Spinor representation of curves and complexified forces on filaments

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We present a theoretical framework to study equilibrium configurations of filaments within a spinor representation of curves. The curve representing the filament is described by a unit two-component spinor field and its charge conjugate satisfying two-dimensional equations coupled by the curvature and torsion. The spinor field replaces the Frenet-Serret frame, whereas its structure equations replace the Frenet-Serret equations. Employing this spinorial description of curves, we derive the Euler-Lagrange equations of curves whose energies depend on their curvature and torsion. We analyze the conservation laws of the spinors representing the balance of the forces and torques along the filament. We illustrate this framework by applying these results to the Euler Elastica, whose bending energy is quadratic in the curvature.

Keywords: Geometric variational principles; spinors; filaments; bending energy; planar Euler elastica.

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1. Introduction

Often the physical properties of filaments are mainly described in terms of their geometry [1, 2]. The most relevant modes of deformation being their bending and twist, penalized by the squares of their curvature and of the sum of the torsion and the arc length derivative of the twist angle. Usually, these degrees of freedom are analyzed employing a vector basis adapted to the curve, being the Frenet-Serret (FS) frame the natural choice, although one can use any material frame instead [2, 3], or even a complexification of the normal curvatures and normal vectors [4, 5]. Furthermore, by means of the homomorphism of the group $\mathrm{SU}(2)$ onto $\mathrm{SO}(3)$, alternatively one can consider a two component spinor basis instead of the FS frame. In this approach a unit two component spinor and its charge conjugate play the role of the FS frame, whereas their structure equations correspond to the spinorization of the FS equations [6–10].

Although this correspondence between vectors and spinors has been studied in detail, their application to the development of variational principles is not so straightforward. As shown by one of the authors for the case of surfaces described within the generalized Weierstrass-Enneper representation, it is necessary to take into account the structure equations in the variational principles [11]. Furthermore, as demonstrated before, for geometric variational principles for curves and surfaces, the introduction of the definition of the tangent vector as the derivative of the embedding functions permits one to identify the vector or tensor representing the forces or stresses on the curve or surface [12–16]. However it is not obvious how to implement the definition of the embedding functions or the tangent vector in the variational principle using these spinors.

In this work, using spinorial quantities we develop variational principles for curves whose associated energy depends on their geometry. Although one could consider energies with additional degrees of freedom such as the twist, stretch or shear, whose equilibrium equations have been presented before using the usual vectorial framework [16–19], to illustrate this spinorial framework it suffices to consider an energy dependent only on the curvature and torsion. We enforce the definition of the curvature and torsion by introducing spinorial Lagrange multipliers implementing the structure equations of the spinor basis. To implement the definition of the tangent vector, we use its associated second rank spinor, as well as the one associated to the embedding functions. The definition of the former as the derivative of the latter spinor is implemented in the variational principle usin another second rank spinorial Lagrange multiplier. The vector associated to this spinorial Lagrange multiplier is identified as the force on the curve and is conserved. The normal projections of the conservation law of this force vector provide the Euler-Lagrange (EL) equations governing the equilibrium configurations of the curve, [14, 16, 17]. Also, the rotational invariance of the energy allows for the identification of the conserved vector and its associated second rank spinor representing the torques along the curve. To exemplify this spinorial framework, we apply it to the planar Euler-Elastica, whose energy is quadratic in the curvature. We determine the components of the complex force and EL equations, and we also determine the components of the spinors corresponding to the solutions describing wavelike curves.

This paper is organized as follows. We begin in Sec. 2 by reviewing the spinor basis associated to the complex FS frame adapted to a curve, along with their structure equations. Using this spinor representation we develop variational principles for curves in Sec. 3. In Sec. 4 we examine the Euclidean invariance of the energy leading to the identification of the force and torque vectors, as well as their associated spinors. These results are applied to the Euler Elastica in Sec. 5. We close with the discussion of our results and potential applications of our framework.

2. FS spinorial representation

The filament is described by a curve in three-dimensional space $\Gamma : s \to \mathbf{Y}(s) = \mathrm{Y}^i(s)\mathbf{E}_i \in \mathbb{E}^3$, parametrized by arc length s. The geometric quantities of the curve are described using the FS frame adapted to Γ, formed by the right-handed basis given by the principal normal, the binormal and the tangent vector, $\{\mathbf{N}, \mathbf{B}, \mathbf{T} = \mathbf{Y}'\}$, $(' = \mathrm{d}/\mathrm{d}s)$. The FS equations, describing the change of the FS basis along the filament in terms of the basis itself, are given by

$$\mathbf{N}' = -\kappa \mathbf{T} + \tau \mathbf{B}\,, \quad \mathbf{B}' = -\tau \mathbf{N}\,, \quad \mathbf{T}' = \kappa \mathbf{N}\,, \quad (1)$$

where κ and τ are the curvature and torsion of Γ [20, 21].

Instead of working with the FS frame, one can use the complexification of the normal vectors

$$\boldsymbol{\nu} = \mathbf{N} + i\mathbf{B}\,, \quad (2)$$

along with its complex conjugate (CC), $\bar{\boldsymbol{\nu}}$ (the overbar denotes complex conjugation) and the tangent vector, for they constitute an orthogonal trihedron satisfying

$$\boldsymbol{\nu} = i\,\mathbf{T} \times \boldsymbol{\nu}\,, \quad \bar{\boldsymbol{\nu}} = i\,\bar{\boldsymbol{\nu}} \times \mathbf{T}\,, \quad \mathbf{T} = \frac{i}{2}\,\boldsymbol{\nu} \times \bar{\boldsymbol{\nu}}\,. \quad (3)$$

Therefore, the complex normal vector is orthogonal to itself and to the tangent vector, while its norm is constant:

$$\boldsymbol{\nu} \cdot \boldsymbol{\nu} = \boldsymbol{\nu} \cdot \mathbf{T} = 0\,; \quad \|\boldsymbol{\nu}\|^2 = \boldsymbol{\nu} \cdot \bar{\boldsymbol{\nu}} = 2\,. \quad (4)$$

The structure equations of this complex basis are

$$\boldsymbol{\nu}' = -\kappa \mathbf{T} - i\tau \boldsymbol{\nu}\,, \quad \mathbf{T}' = \kappa\, \mathrm{Re}\, \boldsymbol{\nu}\,, \quad (5)$$

and the complex conjugate expression of $\boldsymbol{\nu}'$. Thus, the curvature and torsion are given by by

$$\kappa = \mathbf{T}' \cdot \mathrm{Re}\, \boldsymbol{\nu}\,, \quad \tau = \frac{1}{2}\mathrm{Im}\left(\bar{\boldsymbol{\nu}}' \cdot \boldsymbol{\nu}\right)\,. \quad (6)$$

Since the complex normal vector $\boldsymbol{\nu}$ and its complex conjugate are isotropic, we can express them and the tangent vector using a two-component spinor field and its charge conjugate $\psi(s)$ and $\tilde{\psi}(s)$ defined by [6, 9]

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\,, \quad \tilde{\psi} = \begin{pmatrix} -\bar{\psi}_2 \\ \bar{\psi}_1 \end{pmatrix}\,, \quad \psi_1, \psi_2 \in \mathbb{C}\,. \quad (7)$$

In terms of these spinors, the complex FS frame is given by $\boldsymbol{\nu} = \tilde{\psi}^\dagger \boldsymbol{\sigma} \psi$, $\bar{\boldsymbol{\nu}} = \psi^\dagger \boldsymbol{\sigma} \tilde{\psi}$ and $\mathbf{T} = (1/2)\left(\psi^\dagger \boldsymbol{\sigma} \psi - \tilde{\psi}^\dagger \boldsymbol{\sigma} \tilde{\psi}\right)$, where $\boldsymbol{\sigma} = \sigma^i \otimes \mathbf{E}_i$, with σ^i the Pauli matrices and $\mathbf{E}_i$ the canonical Euclidean basis. In full they read

$$\boldsymbol{\nu} = \begin{pmatrix} {\psi_1}^2 - {\psi_2}^2 \\ i\left({\psi_1}^2 + {\psi_2}^2\right) \\ -2\psi_1\psi_2 \end{pmatrix}\,, \quad (8a)$$

$$\mathbf{T} = \begin{pmatrix} 2\,\mathrm{Re}\left(\bar{\psi}_1\psi_2\right) \\ 2\,\mathrm{Im}\left(\bar{\psi}_1\psi_2\right) \\ |\psi_1|^2 - |\psi_2|^2 \end{pmatrix}\,, \quad (8b)$$

Since these spinor fields are orthogonal, $\tilde{\psi}^\dagger \psi = 0$, they are linearly independent so they form a spinor basis. The conditions that $\boldsymbol{\nu}$ has norm 2 and $\mathbf{T}$ is unit, impose the normalization of these spinor fields, that is $|\psi_1|^2 + |\psi_2|^2 = 1$. Thus we can express the completeness of this spinor basis as $\psi\psi^\dagger + \tilde{\psi}\tilde{\psi}^\dagger = \mathrm{I}$, where I is the 2×2 identity matrix.

The spinorial structure equations are obtained by using the spinorial expressions of the FS complex basis (8) in the complex FS Eqs. (5), which read [6, 9]

$$\psi' = \frac{1}{2}\left(-i\tau\psi + \kappa\tilde{\psi}\right)\,, \qquad \tilde{\psi}' = \frac{1}{2}\left(i\tau\tilde{\psi} - \kappa\psi\right)\,, \quad (9)$$

or in components

$$\psi_1' = -i\frac{\tau}{2}\psi_1 - \frac{\kappa}{2}\bar{\psi}_2\,, \quad \psi_2' = -i\frac{\tau}{2}\psi_2 + \frac{\kappa}{2}\bar{\psi}_1\,. \quad (10)$$

Thus, similar to the vectorial case, the curvature and torsion are given by the projections of the derivatives of the spinor basis $\kappa = \tilde{\psi}^\dagger \psi' - \psi^\dagger \tilde{\psi}'$ and $\tau = i\left(\psi^\dagger \psi' - \tilde{\psi}^\dagger \tilde{\psi}'\right)$, or in terms of spinor components

$$\kappa = 2\mathrm{Re}(\psi_1\psi_2' - \psi_2\psi_1')\,, \quad \tau = 2\mathrm{Im}(\psi_1\bar{\psi}_1' + \psi_2\bar{\psi}_2')\,. \quad (11)$$

In the next section we develop variational principles for curves within this spinor representation.

3. Spinorial variational principle: energy and forces

The energy density may depend on the spinor basis, but it should be in a manner that their combination transforms appropriately. To avoid such complications, we consider energies whose dependence on the spinors occurs through the curvature and torsion of the filament, *i.e.* in the form $\mathcal{L}(\kappa, \tau)$. For instance, in the harmonic approximation the principal energetic cost associated to a deformation of a filament is due to bending, which is quadratic in the curvature. Furthermore, one could also consider an energy penalizing the square of the torsion. Thus, up to quadratic order, the local energy ascribed to the filament is $\mathcal{L}(\kappa, \tau) = \mathcal{L}_B(\kappa) + \mathcal{L}_T(\tau)$, with $\mathcal{L}_B$ and $\mathcal{L}_T$, given by [17–19]

$$\mathcal{L}_B(\kappa) = \frac{\mathrm{k}}{2}\left(\kappa - \kappa_0\right)^2\,, \quad \mathcal{L}_T(\tau) = \frac{\mathrm{t}}{2}\left(\tau - \tau_0\right)^2\,, \quad (12)$$

where k and t are the bending and torsion moduli; κ_0 and τ_0 are the spontaneous curvature and torsion [1, 2].

In the following we consider an arbitrary energy density of the form $\mathcal{L}(\kappa,\tau)$, which integrated along the length of the curve provides the total energy,

$$L[\mathbf{Y}] = \int \mathcal{L}\left(\kappa,\tau\right) \mathrm{d}s\,, \tag{13}$$

where $\mathrm{d}s$ is the line element of Γ.

In the calculation of the variation of the total energy L under a change of the filament's embedding functions $\mathbf{Y} \to \mathbf{Y} + \delta\mathbf{Y}$, one has to take into account that the curvature and torsion are related to the spinor basis by the spinorial structure equations. The simplest way would be to implement the definitions of κ and τ given in Eqs. (11) using two real Lagrange multipliers. However, as shown in Appendix A such relations are insufficient, so the spinorial structure Eqs. (9) must be implemented in the variational principle using two spinor Lagrange multipliers, $\lambda = (\psi_1,\psi_2)^{\mathrm{T}}$ and its charge conjugate $\tilde{\lambda} = (-\bar{\psi}_2,\bar{\psi}_1)^{\mathrm{T}}$.

The identification of the forces on the curve in the variational principle involves the implementation of the definition of the tangent vector as the arc length derivative of the embedding functions. To do this we could introduce a term $\mathbf{F}\cdot(\mathbf{T}-\mathbf{Y}')$, where $\mathbf{F}$ is a real vector. However, instead of working with these real vectors, we can use the fact that the scalar product of two vectors is equal to one half of the trace of their associated second rank spinors. Thus the we can express such term in spinorial form as

$$\mathbf{F}\cdot(\mathbf{T}-\mathbf{Y}') = \frac{1}{2}\mathrm{tr}[\Phi(\Theta-\Upsilon')]\,, \tag{14}$$

where $\Theta = \mathbf{T}\cdot\boldsymbol{\sigma}$, $\Phi = \mathbf{F}\cdot\boldsymbol{\sigma}$, and $\Upsilon = \mathbf{Y}\cdot\boldsymbol{\sigma}$, are the second rank spinors associated to $\mathbf{T}$, $\mathbf{F}$ and $\mathbf{Y}$. Spanning these vectors as $\mathbf{F} = \mathrm{F}^i\mathbf{E}_i$ and $\mathbf{Y} = \mathrm{Y}^i\mathbf{E}_i$, $i = 1,2,3$, these matrices can be recast as

$$\Phi = \begin{pmatrix} \mathrm{F}^3 & \bar{\phi} \\ \phi & -\mathrm{F}^3 \end{pmatrix}\,, \quad \phi = \mathrm{F}^1 + i\mathrm{F}^2\,, \tag{15a}$$

$$\Theta = \begin{pmatrix} |\psi_1|^2 - |\psi_2|^2 & 2\psi_1\bar{\psi}_2 \\ 2\bar{\psi}_1\psi_2 & |\psi_2|^2 - |\psi_1|^2 \end{pmatrix}\,, \tag{15b}$$

$$\Upsilon = \begin{pmatrix} \mathrm{Y}^3 & \bar{\upsilon} \\ \upsilon & -\mathrm{Y}^3 \end{pmatrix}\,, \quad \upsilon = \mathrm{Y}^1 + i\mathrm{Y}^2\,. \tag{15c}$$

These matrices are traceless and Hermitian by construction, so they have only three independent components [21, 22].

We also have to impose the normalization of the spinor basis, which implies the normalization of the tangent vector and this in turn implies the parametrization by arc-length. This can be done by introducing a real Lagrange multiplier Λ imposing the unit norm of the spinor basis.

Taking into account these facts, we consider the effective spinorial energy $L_E = \int \mathcal{L}_E \mathrm{d}s$, where the energy density is defined by

$$\begin{aligned}\mathcal{L}_E &= \mathcal{L} + \frac{1}{2}\mathrm{tr}[\Phi(\Theta-\Upsilon')] \\ &\quad + \lambda^\dagger\left(\psi' - \frac{1}{2}\left(-i\tau\psi + \kappa\tilde{\psi}\right)\right) \\ &\quad + \tilde{\lambda}^\dagger\left(\tilde{\psi}' - \frac{1}{2}\left(i\tau\tilde{\psi} - \kappa\psi\right)\right) \\ &\quad + \Lambda\left(\frac{1}{2}\left(\psi^\dagger\psi + \tilde{\psi}^\dagger\tilde{\psi}\right) - 1\right)\,,\end{aligned} \tag{16}$$

In full this effective energy reads

$$\begin{aligned}\mathcal{L}_E &= \mathcal{L} + \mathrm{Re}\left[\phi\left(2\psi_1\bar{\psi}_2 - \bar{\upsilon}'\right)\right] \\ &\quad + \mathrm{F}^3\left(|\psi_1|^2 - |\psi_2|^2 - \mathrm{Y}^{3\prime}\right) \\ &\quad + \mathrm{Re}\left[\bar{\lambda}_1\left(2\psi_1' + i\tau\psi_1 + \kappa\bar{\psi}_2\right)\right] \\ &\quad + \mathrm{Re}\left[\bar{\lambda}_2\left(2\psi_2' + i\tau\psi_2 - \kappa\bar{\psi}_1\right)\right] \\ &\quad + \Lambda\left(|\psi_1|^2 + |\psi_2|^2 - 1\right)\,.\end{aligned} \tag{17}$$

Thus, we can vary independently the embedding functions, the curvature, the torsion, the spinor components and their CC.

The EL equations obtained from the variation of the spinors Φ, λ, and $\tilde{\lambda}$, reproduce the definition of the tangent vector and the structure equations of the spinor basis, given in Eqs. (9).

The EL equations obtained from the variations with respect to the curvature and torsion are

$$\varepsilon_\kappa := \frac{\delta\mathcal{L}_E}{\delta\kappa} = \mathcal{L}_\kappa + \mathrm{Re}(\psi_2\lambda_1 - \psi_1\lambda_2) = 0\,, \tag{18a}$$

$$\varepsilon_\tau := \frac{\delta\mathcal{L}_E}{\delta\tau} = \mathcal{L}_\tau + \mathrm{Im}\left(\bar{\psi}_1\lambda_1 + \bar{\psi}_2\lambda_2\right) = 0\,, \tag{18b}$$

where we have defined the the derivatives of the energy density with respect to the curvature and torsion by

$$\mathcal{L}_\kappa = \frac{\partial\mathcal{L}}{\partial\kappa}\,, \quad \mathcal{L}_\tau = \frac{\partial\mathcal{L}}{\partial\tau}\,. \tag{19}$$

Solving these equations we get $\lambda = \alpha\psi + \beta\tilde{\psi}$ and $\tilde{\lambda} = \bar{\alpha}\tilde{\psi} - \bar{\beta}\psi$, or in components

$$\lambda_1 = \alpha\psi_1 - \beta\bar{\psi}_2\,, \quad \lambda_2 = \alpha\psi_2 + \beta\bar{\psi}_1\,, \tag{20}$$

and their CC expressions, where we have defined

$$\alpha = r_1 - i\mathcal{L}_\tau\,, \quad \beta = \mathcal{L}_\kappa + ir_2\,, \tag{21}$$

and r_1, $r_2 \in \mathbb{R}$ are two real scalar functions to be determined.

Using these results in the EL equation for the spinor components, we obtain

$$\varepsilon_1 := \frac{\delta \mathcal{L}_E}{\delta \psi_1} = \bar{\psi}_1 \left(\mathrm{F}_3 - r_1' + \Lambda - i \left(\mathcal{L}_\tau' + \kappa r_2 \right) \right) + \psi_2 \left(\mathcal{L}_\kappa' - \tau r_2 + i (\kappa \mathcal{L}_\tau - \tau \mathcal{L}_\kappa - r_2') \right) + \phi \bar{\psi}_2 = 0 \,, \tag{22a}$$

$$\varepsilon_2 := \frac{\delta \mathcal{L}_E}{\delta \psi_2} = \bar{\psi}_2 \left(-\mathrm{F}_3 - r_1' + \Lambda - i (\mathcal{L}_\tau' + \kappa r_2) \right) - \psi_1 \left(\mathcal{L}_\kappa' - \tau r_2 + i (\kappa \mathcal{L}_\tau - \tau \mathcal{L}_\kappa - r_2') \right) + \bar{\phi} \bar{\psi}_1 = 0 \,. \tag{22b}$$

The EL equations for the CC spinors, $\bar{\varepsilon}_1 := \delta \mathcal{L}_E / \delta \bar{\psi}_1 = 0$ and $\bar{\varepsilon}_2 := \delta \mathcal{L}_E / \delta \bar{\psi}_2 = 0$ provide the CC of Eqs (22).

The combination $\mathrm{Im}(\psi_1 \varepsilon_1 + \psi_2 \varepsilon_2) = 0$ determines r_2

$$r_2 = - \frac{\mathcal{L}_\tau'}{\kappa} \,. \tag{23}$$

From the variations with respect to the embedding spinors we get that the components of Φ are conserved

$$\frac{\delta \mathcal{L}_E}{\delta \upsilon} = \bar{\phi}' = 0 \,, \qquad \frac{\delta \mathcal{L}_E}{\delta \mathrm{Y}^3} = \mathrm{F}^{3\prime} = 0 \,. \tag{24}$$

along with the CC of the first equation. Thus $\Phi' = 0$ or $\mathbf{F}' = 0$. In terms of these components we can express the vector as $\mathbf{F} = \mathrm{Re}(\bar{\phi} \boldsymbol{\mathcal{E}}) + \mathrm{F}^3 \mathbf{E}_3$, where $\boldsymbol{\mathcal{E}} = \mathbf{E}_1 + i \mathbf{E}_2$. The components ϕ and F_3 can be determined from the combinations $\psi_2 \varepsilon_1 + \bar{\psi}_1 \bar{\varepsilon}_2 = 0$ and $\mathrm{Re} \left(\psi_1 \varepsilon_1 - \psi_2 \varepsilon_2 \right) = 0$, respectively. However, it is more convenient to span the vector $\mathbf{F}$ in the complex FS basis $\{ \boldsymbol{\nu}, \bar{\boldsymbol{\nu}}, \mathbf{T} \}$, as $\mathbf{F} = \mathrm{Re}(\mathrm{F}^\nu \bar{\boldsymbol{\nu}}) + \mathrm{F}^T \mathbf{T}$.

To determine the normal complex components F^ν and its CC, as well as the tangential component F^T, we express them in terms of their Euclidean counterparts as

$$\mathrm{F}^\nu = \phi \psi_1^2 - \bar{\phi} \psi_2^2 - 2 \mathrm{F}^3 \psi_1 \psi_2 \,, \tag{25a}$$

$$\mathrm{F}^T = 2 \mathrm{Re}(\phi \psi_1 \bar{\psi}_2) + \mathrm{F}^3 (|\psi_1|^2 - |\psi_2|^2) \,. \tag{25b}$$

Using these expressions, along with Eq. (23) in the combinations $\psi_1 \bar{\varepsilon}_2 - \psi_2 \bar{\varepsilon}_1 = 0$ and $\mathrm{Re}(\psi_1 \varepsilon_1 + \psi_2 \varepsilon_2) = 0$, where the EL derivatives ε_i, $i = 1, 2$, are defined in Eqs. (22), we get that the complex normal component F^ν and the tangential component F^T are given by

$$\mathrm{F}^\nu = \mathcal{L}_\kappa' + \frac{\tau}{\kappa} \mathcal{L}_\tau' - i \left(\left[\frac{\mathcal{L}_\tau'}{\kappa} \right]' + \kappa \mathcal{L}_\tau - \tau \mathcal{L}_\kappa \right) , \tag{26a}$$

$$\mathrm{F}^T = r_1' - \Lambda \,. \tag{26b}$$

Having determined the complex normal component, we can readily obtain the real components along the principal normal and the binormal, given by $\mathrm{F}^N = \mathbf{F} \cdot \mathbf{N} = \mathrm{Re} \mathrm{F}^\nu$ and $\mathrm{F}^B = \mathbf{F} \cdot \mathbf{N} = \mathrm{Im} \mathrm{F}^\nu$.

At this point r_1 and Λ cannot be determined from a combination of the EL equations, because as as shown in Appendix B, they can be determined from the definition of the torques on the curve, defined below in Sec. 4. Despite this fact, the tangential component F^T can be determined from the structure equations of the complex FS frame. We have that arc-length derivative of $\mathbf{F}$ is $\mathbf{F}' = \mathrm{Re} \left(\varepsilon^\nu \bar{\boldsymbol{\nu}} \right) + \varepsilon^T \mathbf{T} = 0$, where

$$\varepsilon^\nu = \mathrm{F}^{\nu\prime} + i \tau \mathrm{F}^\nu + \kappa \mathrm{F}^T \,, \tag{27a}$$

$$\varepsilon^T = \mathrm{F}^{T\prime} - \kappa \mathrm{Re} \mathrm{F}^\nu \,. \tag{27b}$$

Using the identity

$$\mathcal{L}' = (\kappa \mathcal{L}_\kappa + \tau \mathcal{L}_\tau)' - \kappa \mathcal{L}_\kappa' - \tau \mathcal{L}_\tau' \,, \tag{28}$$

and Eq. (26a) we get that

$$\kappa \mathrm{Re} \mathrm{F}^\nu = (\kappa \mathcal{L}_\kappa + \tau \mathcal{L}_\tau - \mathcal{L})' \,. \tag{29}$$

Thus ε^T is a total derivative, which permits us to determine the tangential component

$$\mathrm{F}^T = \kappa \mathcal{L}_\kappa + \tau \mathcal{L}_\tau - \mathcal{L} - \mu \,, \tag{30}$$

where μ is a constant of integration. This constant of integration can be identified as the Hamiltonian density associated to the energy density, whose conservation stems from the fact that $\mathcal{L}$ does not depend explicitly on s [14,23]. Also, one could consider the energy density $\mathcal{L} + \mu$ and obtain the same EL equations, but in such case μ would be interpreted as a global Lagrange multiplier fixing the total lenght of the curve.

This completes the determination of the components of the spinor Φ.

The EL equations governing the critical points of the energy are given by Eq. (27a) and its CC, which upon substitution of the components given in Eqs. (26a) and (30), read $\varepsilon^\nu = \varepsilon^N + i \varepsilon^B = 0$, where

$$\varepsilon^N = \mathcal{L}_\kappa'' + 2 \tau \left(\frac{\mathcal{L}_\tau'}{\kappa} \right)' + \tau' \left(\frac{\mathcal{L}_\tau'}{\kappa} \right) + (\kappa^2 - \tau^2) \mathcal{L}_\kappa + 2 \kappa \tau \mathcal{L}_\tau - \kappa (\mathcal{L} + \mu) = 0 \,, \tag{31a}$$

$$\varepsilon^B = - \left(\frac{\mathcal{L}_\tau'}{\kappa} \right)'' - \left(\kappa - \frac{\tau^2}{\kappa} \right) \mathcal{L}_\tau' + \tau' \mathcal{L}_\kappa - \kappa' \mathcal{L}_\tau + 2 \tau \mathcal{L}_\kappa' = 0 \,, \tag{31b}$$

are the EL equations corresponding to directions along the FS normal [17]. Thus the vanishing of the real and imaginary parts reproduce the EL for the FS frame, just as in the case where the complex FS frame is used [5]. It is straightforward to verify that these EL equations agree with the EL Eqs. (24).

4. Identification of the force and torque spinors

The change of the energy has two contributions due to the variations of the bulk and the boundary, given by

$$\delta L = \int \left(\frac{1}{2}\mathrm{tr}(\Phi' \delta\Upsilon) + \delta Q \right) , \tag{32}$$

where $\delta Q = -(1/2)\mathrm{tr}(\Phi\delta\Upsilon) + \lambda^\dagger \delta\psi + \tilde{\lambda}^\dagger \delta\tilde{\psi}$. In components, we have

$$\delta L = \int \left[\mathrm{Re}\left(\varepsilon^\nu \delta\bar{\mathrm{Y}}^\nu\right) + \delta Q' \right] \mathrm{d}s\,, \tag{33}$$

where

$$\delta Q = \mathrm{Re}\left[2\left(\bar{\lambda}_1 \delta\psi_1 + \bar{\lambda}_2 \delta\psi_2\right) - \phi\delta\bar{v} \right] - \mathrm{F}^3 \delta\Upsilon^3 . \tag{34}$$

In equilibrium $\varepsilon^\nu = 0$, so the first order variation of the energy is given by the boundary terms.

Taking into account that the energy depends only on the curvature and torsion, we have that it is invariant under translations and rotations.

The translational invariance of the energy applied to the boundary terms reproduces the conservation law of Φ. Instead, we consider a constant infinitesimal translation of a boundary of the curve, $\delta\Upsilon = \delta\Upsilon_0$, the FS frame and its associated spinor basis do not change, $\delta\psi = \delta\tilde{\psi} = 0$. Therefore, the change in the energy due to the boundary change is

$$\delta L = \frac{1}{2}\mathrm{tr}\left(\Phi\delta\Upsilon_0\right) = \mathbf{F}\cdot\delta\mathbf{Y}\,, \tag{35}$$

so the constant vector $\mathbf{F}$ (or its associated spinor Φ) is identified as the force on the boundary [17, 24]. On account of the conservation of $\mathbf{F}$ along the curve, we have that it represents the force exerted by a line element of the curve on its neighbor segment with greater arc lenght. Moreover, its constant norm, $1/2\mathrm{tr}\,\Phi^2 = \mathbf{F}\cdot\mathbf{F} = \mathrm{F}^2$, which in full reads

$$\mathrm{F}^2 = \left(\mathcal{L}'_\kappa + \frac{\tau}{\kappa}\mathcal{L}'_\tau\right)^2 + \left(\left[\frac{\mathcal{L}'_\tau}{\kappa}\right]' + \kappa\mathcal{L}_\tau - \tau\mathcal{L}_\kappa \right)^2 + \left(\kappa\mathcal{L}_\kappa + \tau\mathcal{L}_\tau - \mathcal{L} - \mu\right)^2 , \tag{36}$$

provides a first integral of the EL Eqs. (31).

We now consider the energy invariance under rotations. Under a rotation defined by the constant vector $\boldsymbol{\omega} = \mathrm{Re}\left(\omega^\nu\bar{\boldsymbol{\nu}}\right) + \omega^T\mathbf{T}$, the change in the embedding functions is $\delta\mathbf{Y} = \boldsymbol{\omega}\times\mathbf{Y}$. In terms of the their associated second rank spinors $\Omega = \boldsymbol{\omega}\cdot\boldsymbol{\sigma}$ and Υ, such change is given by their commutator $\delta\Upsilon = 1/(2i)[\Omega,\Upsilon]$. Under a rotation the spinors transform as $\psi \to U\psi$ and $\tilde{\psi} \to U\tilde{\psi}$, where $U = e^{-1/2\boldsymbol{\omega}\cdot\boldsymbol{\sigma}} \in \mathrm{SU}(2)$, is the second rank spinor associated to the rotation. For an infinitesimal constant vector $\delta\boldsymbol{\omega}$, the spinor is given by $\delta U = -\frac{i}{2}\delta\boldsymbol{\omega}\cdot\boldsymbol{\sigma}$, where $\delta\boldsymbol{\omega}\cdot\boldsymbol{\sigma} = \mathrm{Re}(\delta\omega^\nu N^\dagger) + \delta\omega^T\Theta$, with $N = \boldsymbol{\nu}\cdot\boldsymbol{\sigma}$ and $\Theta = \mathbf{T}\cdot\boldsymbol{\sigma}$ as defined by Eq. (15b). Therefore, under a constant infinitesimal rotation of the curve, the changes in the components of the spinor basis are given by

$$\delta\psi_1 = -\frac{i}{2}\left(\delta\omega^T\psi_1 - \delta\omega^\nu\bar{\psi}_2\right) , \tag{37a}$$

$$\delta\psi_2 = -\frac{i}{2}\left(\delta\omega^T\psi_2 + \delta\omega^\nu\bar{\psi}_1\right) . \tag{37b}$$

Under this change of the spinor basis the curvature and torsion do not change, $\delta\kappa = 0$ and $\delta\tau = 0$.

Using these expressions, we have that under a constant infinitesimal rotation of the curve, the change in the energy is given by

$$\delta L = \frac{1}{2}\int \mathrm{tr}\left(\delta\Omega\mathrm{M}'\right)\mathrm{d}s\,, \tag{38}$$

where we have defined the second rank spinor,

$$\mathrm{M} = \frac{1}{2i}\left[\Upsilon,\Phi\right] + \Sigma\,, \tag{39}$$

with

$$\Sigma = \mathrm{Re}\left(\left[\frac{\mathcal{L}'_\tau}{\kappa} + i\mathcal{L}_\kappa\right] N^\dagger \right) + \mathcal{L}_\tau\Theta\,. \tag{40}$$

It follows from Eq. (38) that M is conserved $\mathrm{M}' = 0$. In consequence the derivative of Σ is given by

$$\Sigma' = \frac{1}{2i}[\Phi, \mathrm{E}_3]\,. \tag{41}$$

The vector associated to the spinor M is

$$\mathbf{M} = \mathbf{Y}\times\mathbf{F} + \mathbf{S}\,, \tag{42}$$

where $\mathbf{S} = \mathrm{Re}\left(\mathrm{S}^\nu\bar{\boldsymbol{\nu}}\right) + \mathrm{S}^T\mathbf{T}$, with

$$\mathrm{S}^\nu = \frac{\mathcal{L}'_\tau}{\kappa} + i\mathcal{L}_\kappa\,, \quad \mathrm{S}^T = \mathcal{L}_\tau\,. \tag{43}$$

By an argument similar to the one used to identify the force spinor, the consideration of the rotation of one boundary of the curve leads to identification of $\mathbf{M}$ and M as the vector and spinor describing the torques on the curve.

From the conservation of the force and torque spinors we have a second conserved quantity, $J = (1/2)\,\mathrm{Tr}(\Phi\mathrm{M}) = \mathbf{F}\cdot\mathbf{M} = \mathbf{F}\cdot\mathbf{S} = \mathrm{Re}\left(\mathrm{F}^\nu\bar{\mathrm{S}}^\nu\right) + \mathrm{F}^T\mathrm{S}^T$, which in full reads

$$J = \mathcal{L}'_\kappa\left(\frac{\mathcal{L}'_\tau}{\kappa}\right) - \mathcal{L}_\kappa\left(\frac{\mathcal{L}'_\tau}{\kappa}\right)' + \tau\left(\left[\frac{\mathcal{L}'_\tau}{\kappa}\right]^2 + \mathcal{L}_\kappa^2 + \mathcal{L}_\tau^2 \right) - \mathcal{L}_\tau(\mathcal{L}+\mu)\,. \tag{44}$$

In the derivation of the force and torque spinors it was unnecesary to determine the Lagrange multiplier Λ and the scalar function r_1. However, they can be determined from the definition of the intrinsic torque spinor Σ as shown in Appendix B.

In the next section we apply this framework to derive the complex forces and the equilibrium equations of the Euler Elastica.

5. Planar Euler Elastica

For the classic Euler-Elastica, the associated energy is due to bending, $\mathcal{L} = \mathcal{L}_B$, defined in Eq. (12). In this case we have $\mathcal{L}_\kappa = \mathrm{k}(\kappa - \kappa_0)$ and $\mathcal{L}_\tau = 0$. The scaled components of the complex force vector, $\mathrm{f}^\nu := \mathrm{F}^\nu/\mathrm{k}$ and $\mathrm{f}^T := \mathrm{F}^T/\mathrm{k}$, are

$$\mathrm{f}^\nu = \kappa' + i\tau(\kappa - \kappa_0)\,, \quad \mathrm{f}^T = \frac{1}{2}\left(\kappa^2 - \kappa_0^2\right) - \zeta\,, \tag{45}$$

where $\zeta = \mu/\mathrm{k}$. The complex EL equation reads

$$\frac{\varepsilon^\nu}{\mathrm{k}} = \kappa'' + (\kappa - \kappa_0)\left(\frac{\kappa}{2}(\kappa + \kappa_0) - \tau^2\right) - \zeta\kappa + i\left((\kappa - \kappa_0)\tau' + 2\kappa'\tau\right) = 0\,. \tag{46}$$

The real and the imaginary parts provide the EL associated with deformations along the two normal directions. The rescaled first integrals, $f := \mathrm{F}/\mathrm{k}$ and $j := J/\mathrm{k}$ are

$$f^2 = (\kappa')^2 + \tau^2(\kappa - \kappa_0)^2 + \left(\frac{\kappa^2 - \kappa_0^2}{2} - \zeta\right)^2, \tag{47a}$$

$$j = \tau\left(\kappa - \kappa_0\right)^2\,. \tag{47b}$$

It is straightforward to check that the combination of their derivatives reproduce the real and imaginary parts of the complex EL Eq. (46). The second equation determines the torsion as a function of the curvature, which substituted in the first equation results in a quadrature for the curvature,

$$(\kappa')^2 + \left(\frac{\kappa^2 - \kappa_0^2}{2} - \zeta\right)^2 + \frac{j^2}{\left(\kappa - \kappa_0\right)^2} = f^2\,, \tag{48}$$

whose solutions are given in terms of Jacobi elliptic functions [25, 26]. Having determined the curvature and the torsion one has to solve the Eqs. (9) to determine the spinor basis. For instance, let us consider planar curves of null spontaneous curvature, *i.e.* with $\tau = 0$ (or $j = 0$) and $\kappa_0 = 0$. For $f^2 > \zeta^2$, the solution corresponds to a wavelike Elastica, whose curvature is given by [23, 25, 26]

$$\kappa = 2\sqrt{m}\, q\,\mathrm{cn}(qs, m)\,, \quad 0 \leq m \leq 1\,, \tag{49}$$

where $\mathrm{cn}(u, m)$ is the cosine Jacobi elliptic function of argument u and parameter m [27]. The scaled constant force and the integration constant are given in terms of the wavenumber q and the parameter m by $f^2 = q^2$ and $\zeta = q^2(2m - 1)$.

Since the torsion vanishes, the spinorial structure Eqs. (9) simplify to

$$\psi_1' = -\frac{\kappa}{2}\bar{\psi}_2\,, \quad \bar{\psi}_2' = \frac{\kappa}{2}\psi_1\,, \tag{50}$$

along with their CC expressions. The solutions of these equations for the curvature given in Eq. (49) are

$$\psi_1 = -c_1\sqrt{m}\,\mathrm{sn}(qs, m) + c_2\,\mathrm{dn}(qs, m)\,, \tag{51a}$$

$$\bar{\psi}_2 = c_1\,\mathrm{dn}(qs, m) + c_2\sqrt{m}\,\mathrm{sn}(qs, m)\,, \tag{51b}$$

where $\mathrm{sn}(u, m)$ and $\mathrm{dn}(u, m)$ are the sine and delta Jacobi elliptic functions with argument u and parameter m, related by $\mathrm{dn}^2(u, m) = 1 - m\,\mathrm{sn}^2(u, m)$; $c_1, c_2 \in \mathbb{C}$ are two constants of integration. The other two components are obtained from the CC expressions of these two components. The normalization of the spinors imposes the constraint

$$|c_1|^2 + |c_2|^2 = 1\,. \tag{52}$$

Let us consider that the curve is on the plane $X - Y$. Since the principal normal and the binormal are on and orthogonal to the plane, from Eqs. (8) we have

$$\mathbf{N} \cdot \mathbf{E}_3 = \mathrm{Re}\boldsymbol{\nu} \cdot \mathbf{E}_3 = \mathrm{Re}(\psi_1\psi_2) = 0\,, \tag{53a}$$

$$\mathbf{B} \cdot \mathbf{E}_3 = \mathrm{Im}\boldsymbol{\nu} \cdot \mathbf{E}_3 = \mathrm{Im}(\psi_1\psi_2) = 1\,. \tag{53b}$$

These two conditions imply the following three equations

$$|c_1|^2 = |c_2|^2\,, \quad \mathrm{Re}(c_1\bar{c}_2) = 0\,, \quad \mathrm{Im}(c_1\bar{c}_2) = \frac{1}{2}\,. \tag{54}$$

Combining the first equation with Eq. (52) we have $|c_1|^2 = |c_2|^2 = 1/2$. Using this result and combining the last two equations, we get $c_2 = ic_1$.

The planar curve can be parametrized by the complex coordinate $z = x + iy$ and its CC. From Eq. (8), we have that the components of the tangent vector are $x' = 2\mathrm{Re}(\bar{\psi}_1\psi_2)$ and $y' = 2\mathrm{Im}(\bar{\psi}_1\psi_2)$, so $z' = 2\bar{\psi}_1\psi_2$, or in full

$$z' = -2i\bar{c}_1^2\left(\mathrm{dn}(qs, m) - i\sqrt{m}\mathrm{sn}(qs, m)\right)^2\,. \tag{55}$$

Since the constant $-2i\bar{c}_1^2$ is just a global scale factor, we can set it to one, such that $\bar{c}_1^2 = i/2$. Integrating we get

$$z = \frac{1}{q}\left(2\mathrm{E}(\mathrm{am}(qs, m), m) - qs\right) + i\frac{2}{q}\sqrt{m}\mathrm{cn}(qs, m) + z_0\,. \tag{56}$$

where $\mathrm{E}(u, m)$ and $\mathrm{am}(u, m)$ are the incomplete elliptic integral of the second kind and the Jacobi amplitude with argument u and parameter m [27]; z_0 is a constant of integration. The cartesian coordinates of the curve are given by the real and imaginary parts of z.

6. Discussion and conclusions

We have presented a variational framework in which the spinor basis corresponding to the complexification to the FS frame is used for the examination of the equilibria of curves, thus offering an alternative to the usual framework employing the FS frame or its complexification.

We have shown that in order to obtain the EL equations correctly for energies depending on the curvature and torsion, it suffices to implement the spinorial structure equations in the variational principle, very similar to the variational principle using the generalized Weierstrass-Enneper representation of surfaces [11]. Their introduction allows for their independent variation, so there is no need to calculate how the curvature and torsion vary under a change of the spinors. We also identified the force and torque spinors from the change of the boundary energy under Euclidean motions. One benefit of working with these complex vectors is that their components are the CC one of the other. To illustrate this spinorial framework we determined the spinors corresponding to the wavelike solutions of the planar Euler Elastica. Here we specialized our results to wavelike planar curves, but this framework could also be applied to analyze closed planar curves with constraints such as fixed total area [28, 29], or to the study three dimensional curves [25, 26].

Thre are several directions in which this spinorial framework could be generalized. It could be extended to accomodate more general energies, for instance depending on the material curvatures as in the case of Kirchhoff rods [24, 30], or to include an explicit dependence on the spinor basis, as it would be for paramagnetic filaments [31–33]. Furthermore, this spinorial framework could be employed to study not only the statics, but also the dynamics of filaments. In this work we considered spinors parametrized only by arc-length, but for the study of their temporal evolution they could also be parametrized by time. In such case, besides the spatial structure equations, additional equations, with time derivatives replacing arc length derivatives, will govern the kinematics of the filaments [10, 30]. It would also be interesting to extend this framework to higher dimensions, for instance, for energies of curves representing the worldlines of particles in a four dimensional ambient space, which would be described in terms of a tetrad constructed out of a 4-spinor and its adjoint spinor [8].

Appendix

A. Sufficiency of the Lagrange multipliers

If instead of implementing the spinorial structure equation we implement the definition of the curvature and torsion, as in the vectorial framework [14], in the form

$$\begin{aligned}&\lambda_\kappa \left(2\mathrm{Re}(\psi_1\psi_2' - \psi_2\psi_1') - \kappa\right)\\&\quad + \lambda_\tau \left(2\mathrm{Im}(\psi_1\bar{\psi}_1' + \psi_2\bar{\psi}_2') - \tau\right)\,, \end{aligned} \tag{A.1}$$

we obtain the following EL equations

$$\varepsilon_1 = \left(\xi + \mathrm{F}^3 + \Lambda\right)\bar{\psi}_1 + \eta\psi_2 + \phi\,\bar{\psi}_2 = 0\,, \tag{A.2a}$$

$$\varepsilon_2 = \left(\xi - \mathrm{F}^3 + \Lambda\right)\bar{\psi}_2 - \eta\psi_1 + \bar{\phi}\,\bar{\psi}_1 = 0\,, \tag{A.2b}$$

where we have defined

$$\xi = -i\lambda_\tau' + \tau\lambda_\tau + \kappa\lambda_\kappa\,, \quad \eta = \lambda_\kappa' - i\tau\lambda_\kappa + i\kappa\lambda_\tau\,. \tag{A.3}$$

However, the combination $\mathrm{Im}\left(\bar{\psi}_1\bar{\varepsilon}_1 + \bar{\psi}_2\bar{\varepsilon}_2\right) = \lambda_\tau' = 0$, implies the incorrect result that λ_τ is constant. Hence, the implementation of the definition of the curvature and torsion though Eqs. (A.1) does not suffice for the enforcement of their relation with the spinor basis.

B. Determination of the Lagrange multipliers

The spinor Σ can be written as the outer product of the spinor basis and the Lagrange multiplier λ and its charge conjugate as $\Sigma = (1/2i)\left(\psi\lambda^\dagger + \tilde{\psi}\tilde{\lambda}^\dagger\right)$, or explicitly

$$\Sigma = \frac{1}{2i}\left(\bar{\alpha}\psi\psi^\dagger + \alpha\tilde{\psi}\tilde{\psi}^\dagger + \bar{\beta}\psi\tilde{\psi}^\dagger - \beta\tilde{\psi}\psi^\dagger\right)\,. \tag{B.1}$$

In order to be associated to a real vector, this spinor should be Hermitian, $\Sigma = \Sigma^\dagger$. Calculating the latter spinor we get $\Sigma^\dagger = -(1/(2i))\left(\lambda\psi^\dagger + \tilde{\lambda}\tilde{\psi}^\dagger\right)$, or

$$\Sigma^\dagger = \frac{1}{2i}\left(-\alpha\psi\psi^\dagger - \bar{\alpha}\tilde{\psi}\tilde{\psi}^\dagger + \bar{\beta}\psi\tilde{\psi}^\dagger - \beta\tilde{\psi}\psi^\dagger\right)\,. \tag{B.2}$$

Thus, this requirement imposes the condition $\alpha = -\bar{\alpha}$ or $\mathrm{Re}(\alpha) = r_1 = 0$. Furthermore, from Eq. (26b), we have that $\Lambda = -\mathrm{F}^T$. This Lagrange multiplier did not play a relevant role in the determination of the tangential component of the force, because we obtained it from the reparametrizarion invariance of the energy. On account of this fact, it might seem that we could have omitted the implementation of the normalization of the spinor basis, but in such case we would have obtained from Eqs. (22) the wrong result that the tangential component must vanish.

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