The effect of deformation of special relativity by conformable derivative

Ahmed Al-Jamel[a,*], Mohamed Al-Masaeed[a], Eqab. M. Rabei[a], and Dumitru Baleanu[b,c,d]

[a] *Physics Department, Faculty of Science,*
Al al-Bayt University, P.O. Box 130040, Mafraq 25113, Jordan.
[*] *e-mail: aaljamel@aabu.edu.jo; aaljamel@gmail.com*
[b] *Department of Mathematics, Cankaya University, Ankara, Turkey.*
[c] *Institute of Space Sciences, Magurele-Bucharest, Romania.*
[d] *Department of Medical Research, China Medical University Hospital,*
China Medical University,Taichung, Taiwan.

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In this paper, the deformation of special relativity within the frame of conformable derivative is formulated. Within this context, the two postulates of the theory are re-stated. Then, the addition of velocity laws are derived and used to verify the constancy of the speed of light. The invariance principle of the laws of physics is demonstrated for some typical illustrative examples, namely, the conformable wave equation, the conformable Schrodinger equation, the conformable Klein-Gordon equation, and conformable Dirac equation. The current formalism may be applicable when using special relativity in a nonlinear or dispersive medium.

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1. Introduction

The Einstein's special relativity plays a corner stone in modern physics. As stated in 1905 by Einstein, it is based on two postulates. The first postulate is about the constancy of the speed of light: the speed of light c is the same in all inertial frames of references. The second postulate is about the invariance form of the laws of physics under Lorentz transformations. As a consequence of this, any theory of space and time should be compatible with the theory of special relativity. There are some other aspects that were studied after the emergence of theory of relativity [1,2]. In Ref. [3], the Lorentz transformations were re-stated for an observer in a refracting but non-dispersive medium was proposed, and some physical consequences were discussed. In Ref. [4], Laue and Rosen theories of dielectric special relativity were derived, and argued that both are true but with different range of applicability. In Ref. [5], the non-local special relativity is introduced to overcome the difficulties accompanied the non-local electrodynamics problems.

In the last two decades, the fractional calculus approach to model or resolve various physical problems has attracted many researchers. There are a number of definitions or senses for fractional calculus such as Riemann-Liouville, Caputo, Riesz and Weyl [6-9]. The most important definitions are the Riemann-Liouville and Caputo definitions. These definitions have many applications in various fields [10-17]. The fractional derivative has lately been given a new definition. This is the first definition to use the limits definition, and it is called conformable fractional derivative (CFD) [18]. For a given function $f(t) \in [0,\infty) \rightarrow R$, the conformable derivative of $f(t)$ of order α, denoted as $D_t^\alpha f(t)$ with $0<\alpha\leq 1$, is defined as [18]:

$$D_t^\alpha f(t) = \lim_{\epsilon\rightarrow 0}\frac{f(t+\epsilon t^{1-\alpha})-f(t)}{\epsilon} = t^{1-\alpha}\frac{d}{dt}f(t). \quad (1)$$

This definition is simple in the sense that it meets the general properties and rules of the traditional derivative, whereas the other fractional derivatives do not satisfy them. From these properties the Leibniz, chain rules, and derivative of the quotient of two functions. Because of its ease of use, general features, and preservation of general properties including the locality property, the conformable derivative has a wide range of applications in a variety of fields of science.

In Refs. [19,20], this CFD is re-investigated and new properties similar to these in traditional calculus were derived and discussed. The CFD has been used to study various physical problems with possible nonlinear or diffusive nature. In Ref. [21], the mass spectroscopy of heavy mesons were investigated within the frame of conformable derivative searching for any ordering effect in their spectra that varies with the fractional order. In Ref. [22], the fractional dynamics of relativistic particles was studied, and it was found that fractional dynamics of such particles are described as non-Hamiltonian and dissipative. Possibility of being Hamiltonian system under some conditions was also presented. In Ref. [23], a new conformable fractional mechanics using the fractional addition was proposed and new definitions for the fractional velocity fractional acceleration are given. In Ref. [24], deformation of quantum mechanics due to the inclusion of conformable fractional derivative is presented and investigated with some physical illustrative examples. Recently, Pawar *et.al.* [25] introduced Riemannian geometry through using the conformable fractional derivative in Christoffel index symbols of the first and second kind. The conformable calculus has been used in making an extension of approximation methods to become applicable to conformable quantum mechanics [26-28], and to find solutions of related differential equations such as the conformable Laguerre and associated Laguerre equations [29]. In Ref. [30], the Hamiltonian for the conformable harmonic oscillator is constructed using fractional operators termed α-creation and α-annihilation operators.

Later, in Ref. [31], pointed out the conformable derivative is not fractional but it is an operator. Thus in the present paper we call it conformable derivative.

The purpose of this paper is to investigate the deformation of the theory of special relativity within the frame of conformable fractional derivative. This means that, we will adopt a new set of $\alpha-$Lorentz transformations and use them to re-state the postulates of special relativity, and to verify the validity of the invariance principle to various laws or equations of physics.

2. Theory

Deformation of Lorentz transformations using conformable derivative is reported in Ref. [24].

Definition The $\alpha-$ Lorentz transformations between two inertial frames S and S' are defined as [24]:

$$x'^{\alpha} = \Gamma_{\alpha}(x^{\alpha} - v_{\alpha}t^{\alpha}), \tag{2}$$

$$t'^{\alpha} = \Gamma_{\alpha}(t^{\alpha} - \frac{v_{\alpha}}{c^{2\alpha}}x^{\alpha}), \tag{3}$$

$$y'^{\alpha} = y^{\alpha}, \tag{4}$$

$$z'^{\alpha} = z^{\alpha}, \tag{5}$$

where $\Gamma_{\alpha} = 1/\sqrt{1-(v_{\alpha}^{2}/c^{2\alpha})}$ is the $\alpha-$ deformed Lorentz factor and v_{α} is the $\alpha-$relative velocity between the two frames.

By adjusting the α values, we can see that the influence of α on the α-Lorentz factor has kept its behavior with the gradient of its value and that this effect fades when $\alpha = 1$.

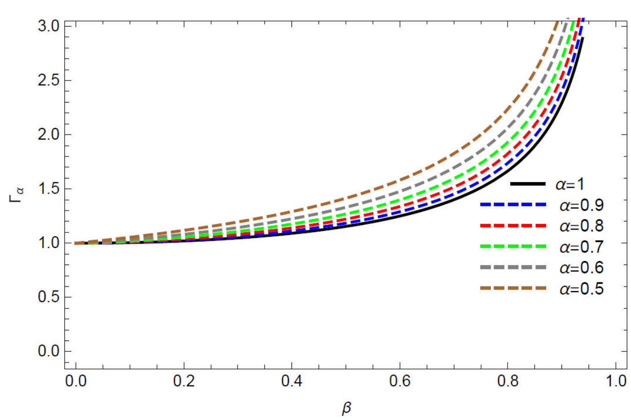

FIGURE 1. Plot of the relation between α-Lorentz factor and β, where $v_{\alpha} = \beta^{\alpha}c^{\alpha}$.

We now state the two postulates of conformable special relativity as follows.

- Postulate 1(Constancy of the speed of light): The speed of light is the same for all $\alpha-$inertial frames of references.
- Postulate 2(Invariance Principle): The laws of physics are invariant under $\alpha-$Lorentz transformations.

The following subsections purpose is to clarify theses two postulates.

2.1. The $\alpha-$velocity addition law

Following [23], we define the $\alpha-$velocity of an event with respect to the S and S' frames as

$$u_{\alpha} \equiv D_{t}^{\alpha}x^{\alpha} = (\frac{t}{x})^{1-\alpha}\frac{dx}{dt}, \tag{6}$$

$$u'_{\alpha} \equiv D_{t'}^{\alpha}x'^{\alpha} = (\frac{t'}{x'})^{1-\alpha}\frac{dx'}{dt'}, \tag{7}$$

respectively. To calculate the velocity using Eqs. (2) and (3), we have

$$\frac{dx'^{\alpha}}{dt'^{\alpha}} = \frac{\Gamma_{\alpha}(dx^{\alpha} - v_{\alpha}dt^{\alpha})}{\Gamma_{\alpha}(dt^{\alpha} - \frac{v_{\alpha}}{c^{2\alpha}}dx^{\alpha})} = \frac{(\frac{dx^{\alpha}}{dt^{\alpha}} - v_{\alpha})}{(1 - \frac{v_{\alpha}}{c^{2\alpha}}\frac{dx^{\alpha})}{dt^{\alpha}}}.$$

By interpreting $dx'^{\alpha}/dt'^{\alpha} = u'_{\alpha}$ and $dx^{\alpha}/dt^{\alpha} = u_{\alpha}$, we thus obtain

$$u'_{\alpha} = \frac{(u_{\alpha} - v_{\alpha})}{(1 - \frac{v_{\alpha}}{c^{2\alpha}}u_{\alpha})}. \tag{8}$$

In case $u_x = c$, we have

$$\begin{aligned}\left(\frac{x'}{t'}\right)^{\alpha-1}u'_{x} &= \frac{((\frac{x}{t})^{\alpha-1}u_{x} - v_{\alpha})}{(1 - \frac{v_{\alpha}}{c^{2\alpha}}(\frac{x}{t})^{\alpha-1}u_{x})} \\ &= \frac{((\frac{x}{t})^{\alpha-1}c - v_{\alpha})}{(1 - \frac{v_{\alpha}}{c^{2\alpha}}(\frac{x}{t})^{\alpha-1}c)},\end{aligned} \tag{9}$$

where we have made use of Eqs. (6) and (7). With the realization $x/t = c$ and $x'/t' = c$, we have

$$\begin{aligned}c^{\alpha-1}u'_{x} &= \frac{(c^{\alpha-1}c - v_{\alpha})}{(1 - \frac{v_{\alpha}}{c^{2\alpha}}c^{\alpha-1}c)} \\ &= \frac{(c^{\alpha} - v_{\alpha})}{(1 - \frac{v_{\alpha}}{c^{\alpha}})} = c^{\alpha}\frac{(c^{\alpha} - v_{\alpha})}{(c^{\alpha} - v_{\alpha})},\end{aligned} \tag{10}$$

from which we obtain

$$c^{\alpha-1}u'_{x} = c^{\alpha} \rightarrow u'_{x} = c^{1-\alpha}c^{\alpha} = c, \tag{11}$$

or

$$u'_{x} = c. \tag{12}$$

This verifies that the $\alpha-$Lorentz transformations proposed in Eqs. (2-5) leads to the constancy of the speed of light.

2.2. Conformable wave equation

Here, we verify the covariance of the wave equation under the $\alpha-$ Lorentz transformation. The $\alpha-$ wave equation in $1+1$ dimension is Ref. [24]

$$D_x^\alpha D_x^\alpha \Psi - \frac{1}{c^{2\alpha}} D_t^\alpha D_t^\alpha \Psi = 0. \tag{13}$$

Using the $\alpha-$Laplacian [32], then

$$\nabla^{2\alpha}\Psi - \frac{1}{c^{2\alpha}} D_t^\alpha D_t^\alpha \Psi = 0, \tag{14}$$

where $\nabla^{2\alpha} = D_x^\alpha D_x^\alpha + D_y^\alpha D_y^\alpha + D_z^\alpha D_z^\alpha$. Using of the chain rule [20]

$$D_x^\alpha \Psi = x'^{\alpha-1} D_x^\alpha x' D_{x'}^\alpha \Psi + t'^{\alpha-1} D_x^\alpha t' D_{t'}^\alpha \Psi,$$

and then using the $\alpha-$Lorentz transformations Eqs. (2) and (3), $x' = \Gamma_\alpha^{(1/\alpha)}(x^\alpha - v_\alpha t^\alpha)^{(1/\alpha)}$, $t' = \Gamma_\alpha^{(1/\alpha)}(t^\alpha - [v_\alpha/c^{2\alpha}]x^\alpha)^{(1/\alpha)}$, we have

$$\begin{aligned}
D_x^\alpha \Psi &= (\Gamma_\alpha^{\frac{1}{\alpha}}(x^\alpha - v_\alpha t^\alpha)^{\frac{1}{\alpha}})^{\alpha-1} x^{1-\alpha} \frac{d}{dx} \Gamma_\alpha^{\frac{1}{\alpha}}(x^\alpha - v_\alpha t^\alpha)^{\frac{1}{\alpha}} D_{x'}^\alpha \Psi \\
&+ (\Gamma_\alpha^{\frac{1}{\alpha}}(t^\alpha - \frac{v_\alpha}{c^{2\alpha}} x^\alpha)^{\frac{1}{\alpha}})^{\alpha-1} x^{1-\alpha} \frac{d}{dx} \Gamma_\alpha^{\frac{1}{\alpha}}(t^\alpha - \frac{v_\alpha}{c^{2\alpha}} x^\alpha)^{\frac{1}{\alpha}} D_{t'}^\alpha \Psi, \\
&= \Gamma_\alpha^{1-\frac{1}{\alpha}}(x^\alpha - v_\alpha t^\alpha)^{1-\frac{1}{\alpha}} x^{1-\alpha} \Gamma_\alpha^{\frac{1}{\alpha}} \frac{1}{\alpha}(x^\alpha - v_\alpha t^\alpha)^{\frac{1}{\alpha}-1} \alpha x^{\alpha-1} D_{x'}^\alpha \Psi \\
&- \Gamma_\alpha^{1-\frac{1}{\alpha}}(t^\alpha - \frac{v_\alpha}{c^{2\alpha}} x^\alpha)^{1-\frac{1}{\alpha}} x^{1-\alpha} \Gamma_\alpha^{\frac{1}{\alpha}} \frac{1}{\alpha}(t^\alpha - \frac{v_\alpha}{c^{2\alpha}} x^\alpha)^{\frac{1}{\alpha}-1} \frac{v_\alpha}{c^{2\alpha}} \alpha x^{\alpha-1} D_{t'}^\alpha \Psi \\
&= \Gamma_\alpha D_{x'}^\alpha \Psi - \Gamma_\alpha \frac{v_\alpha}{c^{2\alpha}} D_{t'}^\alpha \Psi.
\end{aligned} \tag{15}$$

Operating again on $D_x^\alpha \Psi$ by D_x^α, yields

$$\begin{aligned}
D_x^\alpha D_x^\alpha \Psi &= (\Gamma_\alpha D_{x'}^\alpha - \Gamma_\alpha \frac{v_\alpha}{c^{2\alpha}} D_{t'}^\alpha)(\Gamma_\alpha D_{x'}^\alpha \Psi - \Gamma_\alpha \frac{v_\alpha}{c^{2\alpha}} D_{t'}^\alpha \Psi), \\
&= \Gamma_\alpha^2 D_{x'}^\alpha D_{x'}^\alpha \Psi - 2\Gamma_\alpha^2 \frac{v_\alpha}{c^{2\alpha}} D_{x'}^\alpha D_{t'}^\alpha \Psi + \Gamma_\alpha^2 \frac{v_\alpha^2}{c^{4\alpha}} D_{t'}^\alpha D_{t'}^\alpha \Psi.
\end{aligned} \tag{16}$$

From eqs. (4) and (5), it is clear that

$$y'^\alpha = y^\alpha \to \alpha y'^{\alpha-1} dy' = \alpha y^{\alpha-1} dy \to y'^{1-\alpha} \frac{d}{dy'} = y^{1-\alpha} \frac{d}{dy},$$

and thus

$$D_{y'}^\alpha = D_y^\alpha. \tag{17}$$

Therefore,

$$D_{y'}^\alpha D_{y'}^\alpha = D_y^\alpha D_y^\alpha. \tag{18}$$

Same procedure yields,

$$D_{z'}^\alpha = D_z^\alpha, \tag{19}$$

and

$$D_{z'}^\alpha D_{z'}^\alpha = D_z^\alpha D_z^\alpha. \tag{20}$$

For the t dependence of Eq. (16), We implement the chain rule [20]:

$$\begin{aligned} D_t^\alpha \Psi &= x'^{\alpha-1} D_t^\alpha x' D_{x'}^\alpha \Psi + t'^{\alpha-1} D_t^\alpha t' D_{t'}^\alpha \Psi, \\ &= (\Gamma_\alpha^{\frac{1}{\alpha}} (x^\alpha - v_\alpha t^\alpha)^{\frac{1}{\alpha}})^{\alpha-1} t^{1-\alpha} \frac{d}{dt} \Gamma_\alpha^{\frac{1}{\alpha}} (x^\alpha - v_\alpha t^\alpha)^{\frac{1}{\alpha}} D_{x'}^\alpha \Psi \\ &+ (\Gamma_\alpha^{\frac{1}{\alpha}} (t^\alpha - \frac{v_\alpha}{c^{2\alpha}} x^\alpha)^{\frac{1}{\alpha}})^{\alpha-1} t^{1-\alpha} \frac{d}{dt} \Gamma_\alpha^{\frac{1}{\alpha}} (t^\alpha - \frac{v_\alpha}{c^{2\alpha}} x^\alpha)^{\frac{1}{\alpha}} D_{t'}^\alpha \Psi, \\ &= -\Gamma_\alpha^{1-\frac{1}{\alpha}} (x^\alpha - v_\alpha t^\alpha)^{1-\frac{1}{\alpha}} t^{1-\alpha} \Gamma_\alpha^{\frac{1}{\alpha}} \frac{1}{\alpha} (x^\alpha - v_\alpha t^\alpha)^{\frac{1}{\alpha}-1} \alpha v_\alpha t^{\alpha-1} D_{x'}^\alpha \Psi \\ &+ \Gamma_\alpha^{1-\frac{1}{\alpha}} (t^\alpha - \frac{v_\alpha}{c^{2\alpha}} x^\alpha)^{1-\frac{1}{\alpha}} t^{1-\alpha} \Gamma_\alpha^{\frac{1}{\alpha}} \frac{1}{\alpha} (t^\alpha - \frac{v_\alpha}{c^{2\alpha}} x^\alpha)^{\frac{1}{\alpha}-1} \alpha t^{\alpha-1} D_{t'}^\alpha \Psi \\ &= -v_\alpha \Gamma_\alpha D_{x'}^\alpha \Psi - \Gamma_\alpha D_{t'}^\alpha \Psi. \end{aligned} \tag{21}$$

Thus,

$$\begin{aligned} D_t^\alpha D_t^\alpha \Psi &= (-v_\alpha \Gamma_\alpha D_{x'}^\alpha - \Gamma_\alpha D_{t'}^\alpha)(-v_\alpha \Gamma_\alpha D_{x'}^\alpha \Psi - \Gamma_\alpha D_{t'}^\alpha \Psi), \\ &= v_\alpha^2 \Gamma_\alpha^2 D_{x'}^\alpha D_{x'}^\alpha \Psi - 2 v_\alpha \Gamma_\alpha^2 D_{x'}^\alpha D_{t'}^\alpha \Psi + \Gamma_\alpha^2 D_{t'}^\alpha D_{t'}^\alpha \Psi. \end{aligned} \tag{22}$$

Substituting Eqs. (16),(18),(20)and (22) in Eq. (14), we obtain

$$\begin{aligned} &\Gamma_\alpha^2 D_{x'}^\alpha D_{x'}^\alpha \Psi - 2\Gamma_\alpha^2 \frac{v_\alpha}{c^{2\alpha}} D_{x'}^\alpha D_{t'}^\alpha \Psi + \Gamma_\alpha^2 \frac{v_\alpha^2}{c^{4\alpha}} D_{t'}^\alpha D_{t'}^\alpha \Psi + D_{y'}^\alpha D_{y'}^\alpha + D_{z'}^\alpha D_{z'}^\alpha \\ &\quad - \frac{v_\alpha^2}{c^{2\alpha}} \Gamma_\alpha^2 D_{x'}^\alpha D_{x'}^\alpha \Psi + 2 \frac{v_\alpha}{c^{2\alpha}} \Gamma_\alpha^2 D_{x'}^\alpha D_{t'}^\alpha \Psi - \frac{\Gamma_\alpha^2}{c^{2\alpha}} D_{t'}^\alpha D_{t'}^\alpha \Psi = 0. \end{aligned}$$

Rearranging,

$$\Gamma_\alpha^2 \left(1 - \frac{v_\alpha^2}{c^{2\alpha}}\right) D_{x'}^\alpha D_{x'}^\alpha \Psi + D_{y'}^\alpha D_{y'}^\alpha + D_{z'}^\alpha D_{z'}^\alpha - \frac{\Gamma_\alpha^2}{c^{2\alpha}} \left(1 - \frac{v_\alpha^2}{c^{2\alpha}}\right) D_{t'}^\alpha D_{t'}^\alpha \Psi = 0.$$

Using $\Gamma_\alpha^2(1 - [v_\alpha^2/c^{2\alpha}]) = 1$, we finally obtain

$$D_{x'}^\alpha D_{x'}^\alpha \Psi + D_{y'}^\alpha D_{y'}^\alpha \Psi + D_{z'}^\alpha D_{z'}^\alpha \Psi - \frac{1}{c^{2\alpha}} D_{t'}^\alpha D_{t'}^\alpha \Psi = 0,$$

or

$$\nabla'^{2\alpha} \Psi - \frac{1}{c^{2\alpha}} D_{t'}^\alpha D_{t'}^\alpha \Psi = 0,$$

which shows that the $\alpha-$ wave equation is invariant under the $\alpha-$ Lorentz transformations. In the following three subsections, we provide three examples that are in support of the second postulate.

2.3. Conformable Schrödinger equation

The conformable Schrödinger equation [24] is

$$\left(\frac{\hat{p}_\alpha^2}{2m^\alpha} + V_\alpha(\hat{x}_\alpha)\right) \Psi = i\hbar_\alpha^\alpha D_t^\alpha \Psi. \tag{23}$$

In $3+1$-dimensions, we have

$$-\frac{\hbar_\alpha^{2\alpha}}{2m^\alpha} [D_x^\alpha D_x^\alpha + D_y^\alpha D_y^\alpha + D_z^\alpha D_z^\alpha] \Psi + V_\alpha(\hat{x}_\alpha) \Psi = i\hbar_\alpha^\alpha D_t^\alpha \Psi. \tag{24}$$

where $\hat{p}^\alpha = -i\hbar_\alpha^\alpha \nabla^\alpha$ [24]. Applying the $\alpha-$ Lorentz transformation by substituting from Eqs. (16),(18),(20)and (21) in Eq. (24), we obtain

$$\begin{aligned} &-\frac{\hbar_\alpha^{2\alpha}}{2m^\alpha} [\Gamma_\alpha^2 D_{x'}^\alpha D_{x'}^\alpha \Psi - 2\Gamma_\alpha^2 \frac{v_\alpha}{c^{2\alpha}} D_{x'}^\alpha D_{t'}^\alpha \Psi + \Gamma_\alpha^2 \frac{v_\alpha^2}{c^{4\alpha}} D_{t'}^\alpha D_{t'}^\alpha \Psi + D_{y'}^\alpha D_{y'}^\alpha + D_{z'}^\alpha D_{z'}^\alpha \Psi] + V_\alpha(\hat{x}_\alpha) \Psi \\ &\quad = i\hbar_\alpha^\alpha [-v_\alpha \Gamma_\alpha D_{x'}^\alpha \Psi - \Gamma_\alpha D_{t'}^\alpha \Psi]. \end{aligned}$$

Thus, the conformable Schrödinger equation is not invariant under the $\alpha-$ Lorentz transformations.

2.4. Conformable Gordon-Klein equation

We firstly propose the following definition of conformable relativistic energy.

Definition The conformable relativistic energy is defined as

$$E^{2\alpha} = p^{2\alpha}c^{2\alpha} + m^{2\alpha}c^{4\alpha}. \tag{25}$$

Quantization can be achieved by substituting for the conformable operators as $\hat{E}^{\alpha} = i\hbar_{\alpha}^{\alpha}D_t^{\alpha}$ and $\hat{p}^{\alpha} = -i\hbar_{\alpha}^{\alpha}\nabla^{\alpha}$ [24]. The conformable Klein-Gordon equation is then

$$\frac{1}{c^{2\alpha}}D_t^{\alpha}D_t^{\alpha}\Psi - \nabla^{2\alpha}\Psi + \frac{m^{2\alpha}c^{2\alpha}}{\hbar_{\alpha}^{2\alpha}}\Psi = 0. \tag{26}$$

Substituting Eqs. (16), (18),(20) and (22) in Eq. (26), we have

$$\frac{v_{\alpha}^2}{c^{2\alpha}}\Gamma_{\alpha}^2 D_{x'}^{\alpha}D_{x'}^{\alpha}\Psi - 2\frac{v_{\alpha}}{c^{2\alpha}}\Gamma_{\alpha}^2 D_{x'}^{\alpha}D_{t'}^{\alpha}\Psi + \frac{\Gamma_{\alpha}^2}{c^{2\alpha}}D_{t'}^{\alpha}D_{t'}^{\alpha}\Psi + \Gamma_{\alpha}^2 D_{x'}^{\alpha}D_{x'}^{\alpha}\Psi$$
$$+ 2\Gamma_{\alpha}^2\frac{v_{\alpha}}{c^{2\alpha}}D_{x'}^{\alpha}D_{t'}^{\alpha}\Psi - \Gamma_{\alpha}^2\frac{v_{\alpha}^2}{c^{4\alpha}}D_{t'}^{\alpha}D_{t'}^{\alpha}\Psi - D_{y'}^{\alpha}D_{y'}^{\alpha} - D_{z'}^{\alpha}D_{z'}^{\alpha} + \frac{m^{2\alpha}c^{2\alpha}}{\hbar_{\alpha}^{2\alpha}}\Psi = 0.$$

Then,

$$\frac{\Gamma_{\alpha}^2}{c^{2\alpha}}\left(1 - \frac{v_{\alpha}^2}{c^{2\alpha}}\right)D_{t'}^{\alpha}D_{t'}^{\alpha}\Psi - \Gamma_{\alpha}^2\left(1 - \frac{v_{\alpha}^2}{c^{2\alpha}}\right)D_{x'}^{\alpha}D_{x'}^{\alpha}\Psi - D_{y'}^{\alpha}D_{y'}^{\alpha} - D_{z'}^{\alpha}D_{z'}^{\alpha} + \frac{m^{2\alpha}c^{2\alpha}}{\hbar_{\alpha}^{2\alpha}}\Psi = 0.$$

Thus, we have

$$\frac{1}{c^{2\alpha}}D_{t'}^{\alpha}D_{t'}^{\alpha}\Psi - D_{x'}^{\alpha}D_{x'}^{\alpha}\Psi - D_{y'}^{\alpha}D_{y'}^{\alpha}\Psi - D_{z'}^{\alpha}D_{z'}^{\alpha}\Psi + \frac{m^{2\alpha}c^{2\alpha}}{\hbar_{\alpha}^{2\alpha}}\Psi = 0, \tag{27}$$

or,

$$\frac{1}{c^{2\alpha}}D_{t'}^{\alpha}D_{t'}^{\alpha}\Psi - \nabla'^{2\alpha}\Psi - D_{z'}^{\alpha}D_{z'}^{\alpha}\Psi + \frac{m^{2\alpha}c^{2\alpha}}{\hbar_{\alpha}^{2\alpha}}\Psi = 0. \tag{28}$$

Thus, the conformable Klein-Gordon equation is invariant under the $\alpha-$ Lorentz transformations.

2.5. Four vector in conformable form

We firstly present the definition of conformable position.

Definition.

1-The $\alpha-$covariant notation for position x_{μ}^{α} is defined as

$$x_{\mu}^{\alpha} = (x_0^{\alpha}, x_1^{\alpha}, x_2^{\alpha}, x_3^{\alpha}) = (c^{\alpha}t^{\alpha}, -x^{\alpha}, -y^{\alpha}, -z^{\alpha}). \tag{29}$$

2- The $\alpha-$contravariant notation for position $x^{\mu,\alpha}$ is defined as

$$x^{\mu,\alpha} = (x^{0,\alpha}, x^{1,\alpha}, x^{2,\alpha}, x^{3,\alpha}) = (c^{\alpha}t^{\alpha}, x^{\alpha}, y^{\alpha}, z^{\alpha}). \tag{30}$$

So, the relation between x_{μ}^{α} and $x^{\mu,\alpha}$ is given by

$$x_{\mu}^{\alpha} = g_{\mu\nu}x^{\mu,\alpha} \quad or \quad x^{\mu,\alpha} = g^{\mu\nu}x_{\mu}^{\alpha}, \tag{31}$$

where $g_{\mu\nu}$ is the metric tensor which is in cartesian coordinates given as [25]

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Thus, the displacement in conformable four vector is given by

1- The $\alpha-$covariant displacement

$$d^{\alpha}x_{\mu}=(d^{\alpha}x_{0},d^{\alpha}x_{1},d^{\alpha}x_{2},d^{\alpha}x_{3})=(c^{\alpha}d^{\alpha}t,-d^{\alpha}x,-d^{\alpha}y,-d^{\alpha}z). \tag{32}$$

2- The $\alpha-$contravariant displacement

$$d^{\alpha}x^{\mu}=(d^{\alpha}x^{0},d^{\alpha}x^{1},d^{\alpha}x^{2},d^{\alpha}x^{3})=(c^{\alpha}d^{\alpha}t,d^{\alpha}x,d^{\alpha}y,d^{\alpha}z). \tag{33}$$

The conformable differential line element is then given as

$$d^{\alpha}x_{\mu}d^{\alpha}x^{\mu}=(c^{2\alpha}d^{2\alpha}t,-d^{2\alpha}x,-d^{2\alpha}y,-d^{2\alpha}z). \tag{34}$$

Secondly, we present the definition of operators in conformable four vector.

Definition. The dell operator in conformable four vector is defined as

1- The $\alpha-$covariant dell operator is given by

$$\partial_{\mu}^{\alpha}=(\partial_{0}^{\alpha},\partial_{1}^{\alpha},\partial_{2}^{\alpha},\partial_{2}^{\alpha})=\frac{\partial^{\alpha}}{\partial(x^{\mu})^{\alpha}}=\left(\frac{1}{c^{\alpha}}\frac{\partial^{\alpha}}{\partial t^{\alpha}},\nabla^{\alpha}\right). \tag{35}$$

2- The $\alpha-$contravariant dell operator is given by

$$\partial^{\mu,\alpha}=(\partial^{0,\alpha},\partial^{1,\alpha},\partial^{2,\alpha},\partial^{3,\alpha})=\frac{\partial^{\alpha}}{\partial(x_{\mu})^{\alpha}}=\left(\frac{1}{c^{\alpha}}\frac{\partial^{\alpha}}{\partial t^{\alpha}},-\nabla^{\alpha}\right). \tag{36}$$

Thus, the $\alpha-$ D'Alembert operator is given by

$$\partial_{\mu}^{\alpha}\partial^{\mu,\alpha}=\frac{1}{c^{2\alpha}}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}-\nabla^{2\alpha}. \tag{37}$$

So, using $\alpha-$ D'Alembert operator, the conformable wave equation and the conformable Klein-Gorden equation are

$$\partial_{\mu}^{\alpha}\partial^{\mu,\alpha}\Psi=0, \tag{38}$$

$$\left[\partial_{\mu}^{\alpha}\partial^{\mu,\alpha}+\frac{m^{2\alpha}c^{2\alpha}}{\hbar_{\alpha}^{2\alpha}}\right]\Psi=0, \tag{39}$$

respectively. Thus, the energy-momentum four vector in conformable form can be obtained as follows:

1- In $\alpha-$covariant form

$$P_{\mu}^{\alpha}=i\hbar_{\alpha}^{\alpha}\partial_{\mu}^{\alpha}=i\hbar_{\alpha}^{\alpha}\left(\frac{1}{c^{\alpha}}\frac{\partial^{\alpha}}{\partial t^{\alpha}},\nabla^{\alpha}\right). \tag{40}$$

2- In $\alpha-$contravariant form

$$P^{\mu,\alpha}=i\hbar_{\alpha}^{\alpha}\partial^{\mu,\alpha}=i\hbar_{\alpha}^{\alpha}\left(\frac{1}{c^{\alpha}}\frac{\partial^{\alpha}}{\partial t^{\alpha}},-\nabla^{\alpha}\right). \tag{41}$$

In case independent time of the conformable Schrodinger equation [24], we get

$$i\hbar_{\alpha}^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\Psi=E^{\alpha}\Psi. \tag{42}$$

So, Eqs. (40) and (41) become

$$P_{\mu}^{\alpha}=\left(\frac{E^{\alpha}}{c^{\alpha}},-\hat{p}_{\alpha}\right), \tag{43}$$

and

$$P^{\mu,\alpha}=\left(\frac{E^{\alpha}}{c^{\alpha}},\hat{p}_{\alpha}\right), \tag{44}$$

respectively, where $\hat{p}_{\alpha}$ is called $\alpha-$ momentum operator [24], in one dimension is $\hat{p}_{\alpha}=-i\hbar_{\alpha}^{\alpha}D_{x}^{\alpha}$ and in 3-D is $\hat{p}_{\alpha}=-i\hbar_{\alpha}^{\alpha}\nabla^{\alpha}$.

2.6. The α-Lorentz transformation in Minkowski Space

Minkowski space is the most popular mathematical framework on which special relativity is formulated, and it is strongly related with Einstein's theories of special relativity and general relativity. It is also called Minkowski spacetime and it is a combination of three dimensional Euclidean space and time into a four-dimensional manifold [33].

The α-Lorentz transformation in Minkowski Space is given by

1- In the $\alpha-$contravariant form

$$x^{'\mu,\alpha} = {}^{\alpha}\Lambda^{\mu}_{\nu} x^{\nu,\alpha} \tag{45}$$

where ${}^{\alpha}\Lambda^{\mu}_{\nu}$ is the $\alpha-$tensor and defined as

$$ {}^{\alpha}\Lambda^{\mu}_{\nu} = \frac{\partial^{\alpha}}{\partial (x^{\nu})^{\alpha}} \left(\frac{x^{'\mu,\alpha}}{\alpha} \right) = \begin{pmatrix} \Gamma_{\alpha} & -\Gamma_{\alpha}\beta^{\alpha} & 0 & 0 \\ -\Gamma_{\alpha}\beta^{\alpha} & \Gamma_{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, $$

where $\beta^{\alpha} = v^{\alpha}/c^{\alpha}$ and its inverse is $x^{\nu} = ({}^{\alpha}\Lambda^{\mu}_{\nu})^{-1} x^{'\mu,\alpha}$.

2-The $\alpha-$covariant form

$$x^{'\alpha}_{\mu} = {}^{\alpha}\Lambda^{\nu}_{\mu} x^{\alpha}_{\nu}, \tag{46}$$

where ${}^{\alpha}\Lambda^{\nu}_{\mu}$ is given by

$$ {}^{\alpha}\Lambda^{\nu}_{\mu} = \frac{\partial^{\alpha}}{\partial (x^{\mu})^{\alpha}} \left(\frac{x^{'\nu,\alpha}}{\alpha} \right) = \begin{pmatrix} \Gamma_{\alpha} & \Gamma_{\alpha}\beta^{\alpha} & 0 & 0 \\ \Gamma_{\alpha}\beta^{\alpha} & \Gamma_{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. $$

Proof. Using

$$x^{'\alpha}_{\mu} = g_{\mu s} x^{'s,\alpha}, \tag{47}$$

we can write $x^{'s,\alpha}$ using The $\alpha-$Lorentz transformation in contravariant form as Eq. (45)

$$x^{'s,\alpha} = {}^{\alpha}\Lambda^{s}_{\theta} x^{\theta,\alpha}. \tag{48}$$

Substituting in Eq. (47), yields

$$x^{'\alpha}_{\mu} = g_{\mu s}\, {}^{\alpha}\Lambda^{s}_{\theta} x^{\theta,\alpha}. \tag{49}$$

Then, we can write x^{θ} as

$$x^{\theta,\alpha} = g^{\theta\nu} x^{\alpha}_{\nu}. \tag{50}$$

Substituting in Eq. (49), we obtain

$$x^{'\alpha}_{\mu} = g_{\mu s}\, {}^{\alpha}\Lambda^{s}_{\theta} g^{\theta\nu} x^{\alpha}_{\nu}. \tag{51}$$

Thus, $g_{\mu s}\, {}^{\alpha}\Lambda^{s}_{\theta} g^{\theta\nu}$ is the multiplication of three matrices

$$ g_{\mu s}\, {}^{\alpha}\Lambda^{s}_{\theta} g^{\theta\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \Gamma_{\alpha} & -\Gamma_{\alpha}\beta^{\alpha} & 0 & 0 \\ -\Gamma_{\alpha}\beta^{\alpha} & \Gamma_{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \Gamma_{\alpha} & \Gamma_{\alpha}\beta^{\alpha} & 0 & 0 \\ \Gamma_{\alpha}\beta^{\alpha} & \Gamma_{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = {}^{\alpha}\Lambda^{\nu}_{\mu} $$

Taking the inverse of ${}^{\alpha}\Lambda^{\mu}_{\nu}$ yields

$$ ({}^{\alpha}\Lambda^{\mu}_{\nu})^{-1} = \begin{pmatrix} \Gamma_{\alpha} & \Gamma_{\alpha}\beta^{\alpha} & 0 & 0 \\ \Gamma_{\alpha}\beta^{\alpha} & \Gamma_{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = g_{\mu s}\, {}^{\alpha}\Lambda^{s}_{\theta} g^{\theta\nu} = {}^{\alpha}\Lambda^{\nu}_{\mu}. \tag{52}$$

Therefore, Eq. (51) is equivalent to Eq. (46).

2.7. Conformable Dirac Equation

In Mozaffari *et al.*, [34], the Dirac equation using the conformable derivative is investigated and it is introduced as

$$[i\gamma^{\mu}\partial_{\mu}^{\alpha} - m^{\alpha}]\Psi(x^{\mu,\alpha}) = 0, \tag{53}$$

where γ^{μ} are the famous γ matrices of Dirac equation [35]. Similarly, the Dirac equation is Lorentz covariant, namely,

$$[i\gamma^{\nu}\partial_{\nu}^{\prime\alpha} - m^{\alpha}]\Psi'(x^{\prime\nu,\alpha}) = 0. \tag{54}$$

However, when we do a Lorentz transformation, the wave function changes. Because the Dirac equation and Lorentz transformation are linear, we require that the transformation between Ψ and Ψ' be linear too:

$$\Psi'(x^{\prime\alpha}) = S\Psi(x^{\alpha}), \tag{55}$$

where S denotes an $x-$independent matrix whose properties must be found. The Dirac equation in Lorentz covariance indicates that the γ matrices are identical in both frames. Using

$$\partial_{\mu}^{\alpha} = {}^{\alpha}\Lambda_{\mu}^{\nu}\partial_{\nu}^{\prime\alpha}. \tag{56}$$

From Eq. (55) we found $\Psi(x^{\alpha}) = S^{-1}\Psi'(x^{\prime\alpha})$ and substituting in Eq. (53), we obtain

$$[i\gamma^{\mu}\partial_{\mu}^{\alpha} - m^{\alpha}]S^{-1}\Psi'(x^{\prime\alpha}) = 0. \tag{57}$$

Substituting from Eq. (56) and then multiply with S from the left, yields

$$[iS\gamma^{\nu}S^{-1}\ {}^{\alpha}\Lambda_{\mu}^{\nu}\partial_{\nu}^{\prime\alpha} - m^{\alpha}]\Psi'(x^{\prime\alpha}) = 0. \tag{58}$$

Comparing Eq. (58) with Eq. (54), we obtain

$$S\gamma^{\mu}S^{-1}\ {}^{\alpha}\Lambda_{\mu}^{\nu} = \gamma^{\nu}. \tag{59}$$

So, $\gamma^{\mu}\ {}^{\alpha}\Lambda_{\mu}^{\nu} = S^{-1}\gamma^{\nu}S$. The inverse Lorentz transformation must correspond to the inverse of S [36], namely,

$$\gamma^{\mu}(\ {}^{\alpha}\Lambda_{\mu}^{\nu})^{-1} = S\gamma^{\nu}S^{-1}. \tag{60}$$

Therefore, we demonstrated that the conformable Dirac equation is covariant in α-Lorentz transformation. For more information on the S matrix, one can refer to [35,36].

3. Summary and conclusions

In this paper, we have investigated the deformation of Einstein's special relativity using the concept of conformable derivative. Within this frame, the α-Lorentz transformations were defined, and the two postulates of the theory were extended and re-stated. Then, the conformable addition of velocity laws were derived and used to verify the constancy of the speed of light for any fractional order α. The invariance principle of the laws of physics postulate was demonstrated for some typical illustrative partial differential equations of interest, namely, the conformable wave equation, the conformable Schrödinger equation, the conformable Klein-Gordon equation, and conformable Dirac equation. For a wave equation where time and space appeared with the same α-order, it is found that it is invariant under α-Lorentz transformations. Otherwise, it is not.

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