Effect of the orientation distribution of thin highly conductive inhomogeneities on the overall electrical conductivity of heterogeneous material

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Many natural composite materials contain systems of partially oriented thin low-resistivity inclusions (for example, water-saturated microcracks in a double porosity sedimentary formation). We have calculated the components of the electrical conductivity tensor of such materials as a function of crack density. The results were obtained for thin ellipsoidal inclusions with conductivity (electrical or thermal) much larger than the matrix conductivity. To calculate the effective conductivity, we have used the effective field method (EFM). We have obtained the explicit expressions for the effective parameters of inhomogeneous materials. The application of the EFM allows one to describe the influence of the peculiarities in the spatial distribution of inclusions on the effective properties of the medium. General explicit expressions, obtained in this work, are illustrated by calculating examples for inclusions, homogeneously distributed in the sector [-β, β], where β is the disorientation angle, and some continuous angle distribution functions. The calculations have shown that the spatial distribution of the crack-like inclusions strongly affects the conductive properties of the effective medium and the symmetry of their tensor.

Keywords: Heterogeneous medium; highly conductive inhomogeneities; homogenization problem; effective field method; influence of the inclusion orientations.

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1. Introduction

Many microinhomogeneous materials, for instance, sedimentary rocks, contain a system of oriented or partially oriented inhomogeneities (inclusions). Thus, carbonate hydrocarbon reservoirs may be represented as a system of fluid-filled microcracks that are either randomly oriented or oriented along certain directions [1-18]. The presence of such systems of cracks leads to a considerable anisotropy of the physical characteristics of the medium, such as effective electrical or thermal conductivity or elastic moduli. Identification of oriented systems of cracks and the evaluation of their parameters is an important problem that is of interest for many areas of physics of composite materials and petrophysics. The materials, containing high-conductivity inclusions, may be considered as homogeneous with certain effective properties, when their physical properties are determined in a larger scale compared to the characteristic size of inclusions.

During the past one hundred years, starting from the pioneering works by [5,6,19] a number of methodologies of calculation of effective conductivity of a medium with inclusions of different shapes have been developed [25-29]. A review of such methods that includes the works, published before 1990, was given in the papers by [3,26,30]. An overview of the more recent works is given in Refs. [4,29]. As a rule, in the majority of the works, that are based on the effective medium approach, the authors consider the media that contain either random or parallel inclusions. In the case of real media, such models may be frequently considered only as a rough approximation [18]. In the current work, we consider the model of a microinhomogeneous medium with inclusions that are not strictly oriented. Their spatial distribution is described by some angular function. Such models have long been in the interest of researchers. For elastic cracked solids, for example, such models are considered in Refs. [15,28,31]. The solid review about the effect of orientation distribution on the effective properties of the fiber reinforced materials can be found in Ref. [20]. In the works by [8,9,17] the influence of the inhomogeneity orientation on the elastic and conductive properties of microinhomogeneous materials are studied. The results obtained in the last works were based on the so-called non-interaction approximation that slightly reduced the area of possible application. Our calculations are based on the so-named effective field method that allows considering the interaction between the inclusions. According to this method, every inclusion in the inhomogeneous medium is considered as an isolated one, embedded in the homogeneous background medium (matrix). The field that acts on this inclusion (effective field) does not coincide with the "external" field applied to the medium, but it is the sum of this "external" field and disturbances induced by all surrounding inclusions. This method has a long history, and it was mainly used in nuclear physics and the theory of phase transitions for the description of various types of many particle interactions. In application to the mechanics of composite materials, this method was developed by [10,13]. This method has gained popularity for the calculation of effective elastic properties due to its simplicity. The most complete explanation of the method is given in the monographs by [13].

However, this method is not frequently used in application to the problems of thermal or electric conductivity. In our previous work by [16], we have applied this method to calculate the coefficients of the conductivity tensor of randomly oriented and parallel inclusions. In the present work, we apply the effective field method for the calculation of conductivity tensor of microheterogeneous media containing thin (crack-like) high-conductivity inclusions that are distributed in the space, and this distribution is characterized by some distribution density. In the second section of the paper, we discuss the homogenization problem, while in the third one we present the solution of the so-called one-particle problem that is the building block of any homogenization scheme. Then, the effective field method is shortly presented, and the effective conductivity of the material containing a random set of thin high-conductivity inclusions are calculated. After that we study the changes of the overall conductive material symmetry depending on the changes in the parameters of the distribution function over the orientation of the inclusions.

2. Homogenization problem

Let us consider a set of dispersed isotropic particles (inclusions), having the conductivity coefficient C, randomly distributed in an infinitely isotropic homogeneous medium, having the conductivity coefficient C_0. The vector of the local flux $q_i(x)$ and the field $e_i(x)$ in such a medium satisfy the system of equations:

$$\partial_i q_i(x) = 0, \qquad q_i(x) = C_{ij}(x)e_j(x),$$
$$\mathrm{rot}_{ij} e_j(x) = 0, \qquad \partial_i \equiv \frac{\partial}{\partial x_i}. \tag{2.1}$$

where x is an arbitrary point in 3D-space. These equations can describe various physical processes in solids, including stationary thermal conductivity and filtration, static electroconductivity in conductors and dielectrics, as well as, magneto- and electrostriction. All functions in Eqs. (3) are random functions of coordinates. Determination of the relation between the mathematical expectations of the fields $q_i(x)$ and $e_i(x)$ that generally has the form:

$$\langle q_i(x)\rangle = C^*_{ij}\langle e_j(x)\rangle\,, \tag{2.2}$$

where C^*_{ij} is the tensor of effective (or overall) conductivity. The determination of this tensor is the central problem of micromechanics (the homogenization problem). For the random set of inclusions, the exact solution to this problem is impossible, and only approximate methods are available. There are several such approximate homogenization schemes that are named self-consistent methods [13]. The main distinguishing feature of these methods is the reduction of the problem for many randomly placed particles to the problem for only one separate particle (one-particle problem), that is the building block of these methods. In what follow, we consider a special kind of particle: the thin inclusion with high conductivity. Such inclusion can be used for example as a model of crack, filled with saltwater in a rock formation.

3. One-particle problem for a thin high-conductivity inclusion in a homogeneous medium

This problem has been solved by [16]. Here, we present this solution only briefly, the details can be seen in the mentioned publication. Let us examine a homogeneous isotropic medium having the conductivity coefficient C_0, containing a single inclusion with the conductivity coefficient C, occupying the region v. The fields This problem has been solved by [16]. Here, we present this solution only briefly, the details can be seen in the mentioned publication. Let us examine a homogeneous isotropic medium having the conductivity coefficient C_0, containing a single inclusion with the conductivity coefficient C, occupying the region v. The fields $q_i(x)$ and $e_i(x)$ satisfy the integral equations [16]:

$$e_i(x) = e^0_i(x) - (C - C_0)\int_v P_{ij}(x-x')e_J(x')dx',$$
$$q_i(x) = q^0_i(x) + \left(\frac{1}{C} - \frac{1}{C_0}\right)\int_v Q_{ij}(x-x')q_j(x')dx', \tag{3.1}$$

where it is denoted

$$P_{ij}(x) = \partial_i\partial_j\left(\frac{1}{4\pi C_0|x|}\right),$$
$$Q_{ij}(x) = C_0\left[C_0 P_{ij}(x) - \delta_{ij}\delta(x)\right], \tag{3.2}$$

$e^0_i(x)$ and $q^0_i(x)$ are the "external" fields that would be in the medium without inclusion; $\delta(x)$ is the 3D-Dirac-delta function.

We assume that one characteristic length h of the region v is smaller than two others of order l. Thus the ratio $\delta_1 = h/l$ is small. A thin inclusion with conductivity greater than that of the surrounding medium is of prime interest in application. In this case, the ratio $\delta_2 = C_0/C$ is also small. The most valuable information about the fields $e_i(x)$ and $q_i(x)$ in the vicinity of the inclusion is contained in the principal terms of the asymptotic expansion of these fields over the parameters δ_1 and δ_2. In order to construct these terms, it is necessary to find the limiting solution of the conductivity problem, when $\delta_1, \delta_2 \to 0$, and the ratio δ_1/δ_2 is of unit order and remains constant.

Let us assume that the middle surface of the inclusion Ω is a smooth enough surface with a given continuous field of its normal vector $n_i(x)$. The surface Ω bounded by the closed contour Γ. We take a point x on Ω and put it in the origin of the local coordinate system with We assume that one characteristic length h of the region v is smaller than two others of order l. Thus the ratio $\delta_1 = h/l$ is small. A thin inclusion with conductivity greater than that of the surrounding medium is of prime interest in application. In this case, the ratio $\delta_2 = C_0/C$ is also small. The most valuable information about the fields $e_i(x)$ and $q_i(x)$ in the vicinity of the inclusion is contained in the principal terms of the asymptotic expansion of these fields over the parameters δ_1 and δ_2. In order to construct these terms, it is necessary to find the limiting solution of the conductivity problem, when $\delta_1, \delta_2 \to 0$, and the ratio δ_1/δ_2 is of unit order and remains constant. Let us assume that the middle surface of the inclusion Ω is a smooth enough surface with a given continuous field of its normal vector We assume that one characteristic length h of the region v is smaller than two others of order l. Thus the ratio $\delta_1 = h/l$ is small. A thin inclusion with conductivity greater than that of the surrounding medium is of prime interest in application. In this case, the ratio $\delta_2 = C_0/C$ is also small. The most valuable information about the fields $e_i(x)$ and $q_i(x)$ in the vicinity of the inclusion is contained in the principal terms of the asymptotic expansion of these fields over the parameters δ_1 and δ_2. In order to construct these terms, it is necessary to find the limiting solution of the conductivity problem, when $\delta_1, \delta_2 \to 0$, and the ratio δ_1/δ_2 is of unit order and remains constant.

Let us assume that the middle surface of the inclusion Ω is a smooth enough surface with a given continuous field of its normal vector $n_i(x)$. The surface Ω bounded by the closed contour Γ. We take a point x on Ω and put it in the origin of the local coordinate system with z-axis directed along the normal $n_i(x)$. We denote by $h(x)$ the transverse size of the inclusion along the normal $n_i(x)$. Then, we denote $h(x) = \delta_1 l$ and take into account that δ_1/δ_2 is $O(1)$. It follows from Eqs. (3.1) that the main terms of the field expansion of $e_i(x)$ and $q_i(x)$ over δ_1 and δ_2 in the medium with thin high-conductivity inclusion can be expressed as:

$$e_i(x) = e_i^0(x) - \int_\Omega P_{ij}(x-x')\eta_j(x')dx',$$
$$q_i(x) = q_i^0(x) + \frac{1}{C_0}\int_\Omega Q_{ij}(x-x')\eta_j(x')dx', \qquad (3.3)$$

where

$$\eta_k(x) = (C - C_0)\int_{-h(x)/2}^{h(x)/2} e_k(x+n(x)z)dz, \qquad (3.4)$$

and it is necessary to consider the main terms of expansion over δ_1 and δ_2 in the expansion for $\eta_i(x)$. To construct these terms (for which we will preserve the same notation $\eta_i(x)$), we will use the method of matched asymptotic expansion. Using the results of [12], where one can find the details of the proof, it is possible to show that $\eta_i(x)$ is the vector of the surface Ω satisfying the equality:

$$\theta_{ij}(x)\eta_j(x) = \eta_i(x),$$
$$\theta_{ij}(x) = \delta_{ij} - n_i(x)n_j(x), \qquad (x \in \Omega). \qquad (3.5)$$

This vector is the solution of the following integral equation:

$$\mu_{ij}(x)\eta_j(x) - \int_\Omega U_{ij}(x-x')\eta_j(x')d\Omega' = \theta_{ij}(x)e_j^0(x), \qquad (3.6)$$

where it is denoted

$$\mu_{ij}(x) = \frac{1}{Ch(x)}\theta_{ij}(x),$$
$$U_{ij}(x) = \theta_{ik}(x)P_{kl}(x-x')\theta_{lj}(x') \qquad (3.7)$$

and the action of the operator with kernel $U_{ij}(x, x^{'})$ on a smooth enough function (regularization of this operator) can be obtained in Ref. [12].

Solving Eq. (3.6) for the function $\eta_i(x)$ and substituting the result in the right-hand side of Eqs. (3.3), we obtain expressions for the field $e_i(x)$ and $q_i(x)$. These fields approximate the real fields in the medium with inhomogeneity except for the small vicinity of the contour Γ. For an inclusion of arbitrary shape, Eq. (3.6) can be solved only numerically. But for ellipsoidal thin inclusion and constant "external" field e_i^0 this equation can be solved in the closed analytical form.

Let the inclusion be a thin ellipsoid with semi-axes $a_1, a_2, h, (h/a_1, h/a_2 \ll 1$. Then Ω becomes a plane elliptical surface. Let n_i denotes its normal. In the system of coordinates with the axes that coincide with the major ellipsoid axes, function $h(x)$ can be written as:

$$h(x) = 2hz(x),$$
$$z(x) = \sqrt{1 - \left(\frac{x_1}{a_1}\right)^2 - \left(\frac{x_2}{a_2}\right)^2}. \qquad (3.8)$$

In the results of calculations, the details of which are presented in Appendix A, we obtain that the fields $e_i(x)$ and $q_i(x)$ outside of the thin inclusion can be expressed as:

$$e_i(x) = e_i^0 - \int_\Omega P_{ij}(x-x')\Lambda_{jk}Z(x')dx' \cdot e_k^0,$$
$$q_i(x) = q_i^0 + \frac{1}{C_0}\int Q_{ij}(x-x,)\Lambda_{jk}Z(x')dx' \cdot e_k^0. \quad (3.9)$$

Here, $Z(x)$ is denoted

$$Z(x) = \frac{2a_1^2}{a_2}z(x), \qquad (3.10)$$

and tensor Λ_{ij} in same system of coordinates with unit vectors n_i^1, n_i^2, n_i^3 (n_i^3 is coincided with the normal n_i of the surface Ω) has the form:

$$\Lambda_{ij} = \Lambda_1(a_1, a_2)n_i^1 n_j^1 + \Lambda_2(a_1, a_2)n_i^2 n_j^2, \qquad (3.11)$$

where

$$\Lambda_1(a_1,a_2)=C_0\left[\frac{a_1\delta_2}{a_2\delta_1}+\frac{K(k)-E(k)}{k^2}\right],$$

$$\delta_1=\frac{h}{a_1},\qquad \delta_2=\frac{C_0}{C},$$

$$\Lambda_2(a_1,a_2)=C_0\left[\frac{a_1\delta_2}{a_2\delta_1}+\frac{E(k)-(1-k^2)K(k)}{(1-k^2)k^2}\right],$$

$$k^2=1-\left(\frac{a_2}{a_1}\right)^2,\qquad a_1\geqslant a_2, \tag{3.12}$$

$K(k)$ and $E(k)$ are the complete elliptic integrals of the first and the second kind, respectively:

$$K(k)=\int\limits_0^{\pi/2}\frac{dt}{\sqrt{1-k^2\sin^2 t}},$$

$$E(k)=\int\limits_0^{\pi/2}\sqrt{1-k^2\sin^2 t}dt. \tag{3.13}$$

4. Random set of thin low-resistivity inclusions in an isotropic homogeneous medium

We consider a microinhomogeneous medium consisting of a homogeneous isotropic host material (matrix) and random set of thin low-resistivity inclusions. The "one-particle" problem, considered in the previous Section, is basic for self-consistent homogenization schemes [11,13]. Here, we use one of these schemes that is named the effective field method (EFM). According to this method, we introduce a local "external" field $e_i^*(x)$ that acts on each inclusion. This field is composed of an "external" field e_i^0 and the fields induced by surrounding inclusions. The main hypothesis of the effective field method is as follows: every inclusion in the composite material can be considered as an isolated one in the homogeneous matrix in a local uniform "external" field e_i^*, which depends on the orientation of the inclusion (vector $\mathbf{n}$). Using this hypothesis, the expressions for the fields $e_i(x)$ and $q_i(x)$ can be represented in a form like Eqs. (3.3), in which:

$$\eta_i(x)=\Lambda_{ij}(x)e_j^*(n(x))Z(x)\Omega(x),$$

$$\Omega(x)=\sum_k\Omega_k(x). \tag{4.1}$$

Here, $\Omega_k(x)$ is a generalized function concentrated on the surface of the k-th inclusion, the function $n(x)$ coincides with the normal $\mathbf{n}$ to the surface Ω_k, when $x\in\Omega_k$. The function $\Lambda_{ij}(x)$ is equal to the constant value $\Lambda_{ij}(a_1^{(k)},a_2^{(k)})$ determined in (3.11), when $x\in\Omega_k$ and:

$$Z(x)=\frac{2(a_1^{(k)})^2}{a_2^{(k)}}\sqrt{1-\left(\frac{x_1}{a_1^{(k)}}\right)^2-\left(\frac{x_2}{a_2^{(k)}}\right)^2}. \tag{4.2}$$

The procedure of the homogenization by the effective field method is presented in Appendix B. The final result is:

$$C_{ij}^*=C_0\delta_{ij}+n_0\left[\delta_{ik}-n_0\left\langle v\Lambda_{im}(n)A_{mk}(n)\right\rangle\right]^{-1}\left\langle v\Lambda_{kj}(n)\right\rangle,$$

$$v=\frac{4}{3}\pi a_1^3, \tag{4.3}$$

where $\langle\cdot\rangle$ means the averaging over ensemble distribution of the sizes and orientation of the inclusions; n_0 is the number concentration of the inclusions. Tensor $A_{ij}(n)$ in Eq. (4.3) is determined by the relations:

$$A_{ij}(n)=A_1n_i^1n_j^1+A_2n_i^2n_j^2+A_3n_i^3n_j^3, \tag{4.4}$$

$$A_k=\frac{\alpha_1\alpha_2\alpha_3}{2C_0}\times\int\limits_0^\infty\frac{d\sigma}{(\alpha_k^2+\sigma)\sqrt{(\alpha_1^2+\sigma)(\alpha_2^2+\sigma)(\alpha_3^2+\sigma)}},$$

$$(k=1,2,3), \tag{4.5}$$

where $\alpha_1,\alpha_2,\alpha_3$ are the semi-axes of the ellipsoidal "correlation hole" (see Appendix B).

The cases of parallel and completely disoriented thin high-conductivity inclusions were considered by [16]. In what follows, our attention is concentrated on the effects of partly oriented inclusions.

5. Description of the inclusion orientation distribution

For the description of the thin inclusion orientation, we introduce a global Cartesian basis $e^i(i=1,2,3)$ of the axes x_1,x_2,x_3, where x_3 is the vertical axis (Fig. 1). The orientation of the basis $\mathbf{n}^k(k=1,2,3)$ that defines the symmetry axes of the elliptical thin inclusion with respect to the global

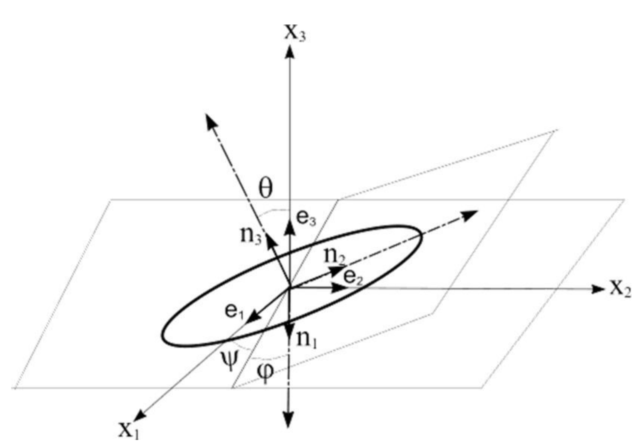

FIGURE 1. The global basis is $\mathbf{e}_i$, and the basis $\mathbf{n}_i$ defines the symmetry axes of the elliptical crack-like inclusion, (ψ,θ,φ) are the Euler angles.

basis e^i is described by the three Euler angles (ψ, θ, φ), and the connection between these bases is given by the relations:

$$\begin{aligned}
\mathbf{n}^1 &= (\cos\varphi\cos\psi - \sin\varphi\sin\psi\cos\theta)\mathbf{e}^1 \\
&\quad + (\cos\varphi\sin\psi + \sin\varphi\cos\psi\cos\theta)e^2 + \sin\varphi\sin\theta\mathbf{e}^3, \\
\mathbf{n}^2 &= -(\sin\varphi\cos\psi + \cos\varphi\sin\psi\cos\theta)\mathbf{e}^1 \\
&\quad + (-\sin\varphi\sin\psi + \cos\varphi\cos\psi\cos\theta)\mathbf{e}^2 + \cos\varphi\sin\theta\mathbf{e}^3, \\
\mathbf{n}^3 &= \sin\psi\sin\theta\mathbf{e}^1 - \cos\psi\sin\theta\mathbf{e}^2 + \cos\theta\mathbf{e}^3.
\end{aligned} \tag{5.1}$$

Substituting these formulas into (4.3), we obtain the presentation of the tensor C^*_{ij} in the global basis $\mathbf{e}^i$.

Let us consider some special cases.

Let the thin inclusions be the same size and shape, but randomly oriented with respect to the global coordinate system. In this case, the formula (4.3) takes the form:

$$\begin{aligned}
C^*_{ij} &= C_0\delta_{ij} + \tau\left[\delta_{ik} - \tau\left\langle \Lambda_{im}(n)A_{mk}(n)\right\rangle\right]^{-1}\left\langle \Lambda_{kj}(n)\right\rangle, \\
\tau &= \frac{4}{3}\pi a_1^3 n_0,
\end{aligned} \tag{5.2}$$

and the Euler angles in Eqs. (5.1) become random variables. The parameter τ in Eq. (5.2) is the so-called crack density.

We introduce the function of distribution $f(\psi, \theta, \varphi)$ of the inclusion orientation over the angles ψ, θ, φ. This function must satisfy the normalization conditions:

$$\frac{1}{8\pi^2}\int_0^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_0^{\pi} f(\psi, \varphi, \theta)\sin\theta d\theta = 1. \tag{5.3}$$

Suppose that the orientations of the inclusions, described by the angles ψ, θ, φ, are statistically independent. Therefore, the function $f(\psi, \theta, \varphi)$ can be represented as:

$$f(\psi, \theta, \varphi) = f_\psi(\psi)f_\varphi(\varphi)f_\theta(\theta). \tag{5.4}$$

The normalization conditions take the form:

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} f_\psi(\psi)d\psi = 1, \qquad \frac{1}{2\pi}\int_0^{2\pi} f_\varphi(\varphi)d\varphi = 1, \\
&\frac{1}{2}\int_0^{\pi} f_\theta(\theta)\sin\theta d\theta = 1.
\end{aligned} \tag{5.5}$$

We consider the case, when $f_\psi(\psi) = 1, \ f_\varphi(\varphi) = 1, \ f_\theta(\theta) \neq 1$. The medium has the macroscopically transversely isotropic conductive properties with symmetry axis x_3:

$$C^*_{ij} = C^*_1(e^1_i e^1_j + e^2_i e^2_j) + C^*_3 e^3_i e^3_j, \tag{5.6}$$

where it is denoted:

$$\begin{aligned}
C^*_1 &= C_0 + \frac{\tau(\Lambda_1 + \Lambda_2)(1 + S_\theta)}{4 - \tau(\Lambda_1 A_1 + \Lambda_2 A_2)(1 + S_\theta)}, \\
C^*_3 &= C_0 + \frac{\tau(\Lambda_1 + \Lambda_2)(1 - S_\theta)}{2 - \tau(\Lambda_1 A_1 + \Lambda_2 A_2)(1 - S_\theta)},
\end{aligned} \tag{5.7}$$

$$S_\theta = \frac{1}{2}\int_0^{\pi} f_\theta(\theta)\cos^2\theta\sin\theta d\theta. \tag{5.8}$$

The case $f_\theta(\theta) = 1$ (the uniform distribution over the angle θ) leads to $S_\theta = 1/3$, that corresponds to the full isotropy of the material:

$$\begin{aligned}
C^*_{ij} &= C^*\left(e^1_i e^1_j + e^2_i e^2_j + e^3_i e^3_j\right) = C^*\delta_{ij}, \\
C^* &= C_0 + \frac{\tau}{3}\left(\Lambda_1 + \Lambda_2\right)\left[1 - \frac{\tau}{3}\left(\Lambda_1 A_1 + \Lambda_2 A_2\right)\right]^{-1}
\end{aligned} \tag{5.9}$$

5.1. System of vertical inclusions

We consider the system of vertical inclusions $(\theta = \pi/2, \quad \varphi = 0)$. In this case

$$\begin{aligned}
&\mathbf{n}^1 = \cos\psi\mathbf{e}^1 + \sin\psi\mathbf{e}^2, \qquad \mathbf{n}^2 = \mathbf{e}^3, \\
&\mathbf{n}^3 = \sin\psi\mathbf{e}^1 + \cos\psi\mathbf{e}^2.
\end{aligned} \tag{5.10}$$

We choose one of the simplest distribution functions over the angle ψ:

$$f_\psi(\psi) = \begin{cases} 1 & \text{when} \quad \psi \in [-\beta, \beta] \\ 0 & \text{when} \quad \psi \notin [-\beta, \beta] \end{cases} \tag{5.11}$$

It means that the horizontal crack-like inclusions homogeneously distributed in the sector $[-\beta, \beta]$, where β is the disorientation angle. For $0 < \beta < \pi/2$ the conductive properties of such material have the orthorhombic symmetry with the following tensor of effective conductivity coefficients:

$$C^*_{ij} = C^*_1 e^1_i e^1_j + C^*_2 e^2_i e^2_j + C^*_3 e^3_i e^3_J, \tag{5.12}$$

where

$$\begin{aligned}
C^*_1 &= C_0 + \tau\Lambda_1 F(\beta)\left[1 - \tau\Lambda_1 A_1 F(\beta)\right]^{-1}, \\
C^*_2 &= C_0 + \tau\Lambda_1(1 - F(\beta))\left[1 - \tau\Lambda_1 A_1(1 - F(\beta))\right]^{-1}, \\
C^*_3 &= C_0 + \tau\Lambda_2\left(1 - \tau\Lambda_2 A_2\right)^{-1},
\end{aligned} \tag{5.13}$$

$$F(\beta) = \frac{1}{2\beta}(\beta + \sin\beta\cos\beta). \tag{5.14}$$

When $\beta \to 0$ (completely aligned vertical inclusions), $F(\beta) \to 1$, and the medium is still orthotropic. The tensor of effective conductivity coefficients is determined by the same formula (5.12), in which:

$$\begin{aligned}
&C^*_1 = C_0 + \tau\Lambda_1\left(1 - \tau\Lambda_1 A_1\right)^{-1}, \qquad C^*_2 = C_0, \\
&C^*_3 = C_0 + \tau\Lambda_2\left(1 - \tau\Lambda_2 A_2\right)^{-1}.
\end{aligned} \tag{5.15}$$

If $\beta \to \pi/2, F(\beta) \to 1/2$, and the medium becomes transversely isotropic with the symmetry axis x_3. Expression for the C_3^* in (5.12) remains the same, but:

$$C_1^* = C_2^* = C_0 + \frac{\tau}{2}\Lambda_1 \left(1 - \frac{\tau}{2}\Lambda_1 A_1\right)^{-1}. \tag{5.16}$$

Suppose that the inclusions have the same spheroidal shape: $a_1 = a_2 = a, a_3$. For such inclusions:

$$\Lambda_1 = \Lambda_2 = \Lambda = C_0 \left(\frac{\delta_2}{\delta_1} + \frac{\pi}{4}\right)^{-1} \tag{5.17}$$

If the shape of the correlation hole is also spheroidal ($\alpha_1 = \alpha_2 = \alpha, \alpha_3$) and coaxial to the inclusion, then:

$$A_1 = A_2 = A = \frac{g(\gamma)}{C_0},$$
$$A_3 = \frac{1}{2C_0}(1 - g(\gamma)), \qquad \gamma = \frac{a}{a_3} > 1,$$
$$g(\gamma) = \frac{\gamma^2}{\gamma^2 - 1}\left[1 - \frac{1}{\sqrt{\gamma^2 - 1}} \arctan \sqrt{\gamma^2 - 1}\right]. \tag{5.18}$$

For spheroidal inclusions and $\beta = 0$ (parallel inclusions), the medium is transversely isotropic with the symmetry axis x_2:

$$C_{ij}^* = C^*(e_i^1 e_j^1 + e_i^3 e_j^3) + C_0 e_i^2 e_j^2, \tag{5.19}$$

where

$$C^* = C_0 + \tau\Lambda \left(1 - \tau\Lambda A\right)^{-1}. \tag{5.20}$$

It has to be noted, that the angle distribution function can be chosen as a continuous one [18]:

$$f_\psi(\psi) = \frac{1}{2\pi}\exp(\cos\psi/\sigma^2), \tag{5.21}$$

where σ is the parameter, which characterizes the inclusion disorientation. With this distribution function the tensor of effective conductivity coefficients is determined by the same formula (5.12), in which:

$$C_1^* = C_0 + \tau\Lambda_1 F_1(\sigma) \left[1 - \tau\Lambda_1 A_1 F_1(\sigma)\right]^{-1},$$
$$C_2^* = C_0 + \tau\Lambda_1 F_2(\sigma) \left[1 - \tau\Lambda_1 A_1 F_2(\sigma)\right]^{-1},$$
$$C_3^* = C_0 + \tau\Lambda_2 \left(1 - \tau\Lambda_2 A_2\right)^{-1}. \tag{5.22}$$

Here, it is denoted

$$F_1(\sigma) = \frac{\sigma^2 I_1(1/\sigma^2) + I_2(1/\sigma^2)}{I_0(1/\sigma^2)},$$
$$F_2(\sigma) = \frac{\sigma^2 I_1(1/\sigma^2)}{I_0(1/\sigma^2)}. \tag{5.23}$$

where $I_n(z)$ is the modified Bessel function.

Note, that the limit $\sigma \to 0$ corresponds to the distribution function in the form:

$$f_\psi(\psi) = \delta(\psi), \tag{5.24}$$

i.e. the inclusions are parallel to the plane x_2x_3. In this case, $F_1(\sigma) = 1, \quad F_2(\sigma) = 0$, and Eqs. (5.22) are transformed to Eqs. (5.15). The other limit $\sigma \to \infty$ corresponds to the homogeneous distribution of inclusions with respect to the angle ψ. For this limit $F_1(\sigma) = F_2(\sigma) = 1/2$, the formula (5.16) is recovered from (5.22), *i.e.* the system becomes transversely isotropic with the symmetry axis x_3.

In a more general case, the distribution function can be represented in the form of series of spherical harmonics. The coefficients of such series can be calculated from the analysis of measuring data.

5.2. System of horizontal thin inclusions with high conductivity.

Now we consider the case, when $\theta = 0$ (horizontal inclusions). It is well known that in this case the Euler angles degenerate (the line of nudes coincides with x_1- axis, and the angles φ and ψ become uncertain). At the same time, it is possible to introduce the only one angle φ between the vectors $\mathbf{n}^1$ and $\mathbf{e}^1$ that determines the orientation of inclusions in the x_1x_2-plane. For this situation we have:

$$\mathbf{n}^1 = \cos\varphi \mathbf{e}^1 + \sin\varphi \mathbf{e}^2,$$
$$\mathbf{n}^2 = -\sin\varphi \mathbf{e}^1 + \cos\varphi \mathbf{e}^2, \qquad \mathbf{n}^3 = \mathbf{e}^3. \tag{5.25}$$

With the same distribution function (5.11) the tensor of the effective conductivity coefficients C_{ij}^* has the form:

$$C_{ij}^* = C_1^* e_i^1 e_j^1 + C_2^* e_i^2 e_J^2 + C_0 e_i^3 e_j^3. \tag{5.26}$$

One of these independent coefficients of conductivity coincides with those of the matrix in the direction of x_3-axis, and two other coefficients are determined by the expressions:

$$C_1^* = C_0 + \tau\left(\Lambda_1 F(\beta) + \Lambda_2(1 - F(\beta)\right)$$
$$\times \left[1 - \tau\left(\Lambda_1 A_1 F(\beta) + \Lambda_2 A_2 (1 - F(\beta)\right)\right]^{-1},$$
$$C_2^* = C_0 + \tau\left(\Lambda_1(1 - F(\beta)) + \Lambda_2 F(\beta)\right)$$
$$\times \left[1 - \tau\left(\Lambda A_1(1 - F(\beta)) + \Lambda_2 A_2 F(\beta)\right)\right]^{-1}. \tag{5.27}$$

where the function $F(\beta)$ is determined in Eq. (5.14).

When $\beta \to 0$ (the symmetry axes of the horizontal ellipsoidal inclusions are completely aligned), the effective medium remains orthorhombic with the effective conductivities (see formula (5.19)):

$$C_1^* = C_0 + \tau\Lambda_1 \left(1 - \tau\Lambda_1 A_1\right)^{-1},$$
$$C_2^* = C_0 + \tau\Lambda_2 \left(1 - \tau\Lambda_2 A_2\right)^{-1}. \tag{5.28}$$

The other limiting case $\beta \to \pi/2$ corresponds to the transversely isotropic material with the following tensor of effective coefficients of conductivity:

$$C_{ij}^* = C^*\left(e_i^1 e_j^1 + e_i^2 e_j^2\right) + C_0 e_i^3 e_j^3, \tag{5.29}$$

where

$$C^* = C_0 + \frac{\tau}{2}(\Lambda_1 + \Lambda_2)\left[1 - \frac{\tau}{2}(\Lambda_1 A_1 + \Lambda_2 A_2)\right]^{-1}. \quad (5.30)$$

If we choose the distribution function in the form (5.21), the formula for the tensor of the effective conductivity coefficients remains the same (5.26), but in this case:

$$C_1^* = C_0 + \tau(\Lambda_1 F_1(\sigma) + \Lambda_2 F_2(\sigma)) \times [1 - \tau(\Lambda_1 A_1 F_1(\sigma) + \Lambda_2 A_2 F_2(\sigma))]^{-1},$$
$$C_2^* = C_0 + \tau(\Lambda_1 F_2(\sigma) + \Lambda_2 F_1(\sigma)) \times [1 - \tau(\Lambda_1 A_1 F_2(\sigma) + \Lambda_2 A_2 F_1(\sigma))]^{-1}. \quad (5.31)$$

In both limit cases $\sigma \to 0$ and $\sigma \to \infty$, the Eqs. (5.28)-(5.30) are recovered from Eqs. (5.31).

If the inclusions are identical spheroids, the medium with such inclusions is transversely isotropic (5.29). In this formula:

$$C^* = C_0 + \tau\Lambda(1 - \tau\Lambda A)^{-1}, \quad (5.32)$$

independently on the choice of distribution functions (Λ and A are determined in Eqs. (5.17) and (5.18)).

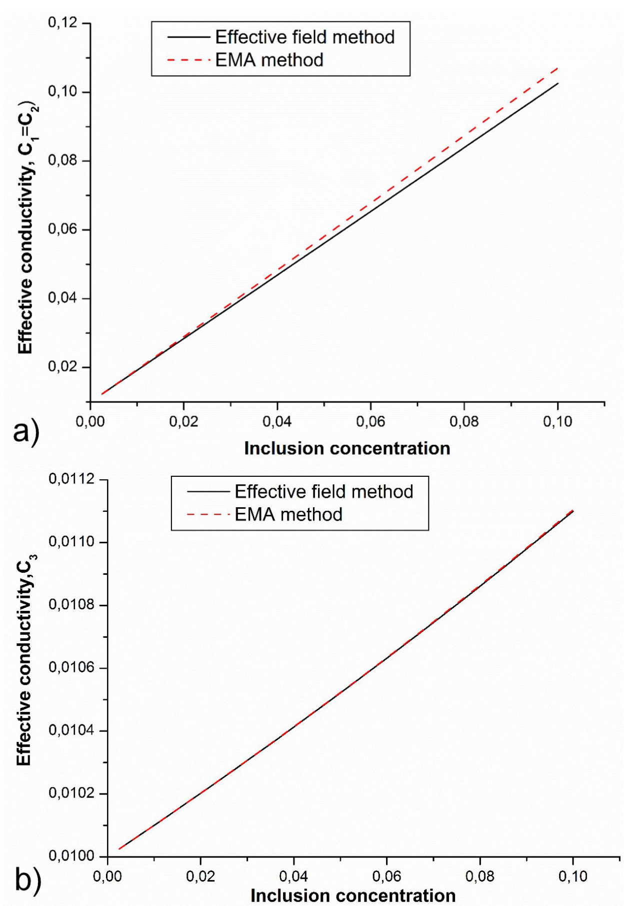

FIGURE 2. Effective conductivity as a function of the inclusion concentration. The calculations are presented for parallel spheroidal inclusions.

6. Numerical examples

To validate our model, we calculated the effective conductivity of the medium, which contains parallel spheroidal inclusions, using the effective field method and the well-known Effective Medium Approximation (EMA) by [3,4,5]. The semi-axes of the inclusions were chosen as $a_1 = a_2 = 1$, $a_3 = 0.001$. The inclusions conductivity is equal to 1, and the matrix conductivity is equal to 0.01. To compare the results obtained with the EMA model and the effective field method, we use the inclusion concentration Φ instead of the crack density τ. For spheroidal inclusions $\tau = 3\Phi/(4\pi\alpha)$, where α is the aspect ratio of the spheroid. The comparison of the results obtained shows that both methods give close results for parallel inclusions (Fig. 2), but to calculate the effective conductivity of the material with partially oriented inclusions by the EMA method is a complicated mathematical problem,while the (EFM) gives the explicit expressions for the effective conductivity coefficients.

Further, we present examples of the effective conductivity calculation for partially oriented inclusions. We assume that the inclusion conductivity is equal to 1, and the matrix conductivity is equal to 0.01. The semi-axes of the inclusions were chosen as $a_1 = 1$, $a_2 = 0.5$, $a_3 = 0.01$. Such parameters are typical, for example, for sedimentary carbonate rocks containing cracks, filled with conductive formation water [1]. We assume that the correlation hole has a spherical shape: $A_1 = A_2 = A_3 = 1$. Figures 3 and 4 show the dependences of the conductivity tensor components of the horizontal and vertical crack-like inclusions, homogeneously distributed in the sector $[-\beta, \beta]$, where β is the disorientation angle. The angle distribution function is described by (5.11). The calcu-

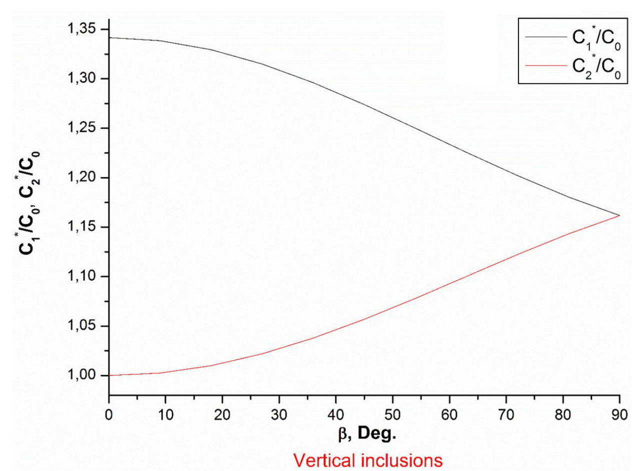

FIGURE 3. Normalized components of the conductivity tensor as a function of the disorientation angle β. The crack density $\tau = 1$. The results are presented for the system of vertical inclusions ($\theta = \pi/2$, $\varphi = 0$) and the first model of the angular distribution (inclusions homogeneously distributed in the sector $[-\beta, \beta]$).

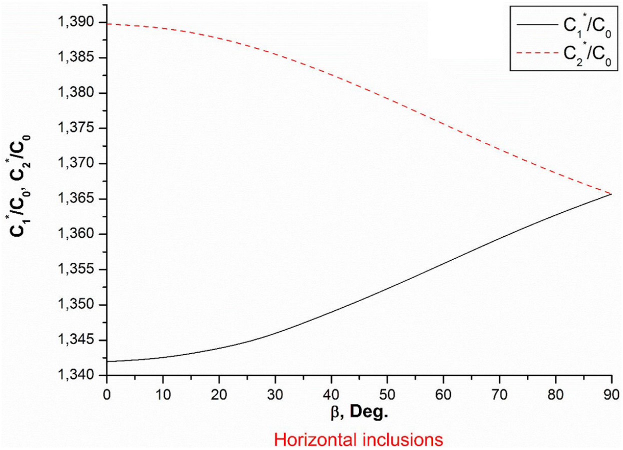

FIGURE 4. Normalized components of the conductivity tensor as a function of the disorientation angle β. The crack density $\tau = 1$. The results are presented for the system of horizontal inclusions ($\theta = 0, \quad \varphi = 0$) and the first model of the angular distribution (inclusions homogeneously distributed in the sector $[-\beta, \beta]$).

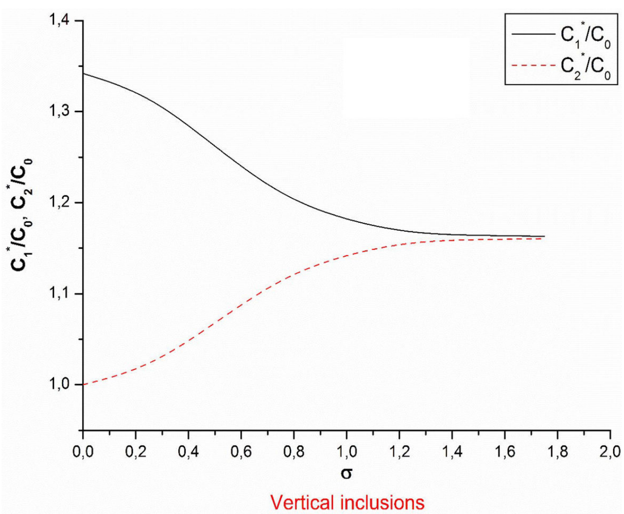

FIGURE 5. Normalized components of the conductivity tensor as a function of the disorientation parameter σ. The crack density $\tau = 1$. The results are presented for the system of vertical inclusions ($\theta = \pi/2, \quad \varphi = 0$) and the second model of the angular distribution of the inclusions (Eq.(5.21)).

lations were fulfilled for the disorientation angle $\theta = \pi/2$ and $\theta = 0 (\varphi = 0)$ for vertical and horizontal inclusions, respectively. The aspect ratio of the correlation hole is close to 1. As expected, the effective medium is orthorhombic ($C_1^* \neq C_2^* \neq C_3^*$), but with increasing of the value of the parameter β the difference between the conductivities C_1^* and C_2^* decreases. When the β value is close to $\pi/2$, the difference between $C_1^*$and C_2^* tends to zero, and the medium becomes transversely isotropic (Figs. 3 and 4).

Figures 5-6 show the dependences of the normalized components of the conductivity tensor as a function of the disorientation parameter σ (the distribution function (5.21). The results are presented for vertical and horizontal crack-like inclusions. In the general case, the effective medium is

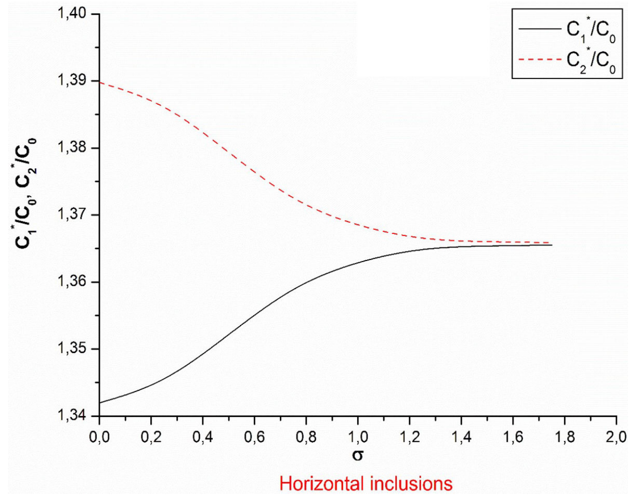

FIGURE 6. Normalized components of the conductivity tensor as a function of the disorientation parameter σ. The crack density $\tau = 1$. The results are presented for the system of vertical inclusions ($\theta = 0, \quad \varphi = 0$) and the second model of the angular distribution of the inclusions (Eq.(5.21)).

orthorhombic, but in the case, when $\sigma \rightarrow \infty$, the effective medium becomes transversely isotropic.

Figure 7 illustrates the dependences of the normalized components of the conductivity tensor as a function of the disorientation angle β for different crack densities. The results are presented for vertical inclusions. In the general case, the effective medium is orthorhombic, but in the case, when $\beta = \pi/2$, the effective medium becomes transversely isotropic.

7. Conclusion

We have presented an approach for calculating the effective conductivity tensor of material containing a system of cracks that are not strictly oriented. Their spatial distribution is described by some angular function. The approach is based on the effective field method. This method is sufficiently general and contains the well-known method of Mori-Tanaka and the Maxwell method as particular cases. The advantage of the effective field method lies in its simple numerical realization compared with other methods. The application of this method permits us to take into account the texture of a microinhomogeneous medium. The general theory was illustrated by numerical results obtained for crack-like inclusions homogeneously distributed in the sector. We have shown that in the case of circular penny-shaped inclusions with the same aspect ratio, the effective medium can be orthorhombic, transversely isotropic, or isotropic depending on the choice of the distribution function of the normal to the crack surface from Euler angles. From our point of view, the dependences, obtained in the framework of this work, may be interesting for many areas of applications, including the rock physics and physics of micro-inhomogeneous (cracked) materials.

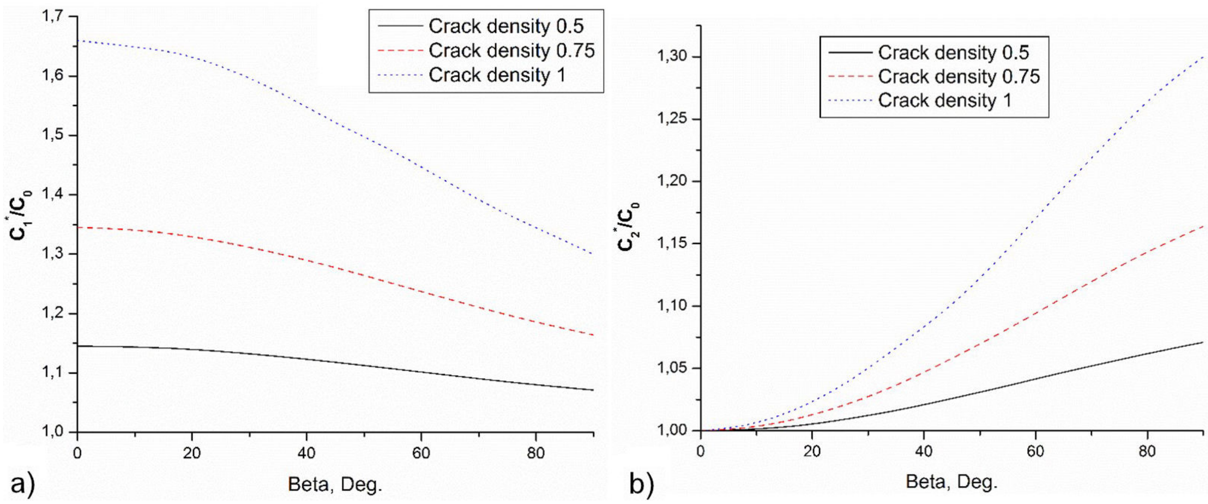

FIGURE 7. Normalized components of the conductivity tensor as a function of the disorientation angle β. Different curves correspond to the different values of crack densities. Figure 7A corresponds to the conductivity in the direction x_1, and Figure 7B corresponds to the conductivity in the direction x_2. The results are presented for the first model of angular distribution (inclusions homogeneously distributed in the sector $[-\beta, \beta]$).

Appendix A.

The kernel of the integral operator in Eq. (3.6) has the form:

$$U_{ij}(x, x')=U_{ij}(x-x')=\theta_{ik}(n)P_{kl}(x-x')\theta_{lj}(n). \quad \text{(A.1)}$$

It can be shown [13] that integral operator with such a kernel transforms the function $z(x)$ into a constant on Ω. It allows us to find the solution of Eq. (3.5) for the elliptic domain Ω with $e_i^0 = \text{const}$ in the form of the constant vector multiplying on $z(x)$:

$$\eta_i(x) = \eta_i z(x). \quad \text{(A.2)}$$

Substituting (A.2) into Eq. (3.5) and taking into account (A.1), we obtain:

$$\eta_i = \frac{2a_1^2}{a_2}\left(\mu_{ik}^0 + U_{ik}^0\right)^{-1} e_k^0, \qquad \mu_{ik}^0 = \frac{1}{2hC}\theta_{ik}, \quad \text{(A.3)}$$

and the constant tensor U_{ik}^0 is expressed in term of absolutely converging integral:

$$U_{ij}^0 = \theta_{ik}(n)P_{kl}^0\theta_{lj},$$
$$P_{kl}^0 = \int P_{kl}(x)[z(x) - 1]d\Omega. \quad \text{(A.4)}$$

Here, the integration is over the plane x_1x_2, and the function $z(x)$ vanishes outside Ω.

The explicit expression for $P_{ij}(x)$ is:

$$P_{kl}(x) = -\frac{1}{4\pi C_0|x|^3}\left(\delta_{kl} - \frac{3x_kx_l}{|x|^2}\right). \quad \text{(A.5)}$$

Substituting this expression into (A.4) and introducing the coordinates r and φ in the plane x_1x_2: $x_1 = ra_1\cos\varphi$, $x_2 = ra_2\sin\varphi$ (a_1, a_2 are the ellipse Ω semi-axes), we obtain

$$P_{kl}^0 = -\frac{a_1a_2}{4\pi C_0}\int_0^\infty \frac{z(r)-1}{r^2}$$
$$\times \int_0^{2\pi}\left[\frac{3m_{kl}(\varphi)}{t^2(\varphi)} - \delta_{kl}\right]\frac{d\varphi}{t^3(\varphi)}, \quad (k, l = 1, 2) \quad \text{(A.7)}$$

$$t^2(\varphi) = a_1^2\cos^2\varphi + a_2^2\sin^2\varphi, \quad z(r) = \sqrt{1-r^2},$$
$$(r \le 1), \quad z(r) = 0, \quad (r > 1)$$
$$m_{11} = a_1^2\cos^2\varphi, \quad m_{22} = a_2^2\sin^2\varphi,$$
$$m_{12} = m_{21} = a_1a_2\sin\varphi\cos\varphi.$$

Calculation of the integral in (A.7) gives

$$U_{11}^0 = \frac{a_2}{2a_1^2C_0}\frac{K(k) - E(k)}{k^2},$$
$$k^2 = 1 - \left(\frac{a_2}{a_1}\right), \quad a_1 \ge a_2,$$
$$U_{12} = 0, \quad U_{22}^0 = \frac{a_2}{2a_1^2C_0}\frac{E(k) - (1-k^2)K(k)}{k^2(1-k^2)}. \quad \text{(A.8)}$$

After the vector $\eta_i(x)$ has been determined, the function $e_i(x)$ and $q_i(x)$ outside the inclusion can be expressed by Eqs. (3.8)-(3.9) of the main text, and tensor Λ_{ij} is determined by Eqs. (3.11)-(3.12).

Appendix B

If we introduce the function:

$$\Omega(x;x') = \sum_{i\neq j}\Omega_i(x'), \quad \text{when } x \in \Omega_j. \tag{B.1}$$

The equation for the local "external" field at the point x located on the middle surface of an arbitrary inclusion can be presented in the following form:

$$e_i^*(x) = e_i^0 - \int P_{ij}(x-x')\Lambda_{jk}(x') \times e_k^*(x')Z(x')\Omega(x;x')dx', \quad x \in \Omega. \tag{B.2}$$

Let us average this equation under the condition that the point x is located on the middle surface of the inclusion with normal $\mathbf{n}$. This averaging is denoted as $\langle\cdot|x,n\rangle$. If the mean $\langle e_i^*(x)|x,n\rangle$ is identified with an effective field acting on the inclusion of orientation $\mathbf{n}$:

$$\langle e_i^*(x)|x,n\rangle = e_i^*(n), \tag{B.3}$$

then we obtain from (B.2)

$$e_i^*(x) = e_i^0 - \int P_{ij}(x-x')\langle\Lambda_{jk}(x') \times e_k^*(x')Z(x')\Omega(x;x')|x,n\rangle dx'. \tag{B.4}$$

Assuming that the conductivity properties of inclusions are statistically independent on their spatial position, one can find the expression for the mean under the integral in (B.4):

$$\begin{aligned}\langle\Lambda_{jk}(x')e_k^*(x')Z(x')\Omega(x;x')|x,n\rangle &= \langle\Lambda_{ij}(n)e_j^*(n)\rangle\Psi_n(x-x'),\\ \Lambda_{ij}(n) &= \langle Z(x)\Omega(x)\Lambda_{ij}(x)\rangle,\\ \Psi_n &= \frac{\langle\Omega(x;x')x,n\rangle}{\langle\Omega(x)\rangle}.\end{aligned} \tag{B.5}$$

The mean $\left\langle\Lambda_{ij}(n)e_j^*(n)\right\rangle$ is calculated over the ensemble of inclusion distribution by orientation. The function $\Psi_n(x)$ characterizes the spatial correlation of the random set of thin inclusions. It follows from definition of the function $\Omega(x;x')$ that

$$\Psi_n(0) = 0, \quad \Psi_n(x) \to 1, \text{ when } |x| \to \infty. \tag{B.6}$$

This function defines the shape of the so-called "correlation hole" – the region in the vicinity of each inclusion inside which the existence of the center of some other inclusion is improbable. Let us assume that there exists a linear transformation of x-space rearrange the function $\Psi_n(x)$ into a spatially symmetric one:

$$y_i = \alpha_{ij}(n) = \Psi(|y|). \tag{B.7}$$

In this case, the ellipsoid A with semi-axes $\alpha_1, \alpha_2, \alpha_3$, defined by the equation:

$$|\alpha_{ij}(n)x_j| \leq 1, \tag{B.8}$$

describes the shape of the correlation hole.

After the substitution (B.5) in Eq. (B.4), one can obtain an expression for $e_i^*(n)$ in the form:

$$e_i^*(n) = e_i^0 + A_{ik}(n)\left\langle\Lambda_{kj}(n)e_j^*(n)\right\rangle, \tag{B.9}$$

where it is denoted:

$$A_{ij}(n) = \int P_{ij}(x)[1-\Psi_n(x)]dx. \tag{B.10}$$

If the correlation hole is an ellipsoid, coaxial with the inclusion having the orientation $\mathbf{n}$, then $A_{ij}(n)$ is defined by the formulas (4.4) of the main text.

Let us multiply both sides of Eq. (B.9) by the tensor $\Lambda_{ij}(n)$ and average the result over the ensemble of random sizes and orientation of the inclusions. Solving the obtained equation for the vector $\left\langle\Lambda_{kj}(n)e_j^*(n)\right\rangle$, we have:

$$\left\langle\Lambda_{kj}(n)e_j^*(n)\right\rangle = [\delta_{il} - \langle\Lambda_{ik}(n)A_{kl}(n)\rangle]^{-1} \times \langle\Lambda_{lj}(n)\rangle\, e_j^0. \tag{B.11}$$

The expression for the effective field $e_i^*(n)$ can be found, if we substitute $\left\langle\Lambda_{ij}(n)e_j^*(n)\right\rangle$ from (B.11) into the right-hand side of Eq. (B.9).

Let us average the Eqs. (4.1) over the ensemble of the random set of the inclusions. Taking into account the relation:

$$\langle\Lambda_{ij}(x)e_i^*(x)Z(x)\Omega(x)\rangle = \left\langle\Lambda_{ij}(n)e_j^*(n)\right\rangle, \tag{B.12}$$

we obtain:

$$\begin{aligned}\langle e_i\rangle &= e_i^0 - \int P_{ij}(x-x')\langle\Lambda_{jk}(n)e_k^*(n)\rangle dx',\\ \langle q_i\rangle &= q_i^0 + \frac{1}{C_0}\int Q_{ij}(x-x')\langle\Lambda_{jk}(n)e_k^*(n)\rangle dx'.\end{aligned} \tag{B.13}$$

Because the "external" field e_i^0 is fixed in the homogenization problem [13], Eqs. (B.13) yield:

$$\langle e_i\rangle = e_i^0, \quad \langle q_i\rangle = C_{ij}^*\langle e_j\rangle, \tag{B.14}$$

where

$$C_{ij}^* = C_0\delta_{ik} + [\delta_{ik} - \langle\Lambda_{il}(n)A_{lk}(n)\rangle]^{-1}\langle\Lambda_{kj}(n)\rangle, \tag{B.15}$$

is the tensor of the effective conductivity coefficients of the composite material with a random set of thin high conductive inclusions.

Note that the Mori-Tanaka approach [2,21] gives the same (B.15) expression for the effective conductive coefficients only if the shape of the correlation hole coincides with the shape of the typical inclusion. In the general case, this shape can be different [16].

The ergodic properties of the functions considered allow replacing the ensemble averaging over random set of inclusions by the volume averages over the fixed realization of this set, so that ($a_1 \geq a_2$):

$$\langle \Lambda_{ij}(n) \rangle = \lim_{W \to \infty} \frac{1}{W} \int\limits_{W} \Lambda_{ij}(x) Z(x) \Omega(x) dx = n_0 \langle v \Lambda_{ij}(a_1, a_2) \rangle , \tag{B.16}$$

with analogous determination of the average $\langle \Lambda_{ik}(n) A_{kj}(n) \rangle$.

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