A new computation method of minimum dwell time for the global asymptotic stability of switched linear differential systems

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In this paper, switched linear systems are considered dwell and average dwell time for their global asymptotic stability is examined. Dwell and average dwell time are determined based on the condition number for the global asymptotic stability of switched linear differential systems. Numerical examples which show the effect of the results obtained are given with the new dwell and average dwell times.

Keywords: Average dwell time; global asymptotic stability; restricted dwell time; switched linear systems.

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1. Introduction

It is important to study the stability of the switched differential systems, since they are used in mathematical physics of many fields such as power systems, gravity, motor engine control, network control systems, constrained robotics, automotive engineering [1-15]. One of the ways to examine the stability of switched systems is to study the dwell and average dwell time of the system.

We consider the linear switched system described by

$$\dot{x}(t) = A_{\sigma(t)}x(t),\ \sigma \in \mathcal{S},\ t \geq 0, \tag{1}$$

where $x(t) = (x_i(t))$ is l dimensional vector, $x_i(t)$ $(i = 1, 2, ..., l)$ are differentiable functions, $\mathcal{P} = \{1, 2, \ldots, N\}$, $\{A_p \in \mathbb{C}^{l\times l}, p \in \mathcal{P}\}$ is matrix family, $\mathcal{S} = \{\sigma|\sigma : [0, \infty) \to \mathcal{P}, \sigma \text{ switching signal}\}$. The amount of time passed between the consecutive switching events is called dwell time of system (1).

Let us give the definition of globally asymptotically stability (GAS) for the point $x(t) = 0$, which is the trivial solution and the equilibrium point of the switched system (1).

The trivial solution of the system (1) is GAS for a given switching signal σ if (1) is

– Lyapunov stable, and

– uniformly globally asymptotically convergent, *i.e.*, for all $r, \varepsilon > 0$ there exists $T(r, \varepsilon) > 0$ such that $\|x(t)\| < \varepsilon$ for all $t > T(r, \varepsilon)$ whenever $\|x_0\| < r$.

If each subsystems are GAS then there exists a minimum dwell time that guarantees GAS of the system (1). For the system (1), let the following sets of switching signals be defined, where t_i's are successive switching time instants and $N_{\sigma(t)}$ is the number of switchings before time t:

$$\mathcal{S} = \mathcal{S}_{\text{dwell}}[\tau] = \{\sigma|t_{k+1} - t_k \geq \tau\},$$

$$\mathcal{S} = \mathcal{S}_{\text{average}}[\overline{\tau}, N_0] = \left\{\sigma|N_\sigma(t) \leq N_0 + \frac{t}{\overline{\tau}}\right\}.$$

Determination of the dwell or average dwell time is based on the calculation of the infimum of the numbers τ or $\overline{\tau}$ that makes the switched system GAS [16-18].

There are many studies on the dwell and average dwell time for the GAS of the system (1) [17-22]. These studies are generally used the eigenvalues of the coefficient matrices of the given system. It is well known that the eigenvalue problem is an ill-posed problem for non-symmetric matrices [23-25]. Moreover, if a matrix has multiple eigenvalues, or is close to a matrix with multiple eigenvalues, then its Jordan normal form is very sensitive to perturbations. This ill conditioning makes it difficult to develop a robust numerical algorithm for the Jordan normal form. So, the Jordan normal form is usually avoided in numerical computations [24-25].

A new method is proposed to determine the dwell and average dwell time without calculating the eigenvalue, in this paper. The proposed method depends on the $\kappa(A)$ parameter, which shows the quality of the GAS of the systems of differential equations [26-30]. "Dwell time" and "average dwell time" have not been studied depending on the $\kappa(A)$ parameter yet, in the literature. Therefore, the results obtained in this study are new and original.

This paper is structured as follows: In Sec. 2, preliminaries are given. In Sec. 3, the dwell time and average dwell time for GAS are determined. Finally, numerical examples are given in Sec. 4.

2. Preliminaries

2.1. Criterions of global asymptotic stability

Let $A \in \mathbb{C}^{l\times l}$, $x(t) = (x_i(t))$ is l dimensional vector and $x_i(t)$ $(i = 1, 2, ..., l)$ be differentiable functions. Consider the following differential equation system:

$$\dot{x}(t) = Ax(t),\ t \geq 0. \tag{2}$$

The differential equation system (2) is stable if for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$ such that $\|x(t)\| \leq \epsilon$ for $t \in [0, \infty)$ whenever for $\|x(0)\| \leq \delta$. Further, the system (2) is GAS if it is stable and $\|x(t)\| \to 0$ with increase t to infinity for all $x(0)$.

If the real parts of all eigenvalues of the matrix A in the system (2) is less than zero, then the matrix A is called a GAS matrix and the system (2) is also called a GAS system. This criterion is known as the "spectral criterion" in the literature [26-31].

Lyapunov theorem, another criterion for GAS, is as follows.

"The matrix A (trivial solution of the system (2)) is GAS if and only if there is a solution $H = H^* > 0$ of the Lyapunov matrix equation $A^*H + HA = -I$".

It means that if such H does exist then all the eigenvalues of matrix A lie strictly in the left-hand half-plane [26-30].

2.2. Global asymptotic stability parameter

As it is known, the eigenvalue problem is an ill-possed problem [23-25]. Therefore, instead of calculating eigenvalues, it should be preferred to study with parameters revealing the quality of the GAS.

GAS parameter of the system (2) is represented by $\kappa(A)$ and defined as:

$$\kappa(A) = 2\|A\|\|H\|,$$

where

$$H = \int_0^\infty e^{tA^*} e^{tA} dt$$

is the solution of Lyapunov matris equation,

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

is the spectral norm of the matrix A and $\|x\|$ is Euclidean norm for the vector $x = (x_1, x_2, ..., x_l)^T$. If $\kappa(A)$ is finite, then the system (2) is GAS. Otherwise, the system (2) is not GAS and we set $\kappa(A) = \infty$ [26-30].

Now let's consider the matrices

$$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & -7 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} -1 & 9 \\ 0 & -7 \end{pmatrix}$$

and illustrate that the parameter $\kappa(A)$ represents the quality of GAS. The eigenvalues of both matrices are "-1 and -7" and it can be easily seen that both matrices are GAS. But knowledge of the eigenvalues does not give information about the quality of the stability. However, since $\kappa(A_1) = 7 < \kappa(A_2) = 28.0881$, it is seen that the quality of GAS of matrix A_1 is better than the quality of GAS of matrix A_2. This means that the GAS of the matrix A_2 deteriorates than the GAS of matrix A_1 with less perturbation. For example; when A_1 and A_2 matrices are perturbed with matrix

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

matrix $A_1 + B$ is GAS, while $A_2 + B$ is not GAS.

As can be seen, while eigenvalues do not give an idea about the quality of GAS of a matrix, the parameter $\kappa(A)$ calculates the quality of GAS.

Now, let's give the upper bound of the matrix e^{At}, which depends on the parameter $\kappa(A)$ given in [27-28].

Theorem 1. *The following inequalitiy*

$$\left\|e^{At}\right\| \leq \sqrt{\kappa(A)} e^{-\frac{t\|A\|}{\kappa(A)}}, \tag{3}$$

is valid for the GAS matrix A [27-28].

2.3. Switching graph for switched linear differential systems

Let $\mathcal{D}$ be a digraph whose nodes are the subsystems of (1) and arcs are admissible switching. Let $\varepsilon = \{(i,j) \,|\text{switching from } i \text{ to } j \text{ is admissible}\}$ where $\mathcal{P}$ is the index set for system (1). Let the weight functions of the graph $\mathcal{D}$ be $\boldsymbol{w}^+$ and $\boldsymbol{w}^-$. In other words, for each switching, $\boldsymbol{w}^+$ and $\boldsymbol{w}^-$ indicate the switching cost and switching time, respectively on the set ε. A weighted switching graph is represented by notation $\mathcal{D} = \{\mathcal{P}, \varepsilon, \boldsymbol{w}^+, \boldsymbol{w}^-\}$.

The concepts to be used for the $\mathcal{D}$ graph are listed as follows.

$$\mathcal{S}_{\mathcal{D},\text{dwell}}[\tau] = \{\sigma \in \mathcal{S}_{\text{dwell}}[\tau] \,|\, (\sigma_k, \sigma_{k+1}) \in \varepsilon, k = 1, 2, ...\}: \text{ signal set for dwell time} \tag{4}$$

$$\mathcal{S}_{\mathcal{D},\text{average}}[\overline{\tau}, N_0] = \{\sigma \in \mathcal{S}_{average}[\overline{\tau}, N_0] \,|\, (\sigma_k, \sigma_{k+1}) \in \varepsilon, k = 1, 2, ...\}: \text{ signal set for average dwell time} \tag{5}$$

$$W_n = (\sigma_1, \sigma_2), (\sigma_2, \sigma_3), ..., (\sigma_n, \sigma_{n+1}): \text{ cycle (walk, path) in the digraph } \mathcal{D}$$

$$\boldsymbol{w}(W_n) = \sum_{k=1}^{n} \boldsymbol{w}(p_k, p_{k+1}): \text{ weight of a cycle for a weighted digraph } \mathcal{D}$$

$$\rho(\mathsf{C}) = \frac{\boldsymbol{w}^+(\mathsf{C})}{\boldsymbol{w}^-(\mathsf{C})}: \text{ cycle ratio of } \mathsf{C}$$

$$\mathcal{C}:\ \text{set of all cycles in }\ \mathcal{D}$$

$$\rho^*(\mathcal{D}) = \max_{\mathsf{C}\in\mathcal{C}} \rho(\mathsf{C}):\ \text{maximum cycle ratio}$$

$$\mu(\mathsf{C}) = \frac{\boldsymbol{w}^+(\mathsf{C})}{\|\mathsf{C}\|}:\ \text{Cycle mean of } \mathsf{C}$$

$$\|\mathsf{C}\|:\ \text{length of } \mathsf{C}$$

$$\mu^*(\mathcal{D}) = \max_{\mathsf{C}\in\mathcal{C}} \mu(\mathsf{C}):\ \text{maximum cycle mean.}$$

These concepts are available in Refs. [18,32-33].

3. Determination of dwell time for GAS

Let us give the following theorem, which gives the upper bound of the solution of the system (1) to use determining the dwell time for GAS.

Theorem 2. *The following equation is provided for the GAS matrix* A_p $(p = 1, 2, ..., N)$ *where* $x(t)$ *is the solution of the system* (1):

$$\|x(t)\| \le \left(\kappa(A_{\sigma_{n+1}})\kappa(A_{\sigma_1})\right)^{\frac{1}{4}} e^{-\frac{\left\|A_{\sigma_{n+1}}\right\|}{\kappa(A_{\sigma_{n+1}})}(t-t_n)} \Phi \|x(0)\| \tag{6}$$

where

$$\Phi = e^{\sum\limits_{i=1}^{n}\left[\frac{1}{4}\ln\left(\kappa(A_{\sigma_{i+1}})\kappa(A_{\sigma_i})\right) - \frac{\|A_{\sigma_i}\|}{\kappa(A_{\sigma_i})}(t_i - t_{i-1})\right]}.$$

Proof. Let the system (1) be given with GAS matrices A_p $(p = 1, 2, ..., N)$. The solution of system (1) is expressed as

$$x(t) = e^{A_{\sigma_{n+1}}(t-t_n)} e^{A_{\sigma_n}(t_n - t_{n-1})} ... e^{A_{\sigma_1}(t_1 - t_0)} x_0,\ t \in [t_n, t_{n+1})$$

or

$$x(t) = e^{A_{\sigma_{n+1}}(t-t_n)} \left(\prod_{i=1}^{n} e^{A_{\sigma i}(t_i - t_{i-1})}\right) x_0,\ t \in [t_n, t_{n+1}) \tag{7}$$

where $x(0) = x_0$ is the initial value of the system (1). By taking the norm of the solution (7) and applying the triangle inequality, the following inequality is obtained

$$\|x(t)\| = \left\| e^{A_{\sigma_{n+1}}(t-t_n)} \left(\prod_{i=1}^{n} e^{A_{\sigma i}(t_i - t_{i-1})}\right) x_0 \right\| \le \left\| e^{A_{\sigma_{n+1}}(t-t_n)} \right\| \prod_{i=1}^{n} \left\| e^{A_{\sigma i}(t_i - t_{i-1})} \right\| \|x_0\|.$$

If we use inequality (3), the upper bound of the solution is obtained as:

$$\begin{aligned}
\|x(t)\| &\le \sqrt{\kappa(A_{\sigma_{n+1}})} e^{-\frac{(t-t_n)\left\|A_{\sigma_{n+1}}\right\|}{\kappa(A_{\sigma_{n+1}})}} \prod_{i=1}^{n} \sqrt{\kappa(A_{\sigma_i})} e^{-\frac{(t_i - t_{i-1})\|A_{\sigma_i}\|}{\kappa(A_{\sigma_i})}} \|x_0\| \\
&= \left(\kappa(A_{\sigma_{n+1}})\kappa(A_{\sigma_1})\right)^{\frac{1}{4}} e^{-\frac{(t-t_n)\left\|A_{\sigma_{n+1}}\right\|}{\kappa(A_{\sigma_{n+1}})}} \prod_{i=1}^{n} \left(\kappa(A_{\sigma_{i+1}})\kappa(A_{\sigma_i})\right)^{\frac{1}{4}} e^{-\frac{(t_i - t_{i-1})\|A_{\sigma_i}\|}{\kappa(A_{\sigma_i})}} \|x(0)\|.
\end{aligned}$$

Therefore,

$$\|x(t)\| \le \left(\kappa(A_{\sigma_{n+1}})\kappa(A_{\sigma_1})\right)^{\frac{1}{4}} e^{-\frac{\left\|A_{\sigma_{n+1}}\right\|}{\kappa(A_{\sigma_{n+1}})}(t-t_n)} e^{\sum\limits_{i=1}^{n}\left[\frac{1}{4}\ln\left(\kappa(A_{\sigma_{i+1}})\kappa(A_{\sigma_i})\right) - \frac{\|A_{\sigma_i}\|}{\kappa(A_{\sigma_i})}(t_i - t_{i-1})\right]} \|x(0)\|,$$

holds.

Theorem 3. *The switched system* (1) *given by* (4) *is GAS for dwell times that provide the inequality* $\tau > \rho^*(\mathcal{D})$, *where* $\boldsymbol{w}^+(i,j) = (1/4)\ln\left(\kappa(A_{\sigma_j})\kappa(A_{\sigma_i})\right)$, $\boldsymbol{w}^-(i,j) = \left\|A_{\sigma_j}\right\| / \kappa(A_{\sigma j})$.

Proof. Suppose that $\sigma(t)$ has infinitely many switching. Because, when the switching signal has finitely switching, the system works in one of the subsystems after the last switching. Thus, since each subsystem is stable, the system (1) is GAS.
Let α be the weight of the walk W_n for the weight function $\boldsymbol{w}(i,j) = \boldsymbol{w}^+(i,j) - \boldsymbol{w}^-(i,j)$ in the switching graph $\mathcal{D}$. Any walk with m nodes consists of cycles and a path with a maximum length of $m-1$. Then it can be written as $\alpha(n) = \alpha_*(n) + \sum_{i=2}^{m} \alpha_i(n)$. Here for $i = 1,2,\ldots,m$, $\alpha_i(n)$ indicate the sum of the weights of all cycles with length i and $\alpha_*(n)$ indicates the weight of the path. Since P is finite, $\alpha_*(n)$ is bounded.
Let take us

$$\gamma = \max_i \left(\kappa(A_{\sigma_i})\kappa(A_{\sigma_1})\right)^{\frac{1}{4}}$$

and

$$\alpha(n) = \sum_{i=1}^{n} \left[\frac{1}{4}\ln\left(\kappa(A_{\sigma_{i+1}})\kappa(A_{\sigma_i})\right) - \frac{\|A_{\sigma_i}\|}{\kappa(A_{\sigma_i})}(t_i - t_{i-1})\right].$$

So, we can write (6) by the equation

$$\|x(t)\| \le \gamma e^{-\frac{\left\|A_{\sigma_{n+1}}\right\|}{\kappa(A_{\sigma_{n+1}})}(t-t_n)} e^{\alpha(n)} \|x(0)\|. \tag{8}$$

By taking $\tau \le t_i - t_{i-1}$ and $e^{-\frac{\left\|A_{\sigma_{n+1}}\right\|}{\kappa(A_{\sigma_{n+1}})}(t-t_n)} \le 1$ in (8), we obtain $\|x(t)\| \le \gamma e^{\alpha(n)} \|x(0)\|$.

Let us consider

$$\alpha(n) = \sum_{i=1}^{n} \left[\frac{1}{4}\ln\left(\kappa(A_{\sigma_{i+1}})\kappa(A_{\sigma_i})\right) - \frac{\|A_{\sigma_i}\|}{\kappa(A_{\sigma_i})}\tau\right] = \sum_{i=1}^{n} \left(\boldsymbol{w}^+(i,i+1) - \boldsymbol{w}^-(i,i+1)\tau\right),$$

for the walk W_n. Since $\tau > \rho^*(\mathcal{D})$ by the assumption, the limit of $\alpha(n)$ as n approaches infinity is $-\infty$. This means that upper bound (8) of the solution approaches zero as $t \to \infty$. Then, in the case of $\tau > \rho^*(\mathcal{D})$, system (1) is GAS.
Theorem 4. *The switched system* (1) *given by* (5) *is GAS for average dwell times that provide the inequality* $\overline{\tau} > \mu^*(\mathcal{D})/w^*$, *where* $w^* = \min_i \{\|A_{\sigma_i}\|/\kappa(A_{\sigma i})\}$.
Proof. Assume that $\sigma(t)$ has infinitely many switching, as in Theorem 3.

Consider W_n as the walk for the weight function $\boldsymbol{w}(\mathrm{i,j}) = \boldsymbol{w}^+(\mathrm{i,j})$ in the switching graph $\mathcal{D}$. Similar to Theorem 3, for any walk with m nodes, it can be written as $\beta(n) = \beta_*(n) + \sum_{i=2}^{m} \beta_i(n)$. Here for $i = 1,2,\ldots,m$, $\beta_i(n)$ indicate the sum of the weights of all cycles with length i and $\beta_*(n)$ indicates the weight of the path. Let us take $\beta(n) = \sum_{i=1}^{n} \ln\left(\kappa(A_{\sigma_{i+1}})\kappa(A_{\sigma_i})\right)^{1/4}$ and write the inequality (6) by the equation

$$\|x(t)\| \le \gamma e^{\beta(n) - w^* t} \|x(0)\|, \tag{9}$$

using assumption $w^* = \min_i\{\|A_{\sigma_i}\|/\kappa(A_{\sigma i})\}$.

If $\overline{\gamma} = \gamma e^{\max_W \boldsymbol{w}^+(W)}$, then the inequality (9) can be written as

$$\|x(t)\| \le \overline{\gamma} e^{\beta_2(n) + \ldots + \beta_m(n) - w^* t} \|x(0)\|. \tag{10}$$

Since $\beta_i(n)$ are cycles for $i = 1,2,\ldots,m$, we get $\sum_{i=2}^{m} \beta_i(n) \le N_\sigma(t)\,\mu^*(\mathcal{D}) \le N_0\mu^*(\mathcal{D}) + t\left(\mu^*(\mathcal{D})/\overline{\tau}\right)$.

Let define us $\overline{\overline{\gamma}} = \overline{\gamma} e^{N_0\mu^*(\mathcal{D})}$ and rewrite (10). So, the following inequality is obtained:

$$\|x(t)\| \le \overline{\overline{\gamma}} e^{\left(\frac{\mu^*(\mathcal{D})}{\overline{\tau}} - w^*\right)t} \|x(0)\|.$$

Since $(\mu^*(\mathcal{D})/\overline{\tau}) - w^* < 0$, the upper bound (10) of the solution approaches zero as $t \to \infty$. Then, system (1) is GAS.

4. Numerical examples

In this section, we give some numerical examples showing the efficiency of the results in Sec. 3.
Example 1. *Let us consider the following system consisting three GAS subsystems:*

$$A_1 = \begin{pmatrix} -1 & -9 \\ 5 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -3 & -2 \\ 8 & -4 \end{pmatrix} \quad \text{and} \quad A_3 = \begin{pmatrix} -2 & 4 \\ -4 & -10 \end{pmatrix},$$

$$\dot{x}(t) = A_i x(t), \qquad x(0) = [-8,8]^T, \qquad t \ge 0; \qquad i \in \{1,2,3\}, \tag{11}$$

Let $\mathcal{D}$ be the switching graph of the system (1) *given in* Fig. 1.

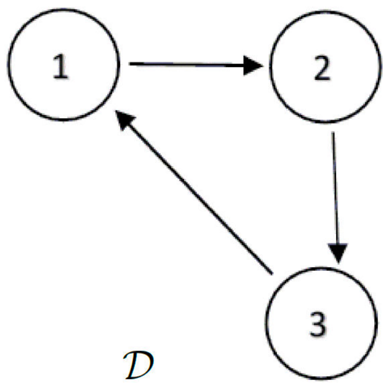

FIGURE 1. Switching graphs of the Cauchy problem consisting of three subsystems with A_1, A_2, and A_3.

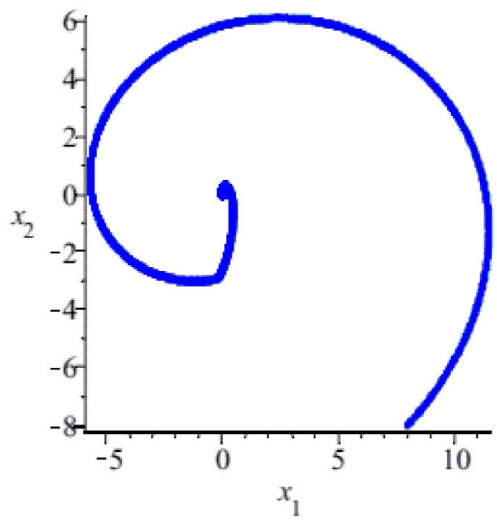

FIGURE 2. State trajectory with $\tau = 0.840262$.

For the graph $\mathcal{D}$, the minimum dwell time calculated in Theorem 3 is obtained as $\tau = 0.840262$. For this minimum dwell time, if the system is switched for graph $\mathcal{D}$, the solution curves given in the graph below are obtained.

Example 2. *Let us consider the systems (1) with four GAS subsystems. For $i \in \{1,2,3,4\}$, let matrices A_i be given as follows*

$$A_1 = \begin{pmatrix} -11 & 0.1 \\ 0 & -10 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -15 & 0.02 \\ 0 & -14 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -9 & 0.3 \\ 0 & -9 \end{pmatrix} \quad \text{and} \quad A_4 = \begin{pmatrix} -8.3 & 1 \\ -0.5 & -8 \end{pmatrix}.$$

For systems (1), *the switching graphs $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ are given in Fig. 4.*

In Table I, computed dwell and average dwell times for the switching graphs $\mathcal{D}_i \quad (i = 1,2,3)$ are given.

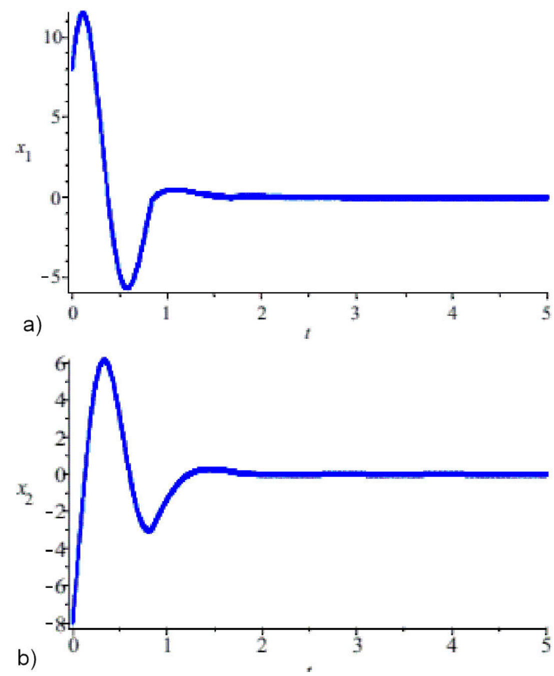

FIGURE 3. State trajectory with x_1 a) and x_2 b) of system (11) .

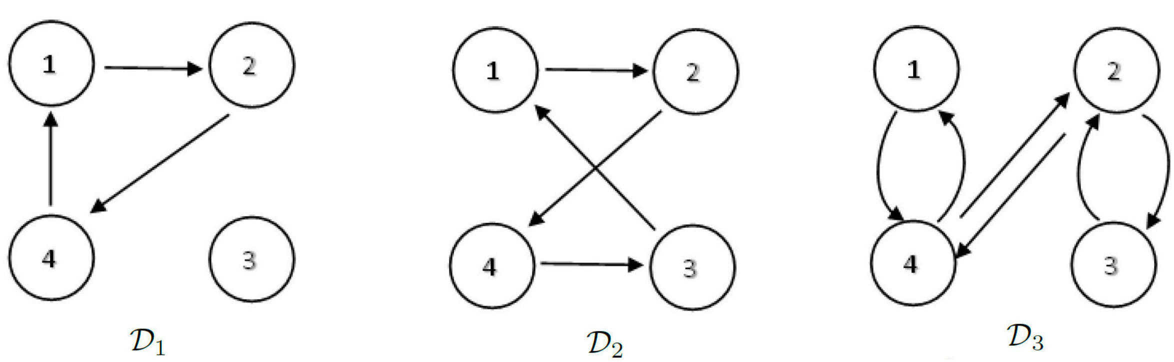

FIGURE 4. Switching graphs of the linear switched systems (1) consisting of four subsystems with A_1, A_2, A_3, and A_4.

TABLE I. Dwell and average dwell times for switching graphs of the linear switched systems (1) consisting of four subsystems with A_1, A_2, A_3 and A_4.

Switching	Dwell time		Average Dwell time	
Graph	Theorem 3	Karabacak [18]	Theorem 4	Karabacak [18]
$\mathcal{D}_1$	0.0037714	0.0300776	0.0050954	0.0395499
$\mathcal{D}_2$	0.00336263	*	0.00435395	*
$\mathcal{D}_3$	0.00479608	*	0.005875524	*

*As illustrated in Table I, for $\mathcal{D}_1$ switching graph, our dwell time (see Theorem 3) and average dwell time (see Theorem 4) values are better than the ones obtained by Karabacak [18]. Moreover, for $\mathcal{D}_2$ and $\mathcal{D}_3$ switching graphs, we are able to calculate dwell time from Theorem 3 and average dwell time from Theorem 4 although these values could not be calculated in Ref. [18] (because the A_3 is defective matrix). The values which we cannot calculate are denoted by the symbol * in Table I.*

5. Conclusion

In this paper, dwell and average dwell time, which make differential equation systems (1) GAS, are calculated in terms of the $\kappa(A)$ parameter without using eigenvalue. As far as we know, "dwell time" and "average dwell time" have not been studied depending on the $\kappa(A)$ parameter in the literature. Therefore, the results obtained in this paper are new and original.

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