the resistance R. The solution of Eq. (18) is

$$q_1(t) = Be^{-Rt/2L}\sin(\Omega_1 t + \xi), \tag{19}$$

where B and ξ are arbitrary constants and Ω_1 is given by

$$\Omega_1 = \sqrt{\omega_1^2 - \left(\frac{R}{2L}\right)^2}, \tag{20}$$

with $\Omega_1 > 0$. Eq.(18) can be generated from the Hamiltonian

$$H_1(t) = e^{-Rt/L}\frac{\Phi_1^2}{2L} + \frac{1}{2}e^{Rt/L}L\omega_1^2 q_1^2, \tag{21}$$

with the magnetic flux

$$\Phi_1 = Le^{Rt/L}\dot{q_1}. \tag{22}$$

We can rewrite this relation as

$$\Phi_1 = L_1(t)i_1, \tag{23}$$

with $L_1(t)$ given by

$$L_1(t) = Le^{Rt/L}. \tag{24}$$

By using Eq.(23) we get that

$$\epsilon_1 = -\frac{d\Phi_1}{dt} = L_1(t){\omega_1}^2 q_1, \tag{25}$$

which represents the Faraday's law for the RLC circuit.

Here, we can confirm, from the results above, that the mathematical framework to study the classical dynamics of the London superconductor and of the mesoscopic RLC circuit is identical. In fact, these systems are described in terms of *bona fide* damped harmonic oscillators [Eqs. (10) and (18)] which are derived from similar Hamiltonians [Eqs. (13) and (21)]. They also share amazing similarities in the expressions for the magnetic flux [Eqs. (15) and (23)], inductance [Eqs. (16) and (24)] and Faraday's law [Eqs. (17) and (25)]. This analogy can be carried still further by making the following correspondence: $\sigma \Leftrightarrow R$ and $\varepsilon \Leftrightarrow L$. However, it is worth mentioning that the equivalence of these two completely different systems is not fully complete. As a matter of fact, the dispersion relations [Eqs. (12) and (20)], which are intrinsic to each physical system, are not identical and consequently the electromagnetic dynamics of these systems is different.

3. Quantum dynamics analysis

In order to obtain the quantum description of the mesoscopic RLC circuit (or, equivalently, of the London superconductor) we must solve the Schrödinger equation associated with the Hamiltonian (21) which is given by

$$H_1|\Psi, t\rangle = i\hbar\frac{\partial}{\partial t}|\Psi, t\rangle, \tag{26}$$

where q_1 and Φ_1 are now operators with $[q_1, \Phi_1] = i\hbar$ and $\Phi_1 = -i\hbar\partial/\partial q_1$. In this paper we use the invariant theory introduced by Lewis and Riesenfeld [29] to find the solutions of the time-dependent Schrödinger (26). Thus, if the system described by the Hamiltonian (21) admits an exact dynamical invariant operator $I(t)$ that satisfies the relation

$$\frac{dI}{dt} = \frac{1}{i\hbar}[I, H_1] + \frac{\partial I}{\partial t} = 0. \tag{27}$$

the solutions of the equation (26) can be written in terms of the eigenstates of $I(t)$ and a time-dependent phase. Then, we can write the solutions of (26) in the form

$$|\psi_n, t\rangle = e^{i\beta_n(t)}|\phi_n, t\rangle, \tag{28}$$

with

$$I(t)|\phi_n, t\rangle = \lambda_n|\phi_n, t\rangle \quad , \quad \lambda_n = const, \tag{29}$$

where the $|\phi_n, t\rangle$ form a complete orthonormal set $\langle\phi_{n'}, t|\phi_n, t\rangle = \delta_{n'n}$. The phase functions $\beta_n(t)$ are obtained from

$$\hbar\frac{d\beta_n(t)}{dt} = \left\langle \phi_n, t \left| i\hbar\frac{\partial}{\partial t} - H_1(t) \right| \phi_n, t \right\rangle. \tag{30}$$

Now, it is known that the Hamiltonian (21) admits an invariant of the form [22, 37–39]

$$I(t) = \frac{1}{2}\left[\left(\frac{q_1}{\rho}\right)^2 + \left[\rho\Phi_1 - Z(t)\dot{\rho}q_1\right]^2\right], \tag{31}$$

where $\rho(t)$ is a real function satisfying the equation [38–41]

$$\ddot{\rho}(t) + \frac{R}{L}\dot{\rho}(t) + \omega_1^2\rho(t) = \frac{1}{Z^2\rho^3}, \tag{32}$$

with

$$Z(t) = Le^{Rt/L} = L_1(t). \tag{33}$$

In what follows, let us now return to the eigenvalue equation (29). To obtain the solutions of this equation we will use the Fock states. For this purpose, we define annihilation and creation-type operators $a(t)$ and $a^\dagger(t)$ as [22,29,37]

$$a(t) = \left(\frac{1}{2\hbar}\right)^{1/2}\left[\frac{q_1}{\rho} + i(\rho\Phi_1 - Z\dot{\rho}q_1)\right], \tag{34}$$

$$a^\dagger(t) = \left(\frac{1}{2\hbar}\right)^{1/2}\left[\frac{q_1}{\rho} - i(\rho\Phi_1 - Z\dot{\rho}q_1)\right], \tag{35}$$

with

$$[a(t), a^\dagger(t)] = 1. \tag{36}$$

If we perform the inverse transformations of Eqs. (34) and (35), it can be shown that the invariant (31) can be written in the form

$$I(t) = \hbar\left[a^\dagger(t)a(t) + \frac{1}{2}\right]. \tag{37}$$

Hence, using (36) and (37) we can solve the eigenvalue equation for $I(t)$ just as for the time-independent mechanical harmonic oscillator by employing the Fock states $|n,t\rangle$. Then, writing the Hermitian number operator $N = a^\dagger a$ and $N|n,t\rangle = n|n,t\rangle$, we get

$$I(t)|n,t\rangle = \hbar\left(n+\frac{1}{2}\right)|n,t\rangle, \quad \lambda_n = \hbar\left(n+\frac{1}{2}\right), \tag{38}$$

$$a(t)|n,t\rangle = n^{1/2}|n-1,t\rangle, \tag{39}$$

$$a^\dagger(t)|n,t\rangle = (n+1)^{1/2}|n+1,t\rangle. \tag{40}$$

Consider next, the phase functions (30). By making $|\phi_n,t\rangle$=$|n,t\rangle$ we find, after a little of algebra, that

$$\beta_n(t) = -\left(n+\frac{1}{2}\right)\int_0^t \frac{1}{Z(\tau)\rho^2(\tau)}d\tau. \tag{41}$$

Notice that a particular solution of (32) can be written as

$$\rho(t) = \frac{e^{-Rt/2L}}{(L\Omega_1)^{1/2}}, \tag{42}$$

for which Eq. (41) reduces to

$$\beta_n(t) = -\Omega_1\left(n+\frac{1}{2}\right)t. \tag{43}$$

Then, the solutions of the Schrödinger Eq. (26) are (see Eq. (28))

$$|\psi_n,t\rangle = e^{i\beta_n(t)}|n,t\rangle, \tag{44}$$

with $\beta_n(t)$ given by (43). Therefore, the general Schrödinger state can be written as $|\Psi,t\rangle = \sum_n c_n|\psi_n,t\rangle$, where the c_n are time-independent.

If we now calculate the expectation values and quantum fluctuations of q_1 and Φ_1 in the Fock states, we obtain

$$\langle q_1\rangle = \langle\Phi_1\rangle = 0, \tag{45}$$

$$\langle q_1^2\rangle = \hbar\rho^2\left(n+\frac{1}{2}\right), \tag{46}$$

$$\langle\Phi_1^2\rangle = \hbar\left[\frac{1}{\rho^2} + (Z\dot{\rho})^2\right]\left(n+\frac{1}{2}\right), \tag{47}$$

whence

$$(\Delta q_1)^2 = \langle q_1^2\rangle - \langle q_1\rangle^2 = \hbar\rho^2\left(n+\frac{1}{2}\right), \tag{48}$$

$$(\Delta\Phi_1)^2 = \langle\Phi_1^2\rangle - \langle\Phi_1\rangle^2 = \hbar\left[\frac{1}{\rho^2} + (Z\dot{\rho})^2\right]\left(n+\frac{1}{2}\right). \tag{49}$$

From the above relations we see that the expectation values of the charge and the magnetic flux are zero, but their quantum fluctuations are not zero. Further, it follows from (48) and (49) that

$$(\Delta q_1)(\Delta\Phi_1) = \hbar\left[1 + (Z\rho\dot{\rho})^2\right]^{1/2}\left(n+\frac{1}{2}\right), \tag{50}$$

which, by making use of (33) and (42), becomes

$$(\Delta q_1)(\Delta\Phi_1) = \frac{\hbar\omega_1}{\Omega_1}\left(n+\frac{1}{2}\right). \tag{51}$$

Here, it is worth noticing that the uncertainty principle (51) depends on the electric resistance R. Yet, if the usual dissipation effect due to the electric resistance is null, that is, $R = 0$ the Eq. (51) is converted into

$$(\Delta q_1)(\Delta\Phi_1) = \hbar\left(n+\frac{1}{2}\right), \tag{52}$$

which represents the uncertainty principle of a time-independent mechanical harmonic oscillator with frequency ω_1 (see Eqs. (18) and (21)) with L playing the role of the mass m. We also notice that for this case, *i.e.*, $R = 0$, the particular solution (42) becomes $\rho = 1/(L\omega_1)^{1/2}$ and the Hamiltonian (21) and the annihilation and creation operators (34) and (35) are reduced to that of the standard harmonic oscillator.

We end this section remarking that we can proceed in the same way to discuss the quantum behavior of the London superconductor, since, as we have proved in the previous section, from a mathematical point of view the analysis of the electromagnetic dynamics of these two different systems is identical.

4. Coherent states for the RLC circuit

In this section, we are going to construct coherent states for the mesoscopic RLC circuit. It is well-known that coherent states for time-dependent quantum systems described by Hamiltonians-like (21) are given by [42]

$$|\alpha,t\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_n \frac{(\alpha)^n}{(n!)^{1/2}}\exp\left[i\beta_n(t)\right]|n,t\rangle, \tag{53}$$

where α is an arbitrary complex constant. Of course, the states $|\alpha, t\rangle$ are eigenstates of the annihilation operator $a(t)$with eigenvalue $\alpha e^{2i\beta_0(t)}$ being $\beta_0(t) = -(\Omega_1 t/2)$.

The calculation of the expectation value of q_1 in the coherent states $|\alpha, t\rangle$ yields

$$\langle q_1\rangle = \left(\frac{2\hbar|\alpha|^2}{L\Omega_1}\right)^{1/2} e^{-Rt/2L}\sin(\Omega_1 + \xi), \qquad (54)$$

where ξ is the phase of α [42]. Here, we observe that this result does indeed agree with that of Eq.(19), so that it is according to the Schrödinger quantum-classical correspondence for the coherent states [43].

The quantum fluctuations in q_1 and Φ_1 in the state $|\alpha, t\rangle$ are given by

$$\langle \Delta q_1\rangle^2 = \langle q_1^2\rangle - \langle q_1\rangle^2 = \frac{\hbar}{2}\rho^2, \qquad (55)$$

$$\langle \Delta\Phi_1\rangle^2 = \langle \Phi_1^2\rangle - \langle \Phi_1\rangle^2 = \frac{\hbar}{2}\left[\frac{1}{\rho^2} + (Z\dot{\rho})^2\right], \qquad (56)$$

whence

$$(\Delta q_1)(\Delta\Phi_1) = \frac{\hbar\omega_1}{2\Omega_1}, \qquad (57)$$

where we have used the Eq. (33) and the solution (42). By comparing Eqs. (51) and (57) we see that the uncertainty principle in the coherent states is exactly the same as the minimum value of that in the Fock states. It may be helpful, at this point, to note that these uncertainty principles do not depend on time and that their values become larger when the resistance increases. We also observe that the uncertainty principle (57), in general, does not attain its minimum value. This occurs because the states $|\alpha, t\rangle$ are not minimum-uncertainty states and correspond to the well-known squeezed states [44–47]. Further, it is worth noticing that when the resistance is null, that is, $R = 0$ the uncertainty principle attains its minimum value because in this case the states $|\alpha, t\rangle$ reduce to the coherent states of the ordinary mechanical harmonic oscillator model. In short, we observe that one can follow the same steps of this section to construct coherent states for the London superconductor.

To end this section, let us make some comments. In Ref. [48] we have demonstrated that the electromagnetic dynamics of a London superconductor and a time-dependent LC circuit with inductance and capacitance modulated exponentially at a constant rate are equivalent and both can be described in the same mathematical framework. In this case, the time-derivative of the inductance $\dot{L}$, causes a damping similar to that produced by the conductivity σ. What is more, in Refs. [22, 23] have been shown that a time-dependent mesoscopic LC circuit with inductance and capacitance increasing exponentially with time is equivalent to the time-independent RLC circuit. For this case, the time differentiation of the inductance $\dot{L}$, produces a dissipation similar to that produced by the resistance R. Now, in the present paper, we have demonstrated that the London superconductor and the time-independent RLC circuit are equivalent and can be described through the same mathematical formalism, in both the classical and quantum contexts. Therefore, we can confirm that these three completely different systems, that is, the London superconductor, the time-dependent LC circuit and the time-independent RLC circuit share amazing similarities so that their electromagnetic dynamics can be analyzed in the same mathematical framework. This result is interesting and important since, as we have already mentioned previously, the study of classical and quantum dynamics of physical systems that present similar behavior and properties is a topic of special interest in the context of theoretical physics. Finally, it is worth mentioning that, to the best of our knowledge, the mathematical equivalence between the London superconductor and the time-independent mesoscopic RLC circuit has not reported in the literature yet.

5. Summary

In this work, we have established a simple and elegant connection between the London superconductor and the mesoscopic RLC circuit. We have demonstrated that they share similar behavior, both classically and quantum mechanically and can be described in terms of genuine damped harmonic oscillators which are generated by similar Hamiltonians. Further, by using the invariant method, appropriated annihilation and creation-type operators and Fock states, we have easily solved the time-dependent Schrödinger equation for the RLC circuit (the same procedure can be made for the London superconductor) and employ its solutions to construct coherent states for the quantized RLC circuit. Yet, we have calculated expectation values of the charge and magnetic flux, their quantum fluctuations as well as the uncertainty principle in both states, namely, Fock and coherent states. We also have seen that the uncertainty principle in the coherent states is equal to the minimum value of that in the Fock states. In addition, we have found that the uncertainty principle in the coherent states does not attain its minimum value. This latter result occurs because the coherent state constructed in this work correspond to the well-known squeezed states. Finally, we would like to observe that Schrödinger states for a London superconductor with time-dependent conductivity has been obtained in Ref. [49].

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