The q-deformed heat equation and q-deformed diffusion equation with q-translation symmetry

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In this paper we consider the discrete heat equation with a certain non-uniform space interval which is related to q-addition appearing in the non-extensive entropy theory. By taking the continuous limit, we obtain the q-deformed heat equation. Similarly, we obtain the solution of the q-deformed diffusion equation.

Keywords: q-deformed; q-translation.

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1. Introduction

Heat equation governs how heat diffuses or transfers through a region, which was first introduced by Fourier [1] in 1822. In one dimension, this equation take the form,

$$\frac{\partial}{\partial t}u(x,t) = \kappa\left(\frac{\partial}{\partial x}\right)^2 u(x,t), \tag{1}$$

where $u(x,t)$ is the temperature at position x at time t and κ is thermal diffusivity.

In this paper we are to find a deformed heat equation. To do so we need the discrete version of heat equation where space is discrete but time is continuous. Discrete physics have been studied in various fields [2-15]. If we consider discrete positions denoted by

$$x_n = na, \quad n \in \mathbb{Z}, \tag{2}$$

we have the discrete heat equation,

$$\frac{\partial}{\partial t}u(x_n,t) = \kappa\Delta_x^2 u(x_n,t), \tag{3}$$

where finite difference operators are defined as

$$\Delta_x u(x_n,t) = \frac{u(x_{n+1},t) - u(x_n,t)}{a}\,. \tag{4}$$

If we take the limit $a \to 0$ in Eq. (4), we have Eq. (1). From Eq. (2), we know that

$$x_{n+1} - x_n = a, \tag{5}$$

which implies that the uniform space interval guarantees the heat equation of the form (1). In other words, if we consider a non-uniform discrete position, we will obtain another form of heat equation.

In this paper we consider the discrete heat equation with a certain non-uniform space interval which is related to q-addition or q-subtraction appearing in the non-extensive entropy theory [16-18]. By taking the continuous limit, we obtain the q-deformed heat equation. Similarly, we derive the q-deformed diffusion equation. This paper is organized as follows: In Sec. 2 we discuss the q-deformed heat equation. In Sec. 3 we discuss the solution of q-deformed heat equation. In Sec. 4 we discuss cooling of a rod from a constant initial temperature. In Sec. 5 we discuss the q-deformed diffusion equation.

2. q-deformed heat equation

In this section we discuss the q-deformed heat equation based on the the q-addition and q-subtraction appearing in the non-extensive thermodynamics [16-18]. As is different from the non-extensive thermodynamics, we introduce the parameter q so that it may have a dimension of inverse length. In the non-extensive thermodynamics, the parameter q is dimensionless. Thus, in the q-deformed heat equation, q can be regarded as $1/\xi$ where ξ denotes a length scale.

Now let us introduce the discrete position with non-uniform interval where the distance between adjacent positions are given by

$$x_{n+1} \ominus_q x_n = a, \tag{6}$$

or

$$x_{n+1} = x_n \oplus_q a, \tag{7}$$

where the q-addition and q-subtraction [16-18] are defined as

$$a \oplus_q b = a + b + qab, \tag{8}$$

$$a \ominus_q b = \frac{a-b}{1+qb}. \tag{9}$$

As is different from the uniform lattice, the non-uniform lattice consisting discrete points obeying Eq. (6) can be regarded as an example of the non-homogeneous medium in the continuous limit ($a \to 0$). We think that the discrete positions defined by the different pseudo addition (deformation of the ordinary addition) can give another examples of the non-homogeneous medium in the continuous limit. For example, in Ref. [19], the α-addition was introduced to describe the non-homogeneous medium where anomalous diffusion arose.

The Eq. (6) gives the relation

$$x_{n+1} = (1+qa)x_n + a \tag{10}$$

Solving Eq. (10) we get

$$x_n = \frac{1}{q}([1+qa]^n - 1), \tag{11}$$

When $q > 0$ we have

$$\lim_{n\to\infty} x_n = \infty \tag{12}$$

and

$$\lim_{n\to-\infty} x_n = -\frac{1}{q}\ . \tag{13}$$

When $q < 0$ we get

$$\lim_{n\to\infty} x_n = \frac{1}{|q|}, \tag{14}$$

and

$$\lim_{n\to-\infty} x_n = -\infty. \tag{15}$$

In this case we demand $|q|a < 1$. The discrete position is not symmetric for $x_0 = 0$. Indeed, we have

$$x_{-n} = -\frac{x_n}{(1+qa)^n}, \quad n \geq 1. \tag{16}$$

For the discrete positions obeying Eq. (6), the difference operator becomes

$$\Delta_{x:q} u(x_n,t) = \frac{u(x_{n+1},t) - u(x_n,t)}{x_{n+1} \ominus_q x_n} = (1+qx_n) \times \left(\frac{u(x_{n+1},t) - u(x_n,t)}{x_{n+1} - x_n}\right). \tag{17}$$

Thus, in the continuum limit, we get

$$\Delta_{x:q} u(x_n,t) \to D_x^q = (1+qx)\frac{\partial u}{\partial x}. \tag{18}$$

Here we know that the q-derivative D_x^q remains invariant under the q-translation $x \to x \oplus \delta x$. Recently, quantum theory with q-translation invariance was constructed in [20]. Using Eq. (18), we obtain the q-deformed heat equation with q-translation symmetry in the form,

$$\frac{\partial}{\partial t} u(x,t) = \kappa \left(D_x^q\right)^2 u(x,t). \tag{19}$$

3. Solution of q-deformed heat equation

Consider a rod of length L with the initial condition

$$u(x,0) = f(x), \tag{20}$$

and the boundary condition

$$u(0,t) = u(L,t) = 0. \tag{21}$$

We look for a solution of the form

$$u(x,t) = X(x)T(t). \tag{22}$$

Inserting Eq. (22) into Eq. (19) we get

$$\frac{1}{\kappa T}\frac{dT}{dt} = \frac{1}{X}(D_x^q)^2 X = -\lambda, \quad \lambda > 0. \tag{23}$$

Thus, we have

$$T(t) = e^{-\kappa\lambda t}, \tag{24}$$

and

$$X(x) = A\cos\left(\frac{\sqrt{\lambda}}{q}\ln(1+qx)\right) + B\sin\left(\frac{\sqrt{\lambda}}{q}\ln(1+qx)\right), \tag{25}$$

From the boundary function, we have $A = 0$ and

$$\sin\left(\frac{\sqrt{\lambda}}{q}\ln(1+qL)\right) = 0, \tag{26}$$

which gives

$$\sqrt{\lambda} = \lambda_n = \frac{qn\pi}{\ln(1+qL)}, \quad n = 1, 2, \cdots \tag{27}$$

Thus, the general solution of q-deformed wave equation is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(n\pi\frac{\ln(1+qx)}{\ln(1+qL)}\right) \times \exp\left(-\frac{\kappa q^2 n^2 \pi^2 t}{(\ln(1+qL))^2}\right). \tag{28}$$

Now let us apply the initial condition. Then we have

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(n\pi\frac{\ln(1+qx)}{\ln(1+qL)}\right). \tag{29}$$

If we use the orthogonality relation

$$\int_0^L \sin\left(n\pi\frac{\ln(1+qx)}{\ln(1+qL)}\right)\sin\left(m\pi\frac{\ln(1+qx)}{\ln(1+qL)}\right) \times \frac{dx}{1+qx} = \frac{\ln(1+qL)}{2q}\delta_{nm}, \tag{30}$$

we have

$$B_n = \frac{2q}{\ln(1+qL)} \int_0^L f(x) \times \sin\left(n\pi \frac{\ln(1+qx)}{\ln(1+qL)}\right) \frac{dx}{1+qx}. \tag{31}$$

Here we solved the q-heat equation in a closed form. Our method is to introduce the q-lattice as an example of the non-homogeneous medium, which is not related to the numerical solution methods based on adaptive grids [21-24] because we obtained the exact solution.

4. Cooling of a rod from a constant initial temperature

Suppose the initial temperature distribution $f(x)$ in the rod is constant, *i.e.* $f(x) = u_0$. Now let us consider the case of $L = 1$, $\kappa = 1$. Then we have

$$B_n = -\frac{2u_0}{n\pi}((-1)^n - 1) \tag{32}$$

Thus, we have

$$u(x,t) = \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left((2n-1)\pi \frac{\ln(1+qx)}{\ln(1+q)}\right) \times \exp\left(-\frac{q^2(2n-1)^2\pi^2 t}{(\ln(1+q))^2}\right). \tag{33}$$

In this case, the ratio of the first and second terms in Eq. (33) is

$$\frac{|\text{second term}|}{|\text{first term}|} = \frac{1}{3} e^{-\frac{8q^2\pi^2 t}{(\ln(1+q))^2}} \frac{\left|\sin\left(3\pi \frac{\ln(1+q)}{\ln(1+q)}\right)\right|}{\left|\sin\left(\pi \frac{\ln(1+qx)}{\ln(1+q)}\right)\right|}, \tag{34}$$

$$\leq e^{-\frac{8q^2\pi^2 t}{(\ln(1+q))^2}}, \tag{35}$$

$$\leq e^{-8} \quad \text{for } t \leq t_q, \tag{36}$$

where we used

$$|\sin nt| \leq n|\sin t|, \tag{37}$$

and

$$t_q = \frac{(\ln(1+q))^2}{q^2\pi^2}. \tag{38}$$

Thus, the first term dominates the sum of the rest of the terms, and hence

$$u(x,t) \approx \frac{4u_0}{\pi} \sin\left(\pi \frac{\ln(1+qx)}{\ln(1+q)}\right) \times \exp\left(-\frac{q^2\pi^2 t}{(\ln(1+q))^2}\right). \tag{39}$$

4.1. Spatial temperature profiles

Now let us consider fixed time. Here we consider the time $t = t_q$. Then we have

$$u(x,t) \approx \frac{4u_0}{\pi} e^{-1} \sin\left(\pi \frac{\ln(1+qx)}{\ln(1+q)}\right). \tag{40}$$

This has the maxima at $x = x_0$ where

$$x_0 = \frac{\sqrt{1+q} - 1}{q}. \tag{41}$$

Thus, center of a rod is not a line of symmetry unless $q = 0$. Fig. 1 shows the plot of u versus x with $u_0 = 1$ for $q = 0$ (Red), $q = 0.2$ (Brown), and $q = -0.2$ (Gray). We know that the position for maximum of u is smaller than $1/2$ for $q > 0$ while it is larger than $1/2$ for $q < 0$.

4.2. Temperature profiles in time

Setting $x = x_0$ in the approximate solution, we have

$$u(x,t) \approx \frac{4u_0}{\pi} \exp\left(-\frac{q^2\pi^2 t}{(\ln(1+q))^2}\right). \tag{42}$$

Figure 2 shows the plot of u versus t with $u_0 = 1, x = x_0$ for $q = 0$ (Red), $q = 0.2$ (Brown), and $q = -0.2$ (Gray).

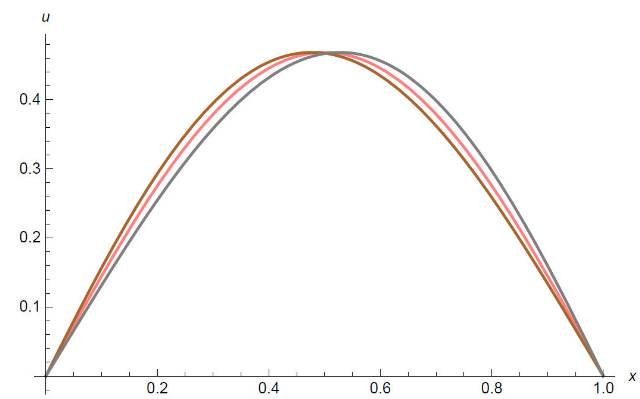

FIGURE 1. Plot of u versus x with $u_0 = 1$ for $q = 0$ (Red), $q = 0.2$ (Brown), and $q = -0.2$ (Gray).

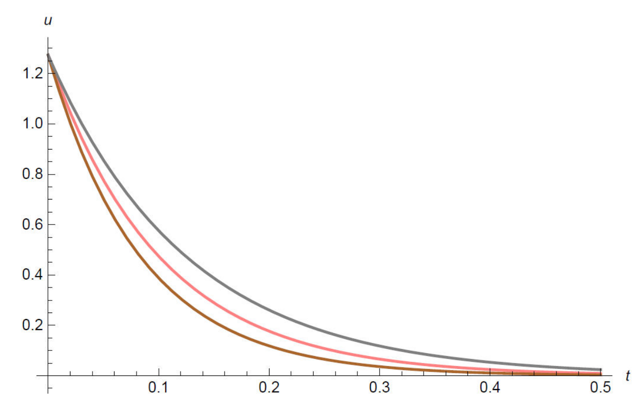

FIGURE 2. Plot of u versus t with $u_0 = 1, x = x_0$ for $q = 0$ (Red), $q = 0.2$ (Brown), and $q = -0.2$ (Gray).

5. q-deformed diffusion equation

The q-deformed diffusion equation has the same form as the q-deformed heat equation,

$$\frac{\partial}{\partial t}u(x,t) = D(D_x^q)^2 u(x,t), \tag{43}$$

where $u(x,t)$ denotes the concentration and D denotes diffusivity. Now let us impose the initial condition

$$u(x,0) = f(x). \tag{44}$$

Now let us introduce the q-deformed Fourier transform as

$$\mathcal{F}(u(x,t)) = U(w,t) = \begin{cases} \frac{1}{2\pi}\int\limits_{-1/q}^{\infty} \frac{dx}{1+qx} u(x,t)(1+qx)^{\frac{iw}{q}} & (q>0) \\ \frac{1}{2\pi}\int\limits_{-\infty}^{1/|q|} \frac{dx}{1+qx} u(x,t)(1+qx)^{\frac{iw}{q}} & (q<0) \end{cases}, \tag{45}$$

and the inverse q-deformed Fourier transform

$$\mathcal{F}^{-1}(U(w,t)) = u(x,t) = \int\limits_{-\infty}^{\infty} dw U(w,t)(1+qx)^{-\frac{iw}{q}}. \tag{46}$$

From the definition of q-deformed Fourier transform, we know

$$\mathcal{F}((D_x^q)^n u(x,t)) = (-iw)^n U(w,t). \tag{47}$$

Taking the q-deformed Fourier transform in Eq. (43) we get

$$\frac{\partial U(w,t)}{\partial t} = -Dw^2 U(w,t), \tag{48}$$

which is solved as

$$U(w,t) = c(w)e^{-Dw^2 t}. \tag{49}$$

Then we have

$$U(w,0) = c(w). \tag{50}$$

Thus we get

$$c(w) = \mathcal{F}(f(x)). \tag{51}$$

Now let us set

$$g(x) = \mathcal{F}^{-1}(e^{-Dw^2 t}), \tag{52}$$

which gives

$$g(x) = \sqrt{\frac{\pi}{Dt}} \exp\left(-\frac{1}{4Dt}\left[\frac{1}{q}\ln(1+qx)\right]^2\right). \tag{53}$$

Then we have

$$U(w,t) = \mathcal{F}(f(x))\mathcal{F}(g(x)). \tag{54}$$

From the convolution theorem we get

$$u(x,t) = \begin{cases} \frac{1}{2\pi}\int\limits_{-1/q}^{\infty} \frac{dx}{1+qx} f(s) g\left(\frac{1}{q}\ln\left(\frac{1+qx}{1+qs}\right)\right) & (q>0) \\ \frac{1}{2\pi}\int\limits_{-\infty}^{1/|q|} \frac{dx}{1+qx} f(s) g\left(\frac{1}{q}\ln\left(\frac{1+qx}{1+qs}\right)\right) & (q<0) \end{cases} \tag{55}$$

If we impose the initial condition

$$f(x) = \delta(x), \tag{56}$$

we have

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{1}{4Dt}\left[\frac{1}{q}\ln(1+qx)\right]^2\right). \tag{57}$$

From Eq. (57), the expectation value of x are

$$E(x) = \frac{1}{q} e^{q^2 Dt}(e^{3q^2 Dt} - 1). \tag{58}$$

For a small q, we get

$$E(x) \approx 3qDt. \tag{59}$$

The variance is then given by

$$V(x) = -\frac{2}{q^2} e^{5q^2 Dt}\big(\cosh(q^2 Dt) + \cosh(3q^2 Dt) - \cosh(4q^2 Dt) - \sinh(q^2 Dt) - 1\big). \tag{60}$$

For a small q, we get

$$V(x) \approx 2Dt + 16q^2(Dt)^2. \tag{61}$$

Figure 3 shows the plot of $u(x,t)$ versus x with $t = 1$ and $D = 1$ for $q = 0$ (Pink), $q = 0.2$ (Brown) and $q = -0.2$ (Gray). We know that the graph is asymmetric unless $q = 0$. Thus Eq. (57) is the asymmetric normal distribution. Figure 4 shows the plot of $u(x,t)$ versus x with $q = 0.2$ and $D = 1$ for $t = 1$ (Pink), $t = 2$ (Brown) and $t = 3$ (Gray). Figure 5 shows the plot of $u(x,t)$ versus x with $q = -0.2$ and $D = 1$ for $t = 1$ (Pink), $t = 2$ (Brown) and $t = 3$ (Gray).

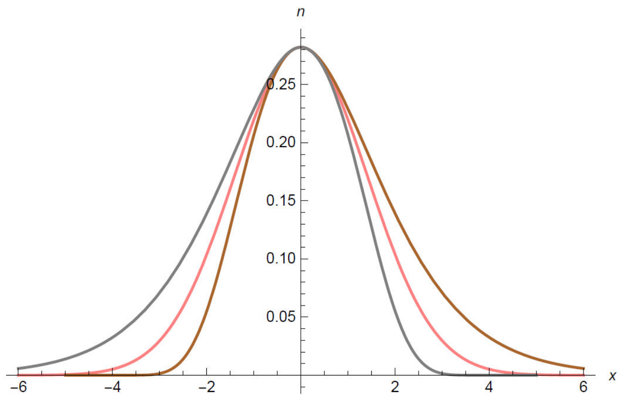

FIGURE 3. Plot of $u(x,t)$ versus x with $t = 1$ and $D = 1$ for $q = 0$ (Pink), $q = 0.2$ (Brown) and $q = -0.2$ (Gray).

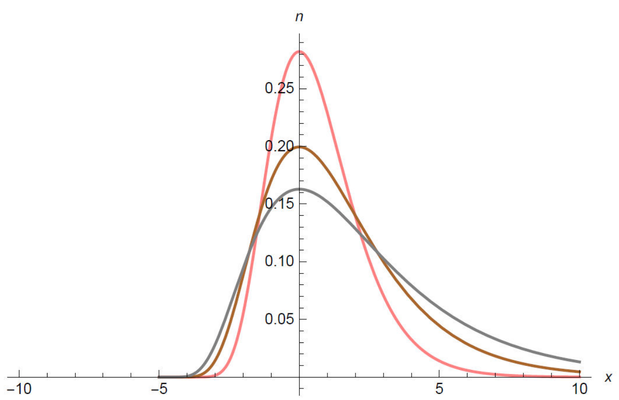

FIGURE 4. Plot of $u(x,t)$ versus x with $q = 0.2$ and $D = 1$ for $t = 1$ (Pink), $t = 2$ (Brown) and $t = 3$ (Gray).

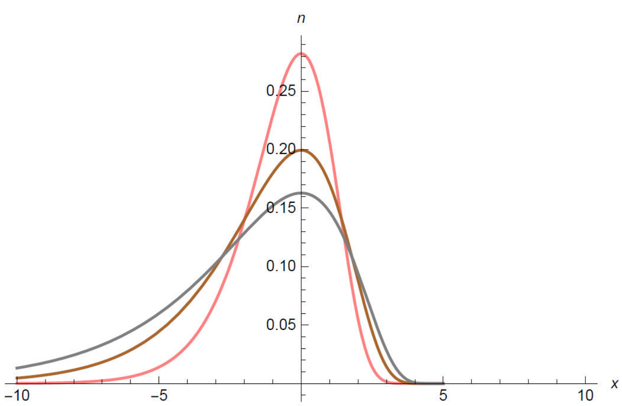

FIGURE 5. Plot of $u(x,t)$ versus x with $q = -0.2$ and $D = 1$ for $t = 1$ (Pink), $t = 2$ (Brown) and $t = 3$ (Gray).

6. Conclusion

In this paper we studied the q-deformed heat equation and q-deformed diffusion equation. From the fact that the ordinary heat equation was obtained by taking continuous limit in the discrete heat equation with a uniform space interval, we considered the discrete heat equation with a certain non-uniform space interval which was related to q-addition or q-subtraction appearing in the non-extensive thermodynamics. By taking the continuous limit, we obtained the q-deformed heat equation. We found that the q-deformed heat equation possessed the q-translation symmetry instead of the ordinary translation. We solved the q-deformed heat equation for a rod of length L. We discussed cooling of a rod from a constant initial temperature. We used the q-deformed Fourier transform to find the solution of the q-deformed diffusion equation. We found that the variance in x takes the form,

$$V(x) = -\frac{2}{q^2} e^{5q^2 Dt}\big(\cosh(q^2 Dt) + \cosh(3q^2 Dt) - \cosh(4q^2 Dt) - \sinh(q^2 Dt) - 1\big). \tag{62}$$

For a small q, we obtained

$$V(x) \approx 2Dt + 16q^2 (Dt)^2. \tag{63}$$

We found that the q-deformed diffusion process is asymmetric.

The q-addition and q-subtraction defined in the non-extensive thermodynamics was rarely used in the deformation of the space-time (q-deformed space time). The application to quantum mechanics was discussed in Ref. [20], application to mechanics was discussed in Ref. [25], and construction of q-lattice and q-Bloch theorem was discussed in Ref. [26].

Besides, we comment the connection of non-homogeneous media with the q-deformation of space briefly. It seems impossible to describe the general non-homogeneous media in an exact way without numerical study. Here, we adopted a special non-homogeneous media related to q-deformed space. Because asymmetry of the q-lattice, the graphs of temperature in the q-heat equation and concentration in the q-deformed equation became asymmetric, which had different feature for positive q and negative q, (See Fig. 1-5).

Finally, we compare the q-deformed diffusion equation with the diffusion model with the effective position dependent diffusion coefficient $\mathcal{D}(x)$ [27-29]. The diffusion equation with the effective position dependent diffusion coefficient $\mathcal{D}(x)$ was given in the form,

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(w(x)\mathcal{D}(x)\frac{\partial}{\partial x}\left[\frac{u(x,t)}{w(x)}\right]\right), \tag{64}$$

where $w(x)$ denotes the variable cross section. Comparing Eq. (43) with Eq. (64), we know

$$w(x) = 1 + qx, \quad \mathcal{D}(x) = D(1+qx)^2. \tag{65}$$

Thus we know that the q-deformed diffusion equation is an example of the diffusion model with the effective position dependent diffusion coefficient.

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