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A complementary covariant approach to gravito-electromagnetism

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From a previous paper where we proposed a description of general relativity within the gravito-electromagnetic limit, we propose an alternative modified gravitational theory. As in the former version, we analyze the vector and tensor equations of motion, the gravitational continuity equation, the conservation of the energy, the energy-momentum tensor, the field tensor, and the constraints concerning these fields. The Lagrangian formulation is also exhibited as an unified and simple formulation that will be useful for future investigation.

Keywords: Classical general relativity; fundamental problems and general formalism; modified theories of gravity.

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1. Gravito-electromagnetism

Is general relativity (GR) a final theory or it will be superseded by another theory in the future? We expect that it will survive while its explanatory power is strong enough to describe the available experimental data. However, even if its explanatory power was not strong enough to understand every known phenomenon, we would keep it in the absence of an alternative theory. At the present time, general relativity is a very successful gravitational theory, but we also know that there are several open questions about it, particularly related to their quantization and to to their cosmological applications. Furthermore, we do not know whether GR is suitable to solve these open questions, or whether a different theory is needed. In this situation, it can be interesting to modify the old theory in order to explain singular data effectively or to introduce a different conceptual idea [1]. From a theoretical point of view, it is interesting to study what kind of modification can possibly be done to a theory and keep its mathematical and physical consistency.

In this article, we propose an exercise concerning a recently published gravitation theory that modifies GR within the gravito-electromagnetic precision order [2]. In summary, we will analyze an alternative to this previously proposed modified theory in order to exhaust the possible alterations that are coherent to the original idea. Before considering the new formulation, we give a brief explanation of the origin of gravito-electromagnetism (GEM) using the Chapter 3 of [3]. As the name announces, in this theory an analogy between GR and electromagnetism is established. GEM comes from the weak field approximation to GR, where the $g_{\mu\nu}$ metric tensor is

$$g^{\mu\nu}=\eta^{\mu\nu}+\kappa h^{\mu\nu},\qquad \kappa=\frac{\sqrt{16\pi G}}{c^2} \tag{1}$$

is a constant in cgs units and $h^{\mu\nu}$ represents the perturbation of the $\eta^{\mu\nu}$ flat space tensor, whose components are respectively $\eta^{00}=1,$ and $\eta^{ii}=-1$ for $i,\,j=\{1,\,2,\,3\}$. Within the weak field limit, $|\kappa h_{\mu\nu}|\ll 1,$ the physical law

$$\frac{d\boldsymbol{v}}{dt}=\boldsymbol{g}+\boldsymbol{v}\times\boldsymbol{b}, \tag{2}$$

describes the motion of a massive particle of velocity $\boldsymbol{v}$ in a gravitational field $\boldsymbol{g}$, and whose gravito-magnetic field is $\boldsymbol{b}$. The boldface characters denote vector quantities in a time-like surface, and the vector product satisfies the usual definition. In terms of components, we have

$$\big(\boldsymbol{v}\times\boldsymbol{b}\big)_i=\epsilon_{ijk}v_jb_k, \tag{3}$$

where ϵ_{ijk} is the Levi-Cività anti-symmetric symbol. The similarity between Eq. (2) and the electro-dynamical Lorentz force is evident. In terms of the perturbation of the metric tensor, the physical law reads

$$g_i=-\frac{\kappa}{2}\frac{\partial h_{00}}{\partial x^i}\quad\text{and}\quad b_i=-\kappa\left(\frac{\partial h_{0k}}{\partial x^j}-\frac{\partial h_{0j}}{\partial x^k}\right), \tag{4}$$

where the space-like components of the space-time index are i, j and k. Hence, one can establish an analogy between the covariant electrodynamics and the field vectors in a tensor formula, so that

$$f_{0i}=g_i\qquad\text{and}\qquad f_{ij}=-\frac{1}{2}\epsilon_{ijk}b_k. \tag{5}$$

However, the quantities of Eq. (4) were obtained from on the zeroth component of $h_{\mu\nu}$, and hence the $f_{\mu\nu}$ tensor obtained from Eq. (5) are in fact the zeroth component of a third rank tensor. Therefore, the analogy between $f^{\mu\nu}$ and the Faraday tensor $F^{\mu\nu}$ of electrodynamics is imperfect because the $f^{\mu\nu}$ tensor is not covariant in the same way that $F^{\mu\nu}$. There are various proposals to determine this third rank gravitational tensor, and we quote [4–8] and references therein. This fact turns the GEM research even more exciting, because it indicates a way to research a more general tensor theory of gravity, where additional space-time indices would be necessary for tensor quantities. Independently of this feature, the conceptual idea of GEM evidences the parallel between electromagnetism and gravitation, and various ideas to implement the gravito-electromagnetic approximation have been elaborated, and a list of references of them can be found in Refs. [2, 9–11].

In Ref. [2], the gravitational field $\boldsymbol{g}$ was decomposed as a sum of two auxiliary fields, the gravito-electric field $\boldsymbol{g}_E$ and the gravito-magnetic field $\boldsymbol{g}_B$, where

$$\boldsymbol{g} = \boldsymbol{g}_E + \boldsymbol{g}_B \qquad \text{constrained with} \qquad \boldsymbol{g}_E \cdot \boldsymbol{g}_B = 0. \quad (6)$$

This decomposition is not usual in gravito-electromagnetism (GEM), and the field equations are also different from the previous formulations. This discussion is already been done in the previous article. However, the previous article does not exhaust the possible formulations, and this paper intends to fill this blank. However, we shall see that this task in not a bureaucratic one. The formulations have a diverse physical content, and the second formulation is necessary for the theoretical comprehension, and for future applications as well.

2. Modified Newtonian gravitation

Modified theories of Newton's gravitation are not a novelty, and we mention [12–14] as a recent conjectures of such kind. In our proposal, the field equations are such as

$$\boldsymbol{\nabla}\cdot\boldsymbol{g} = -4\pi\rho \quad \text{and} \quad \boldsymbol{\nabla}\times\boldsymbol{g} = \frac{4\pi}{c}\boldsymbol{p} - \frac{1}{c}\frac{\partial\boldsymbol{g}}{\partial t}. \quad (7)$$

where $\boldsymbol{g}$ is the gravity field vector, ρ is the density of mass, $\boldsymbol{p} = \rho\boldsymbol{v}$ is the matter flux density vector. Accordingly, the gravity force $\boldsymbol{F}$ acts over a particle of mass m according to the physical law

$$\boldsymbol{F} = m\boldsymbol{g} - \frac{1}{c}\,\boldsymbol{p}\times\boldsymbol{g}. \quad (8)$$

Equations (7) are identical to that proposed in Ref. [2], while (8) has a single difference, a flipped sign on of the second term. The ultimate proposal of the present article is to determine the differences concerning this single difference. Additionally, we will confirm that the gravitational field given by (7) has a physical content comparable to that achieved after the truncation of Eintein's field equations. We remember that truncation of Einstein's equations generates the Newtonian theory at its first approximation, while higher order terms produce (2-4), and this prevision of GR will be recovered from Eq. (7) using a covariant scheme. As first consequence, the continuity equation and the conservation of the mass is obtained from (7)

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{p} = 0. \quad (9)$$

The energy balance is given by

$$\frac{1}{8\pi c}\frac{\partial|\boldsymbol{g}|^2}{\partial t} + \frac{1}{4\pi}\boldsymbol{g}\cdot\boldsymbol{\nabla}\times\boldsymbol{g} = \frac{1}{c}\,\boldsymbol{g}\cdot\boldsymbol{p}, \quad (10)$$

and Eqs. (9-10) are identical to that obtained in Ref. [2]. We observe the self-interacting terms $\boldsymbol{g}\cdot\boldsymbol{\nabla}\times\boldsymbol{g}$, and $\boldsymbol{g}\cdot\boldsymbol{p}$ and a conservative gravitational field is obtained if

$$\boldsymbol{g}\cdot\boldsymbol{\nabla}\times\boldsymbol{g} = 0. \quad (11)$$

Equation (10) is the gravitational equivalent of the Poynting theorem, but a gravitational Poynting vector cannot be obtained. It is interesting to note that every contribution to the energy balance comes from self-interaction. Differently from [2], an expression for the conservation of the linear momentum is not possible in this formulation. Therfore, the field equations, the gravitational force law, the continuity equation and the energy balance encompass all the results that one can obtain from this model. In the following section, the tensor approach will illuminate this physical model from a different standpoint.

3. The gravitational field in the tensor formalism

In this section, we will observe many differences between the model of (7-8) and the previous article. Let us then intoduce the gravitational field tensor

$$C_{\mu\nu} = \begin{bmatrix} 0 & -g_1 & -g_2 & -g_3 \\ g_1 & 0 & -g_3 & g_2 \\ g_2 & g_3 & 0 & -g_1 \\ g_3 & -g_2 & g_1 & 0 \end{bmatrix} \qquad \text{where}$$

$$\boldsymbol{g} = \big(g_1,\, g_2,\, g_3\big). \quad (12)$$

The Minkowskian indices are μ and ν, and the metric tensor $\eta^{\mu\nu}$, is such that $\eta^{00} = 1,\ \eta^{ii} = -1$ and $i,\, j = \{1,\, 2,\, 3\}$. Accordingly,

$$C_{i0} = g_i, \qquad\qquad C_{ij} = -\epsilon_{ijk}g_k, \quad (13)$$

where ϵ_{ijk} is the Levi-Cività anti-symmetric symbol. Using the field tensor, the field equations (7) become

$$\partial_\nu C^{\nu\mu} = \frac{4\pi}{c}p^\mu, \qquad \text{where} \qquad p^\mu = \big(c\rho,\ \boldsymbol{p}\big). \quad (14)$$

We also define the contravariant momentum density $4-$vector p^μ, that can also be called the matter current $4-$vector, and the contravariant coordinate $4-$vector $x^\mu = \big(ct,\ \boldsymbol{x}\big)$. Using this formalism, the continuity equation (9) reads

$$\partial_\mu p^\mu = 0. \quad (15)$$

The gravitational force can as well be obtained in the covariant expression,

$$\frac{dp^\mu}{dt} = \frac{1}{c}C^{\nu\mu}p_\nu. \quad (16)$$

The spacelike $\mu = i$ components of (16) furnish the gravity force, and the timelike $\mu = 0$ component reveals that the energy density of the model obeys

$$c^2\frac{d\rho}{dt} = \boldsymbol{g}\cdot\boldsymbol{p}, \quad (17)$$

On the other hand, using the anti-symmetric feature of the field tensor, we obtain

$$p^\mu \frac{dp_\mu}{dt} = \frac{1}{c} C_{\mu\nu} p^\mu p^\nu = 0 \quad \text{thus}$$
$$\frac{d}{dt}\big(p^\mu p_\mu\big) = 0, \tag{18}$$

and therefore $p_\mu p^\mu$ reveals to be a constant associated to the rest energy density E. Therefore, the four-momentum vector (14) can be interpreted relativistically, so that

$$p_\mu p^\mu \;=\; \rho^2 c^2 - \boldsymbol{p}\cdot\boldsymbol{p} = \frac{E^2}{c^2}. \tag{19}$$

In order to obtain the energy-momentum tensor of this self-interacting gravitational theory, we define the $\tau^{\mu\nu}$ symmetric tensor as

$$\tau_{\mu\nu} = \tau^{\mu\nu} = \begin{cases} 1 & \text{if} \quad \mu = \nu, \\ 0 & \text{if} \quad \mu \neq \nu. \end{cases} \tag{20}$$

which has been introduced in [2] and that satisfies $\eta^{\mu\nu} = \tau^{\mu\kappa}\tau_\kappa^{\ \nu}$. The equations of motion (14) therefore become

$$\partial_\lambda\Big(\tau^\lambda_{\ \mu} C_{\nu\kappa} + \tau^\lambda_{\ \nu} C_{\kappa\mu} + \tau^\lambda_{\ \kappa} C_{\mu\nu}\Big) = \frac{4\pi}{c}\epsilon_{\mu\nu\kappa\lambda} p_\sigma \tau^{\lambda\sigma}, \tag{21}$$

where the anti-symmetric Levi-Cività symbol is $\epsilon_{\mu\nu\kappa\lambda}$. Consequently, using (14) and (16), we get

$$\frac{dp_\mu}{dt} = \frac{1}{4\pi} C_{\mu\nu}\partial_\kappa C^{\nu\kappa}. \tag{22}$$

Additionally, combining (21-22) produces an equation satisfied by the $T_{\mu\nu}$ is the energy-momentum tensor,

$$\frac{dp^\mu}{dt} = \partial_\kappa T^{\kappa\mu} + I^\mu + S^\mu, \tag{23}$$

where I_μ gives the self-interaction and S_μ represents the source. Explicitly,

$$T_{\mu\nu} = \frac{1}{4\pi}\left(C_{\eta\mu} C_\nu^{\ \eta} + \frac{1}{4}\tau_{\mu\nu}\tau^\eta_\kappa C_{\eta\lambda} C^{\kappa\lambda}\right),$$
$$I_\mu = \frac{C^{\nu\kappa}}{8\pi}\Big(\tau^\lambda_\nu \tau^\eta_\kappa \partial_\eta C_{\mu\lambda} + \partial_\nu C_{\mu\kappa}\Big),$$
$$S_\mu = \frac{\epsilon_{\mu\nu\kappa\lambda}}{2c} p_\sigma C^{\kappa\eta}\tau^\nu_\eta \tau^{\lambda\sigma}. \tag{24}$$

At this moment, we point out the major difference of the model presented in this article. This approach is more complicated than the former model [2] where $T_{\mu\nu} = I_\mu = 0$, and consequently the previous approach is probably more realistic if we expect that theoretical simplicity and physical reality are twin brothers. Despite this, we further explore the model, and the energy-momentum tensor and the self-interaction term further simplify to

$$T_{\mu\nu} = \frac{1}{4\pi}\left(C_{\eta\mu} C_\nu^{\ \eta} - \frac{1}{2}\tau_{\mu\nu}|\boldsymbol{g}|^2\right) \qquad \text{and}$$
$$I_\mu = \frac{1}{4\pi}\Big(\partial_0 C_{\mu i} - \partial_i C_{\mu 0}\Big) C^{0i}. \tag{25}$$

Explicitly, the energy-momentum components are

$$T_{00} = \frac{|\boldsymbol{g}|^2}{8\pi}, \qquad T_{ii} = \frac{1}{8\pi}\Big(|\boldsymbol{g}|^2 - 4g_i^2\Big),$$
$$T_{0i} = 0, \qquad T_{ij} = -\frac{g_i g_j}{2\pi}, \tag{26}$$

which generate the scalar quantities

$$T_{\mu\nu}\tau^{\mu\nu} = 0, \qquad T_\mu^{\ \mu} = \frac{|\boldsymbol{g}|^2}{4\pi} \qquad \text{and}$$
$$T_{\mu\nu}T^{\mu\nu} = \frac{3|\boldsymbol{g}|^4}{(4\pi)^2}. \tag{27}$$

Different from electromagnetism, the gravity energy-momentum tensor is not traceless. This result is in fact expected from general relativity, and thus a consistency condition is fulfilled. Furthermore, using the field Eqs. (7), we obtain

$$I^\mu = \frac{1}{4\pi}\left(-\frac{1}{2c}\frac{\partial|\boldsymbol{g}|^2}{\partial t},\ \frac{1}{c}\left[\boldsymbol{g}\times\frac{\partial\boldsymbol{g}}{\partial t}\right]_i + \big[\boldsymbol{g}\cdot\boldsymbol{\nabla}\big] g_i\right),$$
$$S^\mu = \left(\frac{\boldsymbol{g}\cdot\boldsymbol{p}}{c},\, -\rho\boldsymbol{g}\right). \tag{28}$$

Using Eqs. (26) and (28) in (23), the energy conservation and the gravitational force components are recovered, and the physical consistency of the model is assured. We have shown in this section that the gravitation model that (7-8) comprise can be consistently described using a tensor language. However, such a formulation seems unsatisfactory, particularly because the conservation of the energy is not clear in Eq. (10). For the sake of clarity, we develop a potential formulation in the next section.

4. The gravitational potentials in the tensor formalism

Introducing the gravitational scalar potential Φ and the gravitational vector potential $\boldsymbol{\Psi}$, the gravitational field is proposed to be

$$\boldsymbol{g} = -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial\boldsymbol{\Psi}}{\partial t} + \boldsymbol{\nabla}\times\boldsymbol{\Psi}, \tag{29}$$

and the field Eqs. (7) consequently become

$$\nabla^2\Phi + \frac{1}{c}\frac{\partial}{\partial t}\big(\boldsymbol{\nabla}\cdot\boldsymbol{\Psi}\big) = 4\pi\rho$$
$$\nabla^2\boldsymbol{\Psi} - \boldsymbol{\nabla}\big(\boldsymbol{\nabla}\cdot\boldsymbol{\Psi}\big) = -\frac{4\pi}{c}\boldsymbol{p} + \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{\nabla}\Phi + \frac{1}{c^2}\frac{\partial\boldsymbol{\Psi}}{\partial t^2}. \tag{30}$$

Nonetheless, we obtain a simpler formulation after defining auxiliary gravito-electric and gravito-magnetic vector fields, respectively $\boldsymbol{g}_E$ and $\boldsymbol{g}_B$. Therefore,

$$\boldsymbol{g} = \boldsymbol{g}_E + \boldsymbol{g}_B, \qquad \text{where} \qquad \boldsymbol{g}_E = -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial\boldsymbol{\Psi}}{\partial t}$$
$$\text{and} \quad \boldsymbol{g}_B = \boldsymbol{\nabla}\times\boldsymbol{\Psi}. \tag{31}$$

Comparing to the the previous formulation [2], the signs of the third term in Eq. (29) and, consequently, of $\boldsymbol{g}_B$ in (31) are flipped, and the second equation of (30) is simpler than in the previous paper. In consequence, using Eq. (31) in (7) we obtain the gravitational field equations in potential formulation,

$$\begin{aligned} &\boldsymbol{\nabla}\cdot\boldsymbol{g}_E = -4\pi\rho && \boldsymbol{\nabla}\cdot\boldsymbol{g}_B = 0 \\ &\boldsymbol{\nabla}\times\boldsymbol{g}_E = -\frac{1}{c}\frac{\partial\boldsymbol{g}_B}{\partial t} && \boldsymbol{\nabla}\times\boldsymbol{g}_B = \frac{4\pi}{c}\boldsymbol{p} - \frac{1}{c}\frac{\partial\boldsymbol{g}_E}{\partial t}, \end{aligned} \tag{32}$$

that is similar to previous formulations of GEM [2, 4–8], and also similar to the Maxwell electromagnetic field equations. Defining the gravitational potential second rank tensor

$$\mathscr{C}^{\mu\nu} = \tau^{\mu}{}_{\kappa}\tau^{\nu}{}_{\lambda}\Big(\partial^{\kappa}Q^{\lambda} - \partial^{\lambda}Q^{\kappa}\Big) \qquad \text{where} \qquad Q^{\mu} = \big(\,\Phi,\,\boldsymbol{\Psi}\,\big) \tag{33}$$

is the gravitational potential $4-$vector, we directly have

$$\mathscr{C}_{i0} = \big(g_E\big)_i \qquad \text{and} \qquad \mathscr{C}_{ij} = -\epsilon_{ijk}\big(g_B\big)_k. \tag{34}$$

The potential tensor (33-34) enables us to regain the equations of motion (30) using

$$\partial_{\nu}\mathscr{C}^{\nu\mu} = \frac{4\pi}{c}p^{\mu}. \tag{35}$$

Equation (35) contains the non-homogeneous components of (32), and the homogeneous terms come from

$$\partial_{\lambda}\Big(\tau^{\lambda}{}_{\mu}\mathscr{C}_{\nu\kappa} + \tau^{\lambda}{}_{\nu}\mathscr{C}_{\kappa\mu} + \tau^{\lambda}{}_{\kappa}\mathscr{C}_{\mu\nu}\Big) = 0. \tag{36}$$

Manipulating the $4-$vector momentum density, we consequently have

$$\frac{dp^{\mu}}{dt} = \frac{1}{c}\mathscr{C}^{\nu\mu}p_{\nu}, \tag{37}$$

whose components give

$$\frac{dp^0}{dt} = \frac{1}{c}\boldsymbol{p}\cdot\boldsymbol{g}_E \qquad \text{and} \qquad \frac{d\boldsymbol{p}}{dt} = \rho\,\boldsymbol{g}_E - \frac{1}{c}\boldsymbol{p}\times\boldsymbol{g}_B. \tag{38}$$

Analyzing (38) in comparison to Eqs. (8) and (17), two constraints emerge, namely,

$$\boldsymbol{p}\cdot\boldsymbol{g}_B = 0; \qquad c\rho\boldsymbol{g}_B - \boldsymbol{p}\times\boldsymbol{g}_E = \boldsymbol{0}. \qquad \text{Likewise,} \qquad \boldsymbol{g}_E\cdot\boldsymbol{g}_B = 0. \tag{39}$$

Therefore, the linear momentum $\boldsymbol{p}$, the gravito-electric field $\boldsymbol{g}_E$ and the gravitational force vector $d\boldsymbol{p}/dt$ are coplanar and the force law (39) conforms perfectly to (2), and the alternated signs in Eq. (4) may be obtained by a redefinition of $\boldsymbol{b}$. At this moment, we point out the more important drawback of the model. Differently from the previous formulation [2], we cannot obtain a relation expressing the conservation of momentum in the same fashion as the electromagnetic formulation. This does not mean that the momentum is necessarily not conserved, but it may have a more subtle formulation. We may further explain the conservation of momentum by considering the tensor expression of the force law obtained from (35-37), so that

$$\frac{dp^{\mu}}{dt} = \partial_{\kappa}\mathscr{T}^{\kappa\mu} + \mathscr{I}^{\mu}. \tag{40}$$

The energy-momentum tensor is

$$\begin{aligned}\mathscr{T}_{\mu\nu} &= \frac{1}{4\pi}\left(\mathscr{C}_{\lambda\mu}\mathscr{C}_{\nu}{}^{\lambda} + \frac{1}{4}\tau_{\mu\nu}\tau^{\eta}_{\kappa}\mathscr{C}_{\eta\lambda}\mathscr{C}^{\kappa\lambda}\right) \\ &= \frac{1}{4\pi}\left(\mathscr{C}_{\mu\lambda}\mathscr{C}_{\nu}{}^{\lambda} - \frac{1}{2}\tau_{\mu\nu}\big|\boldsymbol{g}_B\big|^2\right),\end{aligned} \tag{41}$$

and the interaction term reads

$$\mathscr{I}^{\mu} = \frac{1}{4\pi}\Bigg(-\frac{1}{2}\partial_0|\boldsymbol{g}_E|^2,\ \Big[\boldsymbol{g}_E\times\big\{\partial_0\boldsymbol{g}_B\big\}\Big]_i + \boldsymbol{g}_E\cdot\boldsymbol{\nabla}\,\big(\boldsymbol{g}_E\big)_i\Bigg). \tag{42}$$

The self-interaction term $\mathscr{I}^{\mu}$ does not appear in the previous formulation [2], and this raises up an hypothesis to explain the non-conservative character of the momentum. In electrodynamics, we have to separate the momentum of the particles and the momentum of the fields, and this works well also in Ref. [2]. In the present theory, we have the additional contribution of self-interaction of the fields in (40), and the four-fource cannot be written as a four-divergence, engendering a more general situation here, because such a terms is not present in previous formulations, and the conservation is recovered if $\mathscr{I}^{\mu} = 0$. Maybe we can impose this as a constraint, but this can be considered as a direction for future research, as well as the whole this discussion of the character of momentum in the present theory.

Explicitly written, the components of (41) are

$$\begin{aligned}\mathscr{T}_{00} &= \frac{1}{4\pi}\left(\big|\boldsymbol{g}_E\big|^2 - \frac{1}{2}\big|\boldsymbol{g}_B\big|^2\right) \\ \mathscr{T}_{ii} &= \frac{1}{4\pi}\left[\,\frac{1}{2}\big|\boldsymbol{g}_B\big|^2 - \big(\boldsymbol{g}_B\big)_i^2 - \big(\boldsymbol{g}_E\big)_i^2\,\right] \\ \mathscr{T}_{0i} &= \frac{1}{4\pi}\Big(\boldsymbol{g}_E\times\boldsymbol{g}_B\Big)_i \\ \mathscr{T}_{ij} &= -\frac{1}{4\pi}\left[\,\big(\boldsymbol{g}_E\big)_i\big(\boldsymbol{g}_E\big)_j + \big(\boldsymbol{g}_B\big)_i\big(\boldsymbol{g}_B\big)_j\,\right].\end{aligned} \tag{43}$$

Accordingly, we obtain the scalar quantities

$$\begin{aligned}&\mathscr{T}_{\mu\nu}\tau^{\mu\nu} = 0, \qquad \mathscr{T}_{\mu}{}^{\mu} = \frac{2\big|\boldsymbol{g}_E\big|^2 - \big|\boldsymbol{g}_B\big|^2}{4\pi} \qquad \text{and} \\ &\mathscr{T}_{\mu\nu}\mathscr{T}^{\mu\nu} = \frac{2}{(4\pi)^2}\Bigg[\frac{\big|\boldsymbol{g}_B\big|^4}{2} + |\boldsymbol{g}_E|^4 - |\boldsymbol{g}_E|^2|\boldsymbol{g}_B|^2 + \big(\boldsymbol{g}_E\cdot\boldsymbol{g}_B\big)^2 - \big|\boldsymbol{g}_E\times\boldsymbol{g}_B\big|^2\Bigg].\end{aligned} \tag{44}$$

By comparing the scalar quantities (44) and (26), the nullity of $\mathscr{T}_{\mu\nu}\tau^{\mu\nu}$ and $T_{\mu\nu}\tau^{\mu\nu}$ fits the role played by the null $T_{\mu}{}^{\mu}$ in electromagnetism. Finally, from Eq. (40) we obtain

$$\frac{dp^0}{dt} = \frac{\partial}{\partial t}\left(\frac{|\boldsymbol{g}_E|^2 - |\boldsymbol{g}_B|^2}{8\pi}\right) + \boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{g}_B \times \boldsymbol{g}_E}{4\pi}\right). \quad (45)$$

Using (38) we generate the energy conservation law that is directly obtained from the field equations (32) and that does not produce additional constraints. Finally, following a formulation of quantum electrodynamics, we use Q^μ from (33), and also $\partial^\mu Q^\mu$, as the independent variable of the gravito-electromagnetic Lagrangian density

$$\mathcal{L} = \frac{1}{8\pi}\partial_\mu Q_\nu \mathscr{C}^{\mu\nu} + \frac{1}{c}p_\mu Q^\mu, \quad (46)$$

and (35) is immediately obtained from Eq. (46). As a final remark, the field equations (32) can also be obtained using

$$\boldsymbol{g} = \boldsymbol{g}_E - \boldsymbol{g}_B, \qquad \text{where} \qquad \boldsymbol{g}_E = -\boldsymbol{\nabla}\Phi + \frac{1}{c}\frac{\partial\boldsymbol{\Psi}}{\partial t}$$
$$\text{and} \qquad \boldsymbol{g}_B = \boldsymbol{\nabla}\times\boldsymbol{\Psi}. \quad (47)$$

However, this formulation flips the sign of $\boldsymbol{p}\times\boldsymbol{g}_B$ in Eq. (38), and so we conclude that (31) is the most suitable choice for the potential. In the next section, we summarize the results of Sec. 3 and 4 into a gravity law that is an alternative to (8).

5. The second gravity force law

Let us consider the force law

$$\boldsymbol{F} = \rho\boldsymbol{g} + \frac{1}{c}\boldsymbol{p}\times\boldsymbol{g}, \quad (48)$$

the field equations

$$\boldsymbol{\nabla}\cdot\boldsymbol{g} = -4\pi\rho, \qquad \boldsymbol{\nabla}\times\boldsymbol{g} = -\frac{4\pi}{c}\boldsymbol{p} + \frac{1}{c}\frac{\partial\boldsymbol{g}}{\partial t}. \quad (49)$$

and the field tensor

$$C_{i0} = g_i; \qquad C_{ij} = \epsilon_{ijk}g_k, \quad (50)$$

where Eqs. (14-17) hold. On the other hand, Eq. (21) changes to

$$\partial_\lambda\Big(\tau^\lambda{}_\mu C_{\nu\kappa} + \tau^\lambda{}_\nu C_{\kappa\mu} + \tau^\lambda{}_\kappa C_{\mu\nu}\Big) = -\frac{4\pi}{c}\epsilon_{\mu\nu\kappa\lambda}p_\sigma\tau^{\lambda\sigma}. \quad (51)$$

The energy-momentum tensor $T_{\mu\nu}$ is identical to (26), and consequently the scalar quantities are also identical (27). In contrast, the source term S^μ is identical to that of Eq. (28), but the spacial components of the self-interaction term I^μ are slightly different, thus,

$$I^\mu = \frac{1}{4\pi}\left(-\frac{1}{2c}\frac{\partial|\boldsymbol{g}|^2}{\partial t},\ \frac{1}{c}\left(\frac{\partial\boldsymbol{g}}{\partial t}\times\boldsymbol{g}\right)_i + (\boldsymbol{g}\cdot\boldsymbol{\nabla})g_i\right). \quad (52)$$

Hence, the second formulation is also consistent, and the proper physical content demands experimental investigation of Eqs. (8) and (48). Additionally, the potential formulation is

$$\boldsymbol{g} = \boldsymbol{g}_E + \boldsymbol{g}_B, \qquad \text{where}$$
$$\boldsymbol{g}_E = -\boldsymbol{\nabla}\Phi + \frac{1}{c}\frac{\partial\boldsymbol{\Psi}}{\partial t} \quad \text{and} \quad \boldsymbol{g}_B = \boldsymbol{\nabla}\times\boldsymbol{\Psi}. \quad (53)$$

and finally the field equations are

$$\boldsymbol{\nabla}\cdot\boldsymbol{g}_E = -4\pi\rho \qquad \boldsymbol{\nabla}\cdot\boldsymbol{g}_B = 0$$
$$\boldsymbol{\nabla}\times\boldsymbol{g}_E = \frac{1}{c}\frac{\partial\boldsymbol{g}_B}{\partial t} \qquad \boldsymbol{\nabla}\times\boldsymbol{g}_B = -\frac{4\pi}{c}\boldsymbol{p} + \frac{1}{c}\frac{\partial\boldsymbol{g}_E}{\partial t}. \quad (54)$$

Additionally,

$$\mathscr{C}_{\mu\nu} = \partial_\lambda\tau_\mu{}^\lambda Q_\nu - \partial_\lambda\tau_\nu{}^\lambda Q_\mu, \quad (55)$$

leads to,

$$\mathscr{C}_{i0} = (g_E)_i, \qquad \mathscr{C}_{ij} = \epsilon_{ijk}(g_B)_k, \quad (56)$$

and Eqs. (35-36) are immediately recovered. From (37), we produce

$$\frac{dp^0}{dt} = \frac{1}{c}\boldsymbol{p}\cdot\boldsymbol{g}_E \qquad \text{and} \qquad \frac{d\boldsymbol{p}}{dt} = \rho\,\boldsymbol{g}_E + \frac{1}{c}\boldsymbol{p}\times\boldsymbol{g}_B. \quad (57)$$

The constraints are

$$\boldsymbol{p}\cdot\boldsymbol{g}_B = 0; \qquad c\rho\boldsymbol{g}_B + \boldsymbol{p}\times\boldsymbol{g}_E = \boldsymbol{0}.$$
$$\text{Likewise} \qquad \boldsymbol{g}_E\cdot\boldsymbol{g}_B = 0. \quad (58)$$

Essentially, both of the formulations are related by the symmetry transformation

$$\boldsymbol{g}_b \to -\boldsymbol{g}_B, \qquad \text{or} \qquad \boldsymbol{\Psi}\to-\boldsymbol{\Psi}$$
$$\text{or} \qquad Q^\mu \to Q^\nu\tau_\nu{}^\mu. \quad (59)$$

Thus, under the alternative gravity law, the equivalents of Eqs. (40-45) are immediately obtained using Eq. (59), and the difference is the alternate sign in the "Pointing vector" of (45), meaning the reversal of the momentum flux in each formulation.

6. Concluding remarks

We examined several formal questions concerning gravito-electromagnetism, and proposed two gravity force laws, namely (8) and (48), and consistent covariant tensor formulations have been built for both of them. It was also verified that both of the formulations are related through a symmetry operation. The results complement the former article [2], where the force law is identical, but the field equations are different different. The results indicate that the energy is conserved in the present formulation, but the momentum is not conserved. Although this seems a negative result, it is in fact a very important piece of information. The force laws (8) and (48) were obtained using a different set of field equations in

Ref. [2], and the choices of the present article introduce the self-interaction terms I^{μ} in (23) and $\mathscr{I}^{\mu}$ in (40), and this kind of interaction does not allow the conservation of the momentum. Only experimental data concerning the deviation of the Newton law can decide which deviation model generate the correct version of GEM. To the best of our knowledge, the state of the art of the experimental research, namely the Gravity Probe B experiment [11, 15], was unable to pick the most suitable GEM model, and therefore the investigation of the formulations of GEM remains an active field of theoretical research.

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