Extended Jacobi elliptic function solutions for general boussinesq systems

Sait San, Rabia Altunay

Eskişehir Osmangazi University, Department of Mathematics-Computer, 26480, Eskişehir-Turkey.
e-mail: ssan@ogu.edu.tr

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In this research paper, we have utilized the Jacobi elliptic function expansion method to obtain the exact solutions of (1+1)- dimensional Boussinesq System (GBQS). The most important difference that distinguishes this method from other methods is the parameters included in the auxiliary equation $F'(\xi) = \sqrt{PF^4(\xi) + QF^2(\xi) + R}$. As far as the authors know, there is no other study in which such a variety of solutions has been given. Depending on P, Q and R, nineteen the solitary wave and periodic wave solutions are obtained at their limit conditions. In addition, 3D and contour plot graphics for the constructed waves are investigated with the computer package program by giving special values to the parameters involved. The validity and reliability of the method is examined by its applications on a class of nonlinear evolution equations of special interest in nonlinear mathematical physics. The results were acquired to verify that the recommended method is applicable and reliable for the analytic treatment of a wide application of nonlinear phenomena.

Keywords: Jacobi elliptic function method; travelling wave solution; boussinesq system.

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1. Introduction

We consider the following family of Boussinesq type systems of water wave theory, model by Zhang *et al*. [1] and Chen *et al*. [2]

$$\begin{aligned} &u_t + uu_x + v_x = c_1 u_{xxt}, \\ &v_t + \left[(1+v)\, u,\right]_x = c_2 u_{xxx}, \end{aligned} \tag{1}$$

where v is the elevation of a water wave and u is the surface velocity of water along x-direction and $c_1 = -(1/2)(\phi^2 - (1/3))$, $c_2 = (1/2)(1-\phi^2)$ and ϕ is a depth of water ($\phi = 0$ is at the bottom, $\phi = 1$ is on the surface), which have the relation $c_1 - c_2 = 1/3$. For the case $c_1 = 0,\ c_2 = -1/3$, Eq. (1) is the classical Boussinesq (cB) system is not linearly well posed in the Hadamard sense [3], it is important because it has an integrable Hamiltonian structure [4] and exact solitary-wave solutions [5-7].

This Eq. (1) is also known as Nwogu's Boussinesq (NB) model is useful for coastal and civil engineering to perform the nonlinear water wave model in a harbour and coastal design. Therefore many scientists studied mathematical properties, such as bifurcation and travelling wave solutions, lie symmetry analysis, single and multiple solitary wave solutions and painleve analysis [8-16].

In recent years, many methods have been developed for the exact solutions of nonlinear evolution equations such as the sub-equation method, the modified trial equation method, the simplest equation method, the generalized Kudryashov method, the symmetry analysis method and so on [17-22]. The main objective of this study is to investigate new traveling wave solutions for NB model by the Jacobi elliptic function method. The effectiveness and efficiency of this method are shown in literature with the various Jacobi elliptic function forms [23-27].

The outline of the present paper is as follows. In Sec. 2, we have a brief description of the Jacobi elliptic function method for solving partial differential equations. In Sec. 3, we apply the Jacobi elliptic function method above mentioned equation. Finally, some conclusions are given the latest section.

2. Jacobi's elliptic function method

In this section, we would like to describe extended Jacobi elliptic function method. Suppose a nonlinear partial differential equation (NPDE) with independent variables x, t and dependent variable u:

$$N(u, u_t, u_x, u_{xx}, ...) = 0. \tag{2}$$

Consider the following travelling wave transformation

$$u(x,t) = u(\xi), \qquad \xi = x - ct, \tag{3}$$

where c is an arbitrary constant to be determined later. By substituting (3) into (2), we have an ordinary differential equation (ODE):

$$N(u, u', u'', u''', ...) = 0. \tag{4}$$

Let us consider the solutions in the form

$$u(\xi) = \sum_{i=0}^{n} a_i F^i(\xi), \tag{5}$$

where F satisfies the Eq. (2) and n is a positive integer which can be evaluated by balancing the highest order partial derivative term and nonlinear term in Eqs. (2) or (4). $F(\xi)$ satisfies the following auxiliary equation:

$$F'(\xi) = \sqrt{PF^4(\xi) + QF^2(\xi) + R}, \tag{6}$$

where P, Q, and R are constants. The last equation hence holds for $F(\xi)$:

$$\begin{aligned} F'' &= 2PF^3 + QF, \\ F''' &= (6PF^2 + Q)F', \\ F''' &= 24P^2F^5 + 20PQF^3 + (12PR + Q^2)F \\ &\vdots \end{aligned} \tag{7}$$

With the help of Maple, substituting (6) into (4) along with Eq. (7) and collecting the coefficients of the same power $F^i(F')^j$ $(j = 0, 1,\ i = 0, 1, 2, ...)$ and setting each of the attained coefficients to be zero we have a set of over determined algebraic equations. And after we solve this by Maple, we find P, Q, R and c. Substituting the attained results into Eq. (6), gives the exponential and periodic solutions. It is well-known that Eq. (6) has families of Jacobi elliptic function solutions as follows [28,29]:

In this Table $sn\xi,\ cn\xi,\ dn\xi$ are respectively Jacobian elliptic sine function, Jacobian elliptic cosine function and the Jacobian elliptic function and the other Jacobian functions can be generated by these three kinds of functions, namely

$$ns\xi = \tfrac{1}{sn\xi},\ \ nc\xi = \tfrac{1}{cn\xi},\ \ nd\xi = \tfrac{1}{dn\xi},\ \ sc\xi = \tfrac{cn\xi}{sn\xi},$$

$$cs\xi = \tfrac{sn\xi}{cn\xi},\ \ ds\xi = \tfrac{dn\xi}{sn\xi},\ \ sd\xi = \tfrac{sn\xi}{dn\xi}$$

Also these functions satisfying the following formulas:

$$\begin{aligned} &sn^2\xi + cn^2\xi = 1, \quad dn^2\xi + m^2sn^2\xi = 1, \\ &ns^2\xi = 1 + cs^2\xi, \quad ns^2\xi = m^2 + m^2ds^2\xi, \\ &sc^2\xi + 1 = nc^2\xi, \quad m^2sd^2\xi + 1 = nd^2\xi. \end{aligned}$$

And addition derivative properties,

$$\begin{aligned} &sn'\xi = cn\xi dn\xi, \qquad cn'\xi = -sn\xi dn\xi, \\ &dn'\xi = -m^2 sn\xi cn\xi. \end{aligned}$$

The Jacobian-elliptic functions degenerate into hyperbolic functions when $m \to 1$ as follows:

$$\begin{aligned} &sn\xi \to \tanh\xi, \quad \{cn\xi, dn\xi\} \to \mathrm{sech}\xi, \\ &\{sc\xi, sd\xi\} \to \sinh\xi, \quad \{ds\xi, cs\xi\} \to \mathrm{csch}\xi, \\ &\{nc\xi, nd\xi\} \to \cosh\xi, \quad ns\xi \to \coth\xi, \\ &\{cd\xi, dc\xi\} \to 1. \end{aligned} \tag{8}$$

$Case$	P	Q	R	$F(\xi)$
1	m^2	$-(1+m^2)$	1	$sn\xi$
2	$-m^2$	$2m^2-1$	$1-m^2$	$cn\xi$
$3-4$	1	$-(1+m^2)$	m^2	$ns\xi$
5	1	$-(1+m^2)$	m^2	$dc\xi$
6	$1-m^2$	$2-m^2$	1	$sc\xi$
$7-8$	1	$2-m^2$	$1-m^2$	$cs\xi$
$9-10$	$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$ns\xi \pm cs\xi$
11	$\frac{(1-m^2)}{4}$	$\frac{(1+m^2)}{2}$	$\frac{(1-m^2)}{4}$	$nc\xi \pm sc\xi$
12	$P>0$	$Q<0$	$\frac{m^2Q^2}{(1+m^2)^2P}$	$\sqrt{\frac{-m^2Q}{(1+m^2)P}}sn\left(\sqrt{\frac{-Q}{1+m^2}}\xi\right)$
13	$P<0$	$Q>0$	$\frac{(1-m^2)Q^2}{(m^2-2)^2P}$	$\sqrt{\frac{-Q}{(2-m^2)P}}dn\left(\sqrt{\frac{Q}{2-m^2}}\xi\right)$
14	1	m^2+2	$1-2m^2+m^4$	$\frac{dn\xi cn\xi}{sn\xi}$
15	$-\frac{4}{m}$	$6m-m^2-1$	$-2m^3+m^4+m^2$	$\frac{mcn\xi dn\xi}{msn^2\xi+1}$
16	$\frac{4}{m}$	$-6m-m^2-1$	$2m^3+m^4+m^2$	$\frac{mcn\xi dn\xi}{msn^2\xi-1}$
$17-18$	$\frac{1}{4}$	$\frac{(1-2m^2)}{2}$	$\frac{1}{4}$	$\frac{sn\xi}{1\pm cn\xi}$
19	$\frac{(1-m^2)}{4}$	$\frac{(1+m^2)}{2}$	$\frac{(1-m^2)}{4}$	$\frac{cn\xi}{1\pm sn\xi}$

The Jacobian-elliptic functions degenerate into trigonometric functions when $m \to 0$ as follows:

$$\begin{aligned}
&\{sn\xi, sd\xi\} \to \sin\xi, \quad \{cn\xi, cd\xi\} \to \cos\xi,\\
&sc\xi \to \tan\xi, \quad \{ns\xi, ds\xi\} \to \csc\xi,\\
&\{nc\xi, dc\xi\} \to \sec\xi, \quad cs\xi \to \cot\xi,\\
&\{dn\xi, nd\xi\} \to 1.
\end{aligned} \tag{9}$$

3. Generalized Boussinesq System (GBQS)

Suppose that the travelling wave solutions for Eq. (11) are of the forms as follows:

$$u(x,t) = u(\xi), \quad v(x,t) = v(\xi), \quad \xi = x - ct, \tag{10}$$

where c is a constant to be determined later and ξ is an arbitrary constant.

$$\begin{aligned}
u_t + uu_x + v_x &= c_1 u_{xxt},\\
v_t + [(1+v)\,u]_x &= c_2 u_{xxx}.
\end{aligned} \tag{11}$$

By substituting (10) into (11), we have an ordinary differential equation (ODE):

$$\begin{aligned}
-cu' + uu' + v' + cc_1 u''' &= 0,\\
-cv' + u' + uv' + vu' - c_2 u''' &= 0.
\end{aligned} \tag{12}$$

where prime denotes differentiation with respect to ξ. Now, balancing the nonlinear terms u''' and $u'u$, we get $m = 2$. Balancing the nonlinear terms u''' and uv', we get $n = 2$. Hence, from (5), we might constitute

$$\begin{aligned}
u(\xi) &= a_0 + a_1 F(\xi) + a_2 F(\xi)^2,\\
v(\xi) &= b_0 + b_1 F(\xi) + b_2 F(\xi)^2,
\end{aligned} \tag{13}$$

in which $a_0, a_1, a_2, b_{0,} b_1$ and b_2 are undetermined constants. Substituting (13) and (6) into (12) and setting the coefficients of $F^i(\xi)F'(\xi)^j = 0,\ i = 0,1,2,3,\ j = 0,1$ to zero yields the following set of algebraic equations for $a_0, a_1, a_2, b_{0,}b_1$,b_2 and c:

$$\begin{aligned}
2a_2^2 + 24cc_1 a_2 P &= 0,\\
3a_1 a_2 + 6cc_1 a_1 P &= 0,\\
-2ca_2 + a_1^2 + 2a_0 a_2 + 2b_2 + 8cc_1 a_2 Q &= 0,\\
-ca_1 + a_0 a_1 + b_1 + cc_1 a_1 Q &= 0,\\
4b_2 a_2 - 24c_2 a_2 P &= 0,\\
3b_2 a_1 + 3b_1 a_2 - 6c_2 a_1 P &= 0,\\
2a_2 - 2cb_2 + 2b_2 a_0 &\\
+2b_1 a_1 + 2b_0 a_2 - 8c_2 a_2 Q &= 0,\\
-cb_1 - c_2 a_1 Q + a_1 + b_1 a_0 + b_0 a_1 &= 0.
\end{aligned} \tag{14}$$

Solving the set of nonlinear algebraic equations by help of Maple program, the following results are attained.

$$\begin{aligned}
&a_0 = -\frac{1}{2}\frac{-2c^2c_1 + 8c^2c_1^2Q - c_2}{cc_1},\\
&a_1 = 0, \quad a_2 = -12cc_1P,\\
&b_0 = \frac{1}{4}\frac{-4c^2c_1^2 + 8c_2Qc^2c_1^2 + c_2^2}{c^2c_1^2},\\
&b_1 = 0, \quad b_2 = 6c_2P, \quad c = c.
\end{aligned} \tag{15}$$

Substituting these results into (13), we have the following solution of Eq. (16):

$$\begin{aligned}
u(\xi) &= -\frac{1}{2}\frac{-2c^2c_1 + 8c^2c_1^2Q - c_2}{cc_1} - 12cc_1PF(\xi)^2,\\
v(\xi) &= \frac{1}{4}\frac{-4c^2c_1^2 + 8c_2Qc^2c_1^2 + c_2^2}{c^2c_1^2} + 6c_2PF(\xi)^2.
\end{aligned} \tag{16}$$

With the means of table and from the above solution (16), one might induce more general united Jacobian-elliptic function solutions of Eq. (12). Hereby, we attain the following exact solutions.

In the limit case when $m \to 1$,we get the solitary wave solutions of Eq. (12). In the limit case when $m \to 0$, we acquire the traveling wave solutions of Eq. (12).

Case 1. If we take $P : m^2,\ Q : -(1+m^2)$, then $F(\xi) = sn\xi$, thus

$$\begin{aligned}
u_1 &= -\frac{1}{2}\frac{-2c^2c_1 + 8c^2c_1^2Q - c_2}{cc_1} - 12cc_1Psn^2\xi,\\
v_1 &= \frac{1}{4}\frac{-4c^2c_1^2 + 8c_2Qc^2c_1^2 + c_2^2}{c^2c_1^2} + 6c_2Psn^2\xi.
\end{aligned}$$

In the limit case when $m \to 1$, then $F(\xi) = \tanh\xi$, and we attain one of the solitary wave solutions of Eq. (12) as

$$\begin{aligned}
u_1(x,t) &= -\frac{1}{2}\frac{-2c^2c_1 - 16c^2c_1^2 - c_2}{cc_1} - 12cc_1\tanh(x-ct)^2,\\
v_1(x,t) &= \frac{1}{4}\frac{-4c^2c_1^2 - 16c_2c^2c_1^2 + c_2^2}{c^2c_1^2} + 6c_2\tanh(x-ct)^2.
\end{aligned}$$

Case 2. When $P : -m^2,\ Q : (2m^2 - 1)$ are chosen, then $F(\xi) = cn\xi$, therefore

$$\begin{aligned}
u_2 &= -\frac{1}{2}\frac{-2c^2c_1 + 8c^2c_1^2Q - c_2}{cc_1} - 12cc_1Pcn^2\xi,\\
v_2 &= \frac{1}{4}\frac{-4c^2c_1^2 + 8c_2Qc^2c_1^2 + c_2^2}{c^2c_1^2} + 6c_2Pcn^2\xi.
\end{aligned}$$

Considering $m \to 1$, then $F(\xi) = \sec h\xi$, and one of the solitary wave solutions of Eq. (12) has been obtained as

$$\begin{aligned}
u_2(x,t) &= -\frac{1}{2}\frac{-2c^2c_1 + 8c^2c_1^2 - c_2}{cc_1} + 12cc_1\,\mathrm{sech}\,(x-ct)^2,\\
v_2(x,t) &= \frac{1}{4}\frac{-4c^2c_1^2 + 8c_2c^2c_1^2 + c_2^2}{c^2c_1^2} - 6c_2\,\mathrm{sech}\,(x-ct)^2.
\end{aligned}$$

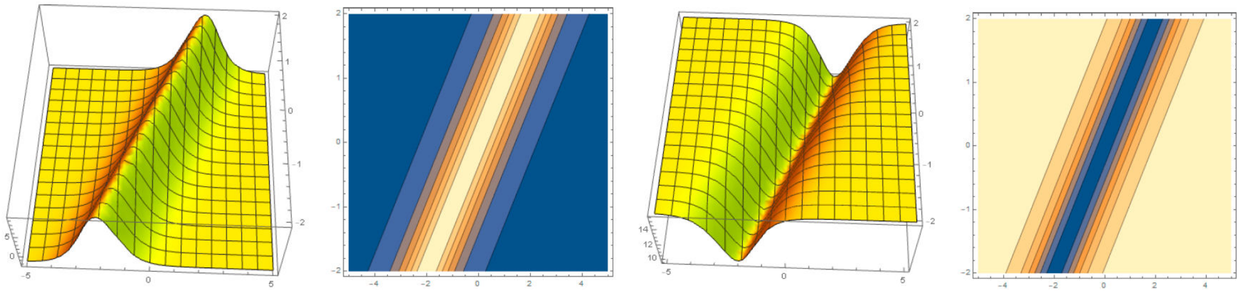

FIGURE 1. The figures represent the single soliton solutions $u_1(x,t)$ and $v_1(x,t)$ with respectively 3-dimensional plots and contour plots when $c = c_1 = c_2 = 1$ and $(x,t) = [-5,5] \times [-2,2]$.

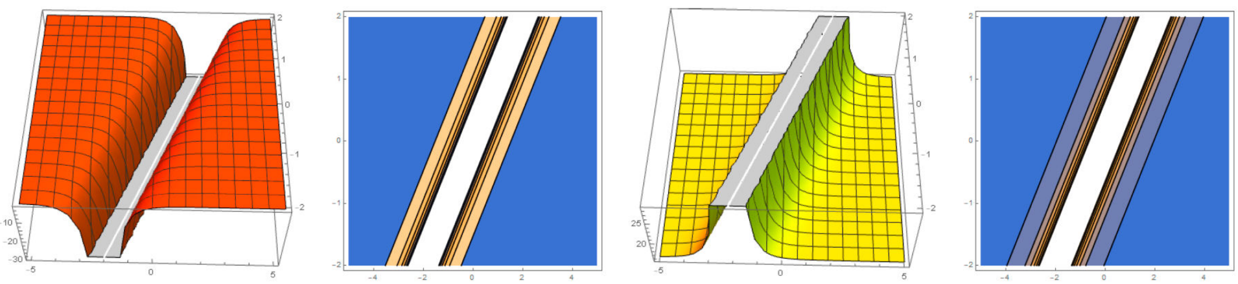

FIGURE 2. The figures represent the shock wave soliton solutions $u_3(x,t)$ and $v_3(x,t)$ with respectively 3-dimensional plots and contour plots, when $c = c_1 = c_2 = 1$ and $(x,t) = [-5,5] \times [-2,2]$.

Case 3. Choosing $P: 1,\ Q: -(1+m^2)$, it may be denoted from table $F(\xi) = ns\xi$, hence

$$u_3 = -\frac{1}{2}\frac{-2c^2c_1 + 8c^2c_1^2Q - c_2}{cc_1} - 12cc_1Pns^2\xi,$$

$$v_3 = \frac{1}{4}\frac{-4c^2c_1^2 + 8c_2Qc^2c_1^2 + c_2^2}{c^2c_1^2} + 6c_2Pns^2\xi.$$

As $m \to 1$, then $F(\xi) = \coth\xi$, and one of the solitary wave solutions of Eq. (12) can be stated as

$$u_3(x,t) = -\frac{1}{2}\frac{-2c^2c_1 - 16c^2c_1^2 - c_2}{cc_1} - 12cc_1\coth(x-ct)^2,$$

$$v_3(x,t) = \frac{1}{4}\frac{-16c_2c^2c_1^2 - 4c^2c_1^2 + c_2^2}{c^2c_1^2} + 6c_2\coth(x-ct)^2.$$

Case 4. Supposing $P: 1,\ Q: -(1+m^2)$ from table this choices correspond to $F(\xi) = ns\xi$, hence

$$u_3 = -\frac{1}{2}\frac{-2c^2c_1 + 8c^2c_1^2Q - c_2}{cc_1} - 12cc_1Pns^2\xi,$$

$$v_3 = \frac{1}{4}\frac{-4c^2c_1^2 + 8c_2Qc^2c_1^2 + c_2^2}{c^2c_1^2} + 6c_2Pns^2\xi.$$

For $m \to 0$ from (9), $F(\xi) = \csc\xi$, and we acquire one of the periodic solutions of Eq. (12) as

$$u_4(x,t) = -\frac{1}{2}\frac{-2c^2c_1 - 8c^2c_1^2 - c_2}{cc_1} - 12cc_1\csc(x-ct)^2,$$

$$v_4(x,t) = \frac{1}{4}\frac{-4c^2c_1^2 - 8c_2c^2c_1^2 + c_2^2}{c^2c_1^2} + 6c_2\csc(x-ct)^2.$$

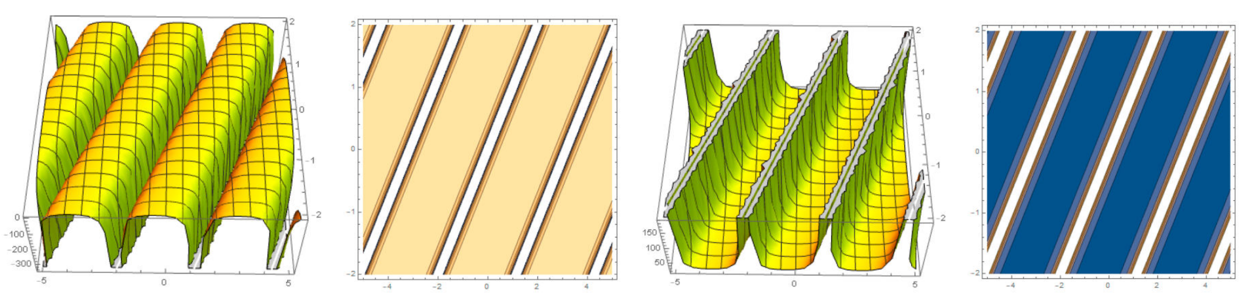

FIGURE 3. The figures represent the periodic wave solutions $u_4(x,t)$ and $v_4(x,t)$ with respectively 3-dimensional plots and contour plots, when $c = c_1 = c_2 = 1$ and $(x,t) = [-5,5] \times [-2,2]$.

Case 5. Considering $P:1,\ Q:-(1+m^2)$ from (9) $F(\xi)=dc\xi$, hence

$$u_5=-\frac{1}{2}\frac{-2c^2c_1+8c^2c_1^2Q-c_2}{cc_1}-12cc_1Pdc^2\xi,$$
$$v_5=\frac{1}{4}\frac{-4c^2c_1^2+8c_2Qc^2c_1^2+c_2^2}{c^2c_1^2}+6c_2Pdc^2\xi.$$

If $m\to 0$, then $F(\xi)=\sec\xi$, and one of the periodic solutions of Eq. (12) has been attained as

$$u_5(x,t)=-\frac{1}{2}\frac{-2c^2c_1-8c^2c_1^2-c_2}{cc_1}-12cc_1\sec(x-ct)^2,$$
$$v_5(x,t)=\frac{1}{4}\frac{-8c_2c^2c_1^2-4c^2c_1^2+c_2^2}{c^2c_1^2}+6c_2\sec(x-ct)^2.$$

Case 6. If we get $P:1-m^2,\ Q:2-m^2,$ then $F(\xi)=sc\xi$, therefore

$$u_6=-\frac{1}{2}\frac{-2c^2c_1+8c^2c_1^2Q-c_2}{cc_1}-12cc_1Psc^2\xi,$$
$$v_6=\frac{1}{4}\frac{-4c^2c_1^2+8c_2Qc^2c_1^2+c_2^2}{c^2c_1^2}+6c_2Psc^2\xi.$$

As long as $m\to 0, F(\xi)=\tan\xi$, and we obtain one of the traveling wave solutions of Eq. (12) as

$$u_6(x,t)=-\frac{1}{2}\frac{-2c^2c_1+16c^2c_1^2-c_2}{cc_1}-12cc_1\tan(x-ct)^2,$$
$$v_6(x,t)=\frac{1}{4}\frac{-4c^2c_1^2+16c_2c^2c_1^2+c_2^2}{c^2c_1^2}+6c_2\tan(x-ct)^2.$$

Case 7. For choices $P:1, Q:(2-m^2)$ from table, F is obtained as $F(\xi)=cs\xi$, in this way the solution may be expressed as

$$u_7=-\frac{1}{2}\frac{-2c^2c_1+8c^2c_1^2Q-c_2}{cc_1}-12cc_1Pcs^2\xi,$$
$$v_7=\frac{1}{4}\frac{-4c^2c_1^2+8c_2Qc^2c_1^2+c_2^2}{c^2c_1^2}+6c_2Pcs^2\xi.$$

Moreover, for $m\to 1$ from(8), $F(\xi)=\csc h\xi$, and one of the solitary wave solutions of Eq. (12) can be found as

$$u_7(x,t)=-\frac{1}{2}\frac{-2c^2c_1+8c^2c_1^2-c_2}{cc_1}-12cc_1\csc h(x-ct)^2,$$
$$v_7(x,t)=\frac{1}{4}\frac{-4c^2c_1^2+8c_2c^2c_1^2+c_2^2}{c^2c_1^2}+6c_2\csc h(x-ct)^2.$$

Case 8. Setting $P:1, Q:(2-m^2)$, then $F(\xi)=cs\xi$, due to this settings,

$$u_8=-\frac{1}{2}\frac{-2c^2c_1+8c^2c_1^2Q-c_2}{cc_1}-12cc_1Pcs^2\xi,$$
$$v_8=\frac{1}{4}\frac{-4c^2c_1^2+8c_2Qc^2c_1^2+c_2^2}{c^2c_1^2}+6c_2Pcs^2\xi.$$

Furthermore, for $m\to 0$ by using from (9), $F(\xi)=\cot\xi$, and we attain one of the traveling wave solutions of Eq. (12) as

$$u_8(x,t)=-\frac{1}{2}\frac{-2c^2c_1+16c^2c_1^2-c_2}{cc_1}-12cc_1\cot(x-ct)^2,$$
$$v_8(x,t)=\frac{1}{4}\frac{-4c^2c_1^2+16c_2c^2c_1^2+c_2^2}{c^2c_1^2}+6c_2\cot(x-ct)^2.$$

Case 9. If we take $P=(1/4), Q=(1-2m^2/2)$ it may be deducted from table, $F(\xi)=ns\xi\pm cs\xi$, therefore

$$u_9=-\frac{1}{2}\frac{-2c^2c_1+8c^2c_1^2Q-c_2}{cc_1}-12cc_1P\left(ns\xi\pm cs\xi\right)^2,$$
$$v_9=\frac{1}{4}\frac{-4c^2c_1^2+8c_2Qc^2c_1^2+c_2^2}{c^2c_1^2}+6c_2P\left(ns\xi\pm cs\xi\right)^2.$$

In this case for $m\to 1, F(\xi)=\coth\xi\pm\csc h\xi$, and one of the solitary wave solutions of Eq. (12) can be shown as

$$u_9(x,t)=-\frac{1}{2}\frac{-2c^2c_1-4c^2c_1^2-c_2}{cc_1}$$
$$-3cc_1(\coth(x-ct)\pm\csc h(x-ct))^2,$$
$$v_9(x,t)=\frac{1}{4}\frac{-4c^2c_1^2-4c_2c^2c_1^2+c_2^2}{c^2c_1^2}$$
$$+\frac{3}{2}c_2(\coth(x-ct)\pm\csc h(x-ct))^2.$$

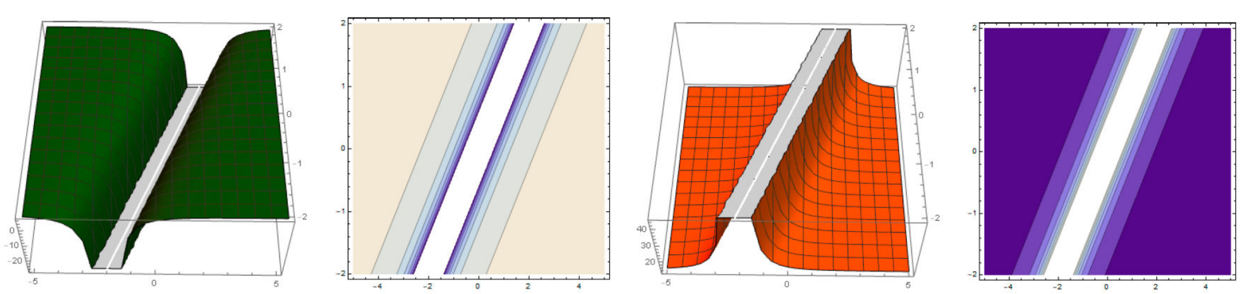

FIGURE 4. The figures represent the shock wave soliton solutions $u_9(x,t)$ and $v_9(x,t)$ with respectively 3-dimensional plots and contour plots, $c=c_1=c_2=1$ and $(x,t)=[-5,5]\times[-2,2]$.

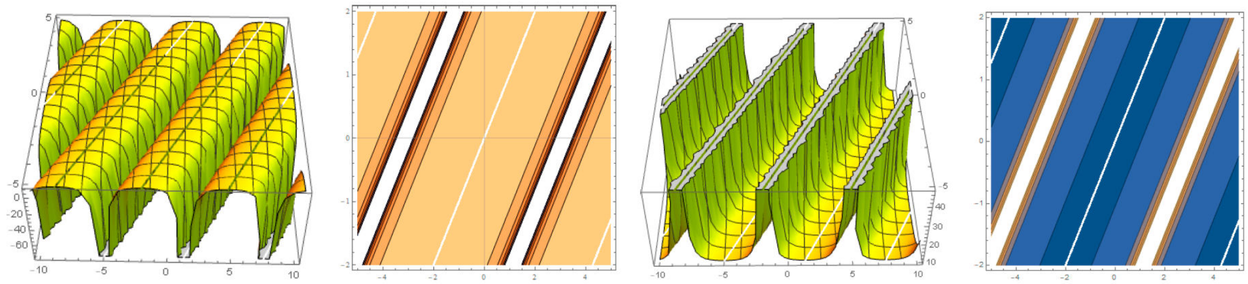

FIGURE 5. The figures represent the periodic wave solutions $u_{10}(x,t)$ and $v_{10}(x,t)$ with respectively 3-dimensional plots and contour plots, when $c = c_1 = c_2 = 1$ and $(x,t) = [-5,5] \times [-2,2]$.

Case 10 . Regarding $P = 1/4$, $Q = 1 - 2m^2/2$, then $F(\xi) = ns\xi \pm cs\xi$, so

$$u_{10} = -\frac{1}{2}\frac{-2c^2c_1 + 8c^2c_1^2Q - c_2}{cc_1} - 12cc_1P\left(ns\xi \pm cs\xi\right)^2,$$
$$v_{10} = \frac{1}{4}\frac{-4c^2c_1^2 + 8c_2Qc^2c_1^2 + c_2^2}{c^2c_1^2} + 6c_2P\left(ns\xi \pm cs\xi\right)^2.$$

In addition, for $m \to 0, F(\xi) = \csc\xi \pm \cot\xi$, and the periodic solution of Eq. (12) can be obtained as

$$\begin{aligned} u_{10}(x,t) &= -\frac{1}{2}\frac{-2c^2c_1 + 4c^2c_1^2 - c_2}{cc_1} \\ &\quad - 3cc_1(\csc(x-ct) \pm \cot(x-ct))^2, \\ v_{10}(x,t) &= \frac{1}{4}\frac{-4c^2c_1^2 + 4c_2c^2c_1^2 + c_2^2}{c^2c_1^2} \\ &\quad + \frac{3}{2}c_2(\csc(x-ct) \pm \cot(x-ct))^2. \end{aligned}$$

Case 11. Assigning $P = (1-m^2)/4, Q = (1+m^2)/2$, then $F(\xi) = nc\xi \pm sc\xi$, hence

$$u_{11} = -\frac{1}{2}\frac{-2c^2c_1 + 8c^2c_1^2Q - c_2}{cc_1} - 12cc_1P\left(ns\xi \pm cs\xi\right)^2,$$
$$v_{11} = \frac{1}{4}\frac{-4c^2c_1^2 + 8c_2Qc^2c_1^2 + c_2^2}{c^2c_1^2} + 6c_2P\left(ns\xi \pm cs\xi\right)^2.$$

In the limit case when $m \to 0, F(\xi) = \sec\xi \pm \tan\xi$, and the periodic solution of Eq. (12) can be written as

$$\begin{aligned} u_{11}(x,t) &= -\frac{1}{2}\frac{-2c^2c_1 + 4c^2c_1^2 - c_2}{cc_1} \\ &\quad - 3cc_1\sec((x-ct) \pm \tan(x-ct))^2, \\ v_{11}(x,t) &= \frac{1}{4}\frac{-4c^2c_1^2 + 4c_2c^2c_1^2 + c_2^2}{c^2c_1^2} \\ &\quad + \frac{3}{2}c_2\sec((x-ct) \pm \tan(x-ct))^2. \end{aligned}$$

Case 12. If we choose $P > 0$, $Q < 0$ from table, $F(\xi) = \sqrt{-m^2Q/(1+m^2)P}\, sn\left(\sqrt{-Q/(1+m^2)}\xi\right)$, so

$$\begin{aligned} u_{12} &= -\frac{1}{2}\frac{-2c^2c_1 + 8c^2c_1^2Q - c_2}{cc_1} \\ &\quad - 12cc_1\frac{-m^2Q}{(1+m^2)}sn^2\left(\sqrt{\frac{-Q}{1+m^2}}\xi\right), \\ v_{12} &= \frac{1}{4}\frac{-4c^2c_1^2 + 8c_2Qc^2c_1^2 + c_2^2}{c^2c_1^2} \\ &\quad + 6c_2\frac{-m^2Q}{(1+m^2)}sn^2\left(\sqrt{\frac{-Q}{1+m^2}}\xi\right). \end{aligned}$$

In the limit case when $m \to 1, F(\xi) = \sqrt{(-m^2Q)/([1+m^2]P)}\tanh\left(\sqrt{-Q/(1+m^2)}\xi\right)$, and the solitary wave solution of Eq. (12) can be stated as

$$\begin{aligned} u_{12}(x,t) &= -\frac{1}{2}\frac{-2c^2c_1 + 8c^2c_1^2Q - c_2}{cc_1} \\ &\quad + 6cc_1Q\tanh\left(\frac{1}{2}\sqrt{-2Q}(x-ct)\right)^2, \\ v_{12}(x,t) &= \frac{1}{4}\frac{-4c^2c_1^2 + 8c_2c^2Qc_1^2 + c_2^2}{c^2c_1^2} \\ &\quad - 3c_2Q\tanh\left(\frac{1}{2}\sqrt{-2Q}(x-ct)\right)^2. \end{aligned}$$

Case 13. For choices $P < 0$, $Q > 0$,then $F(\xi) = \sqrt{-Q/(2-m^2)P}\, dn\left(\sqrt{Q/(2-m^2)}\xi\right)$, hence

$$\begin{aligned} u_{13} &= -\frac{1}{2}\frac{-2c^2c_1 + 8c^2c_1^2Q - c_2}{cc_1} \\ &\quad - 12cc_1\frac{-Q}{(2-m^2)}dn^2\left(\sqrt{\frac{Q}{2-m^2}}\xi\right), \\ v_{13} &= \frac{1}{4}\frac{-4c^2c_1^2 + 8c_2Qc^2c_1^2 + c_2^2}{c^2c_1^2} \\ &\quad + 6c_2\frac{-Q}{(2-m^2)}dn^2\left(\sqrt{\frac{Q}{2-m^2}}\xi\right). \end{aligned}$$

In the limit case when $m \to 1$, $F(\xi) = \sqrt{-Q/([2-m^2]P)}\,\mathrm{sech}\left(\sqrt{Q/(2-m^2)}\xi\right)$, and we get one of the solitary wave solutions of Eq. (12) as

$$u_{13}(x,t) = -\frac{1}{2}\frac{-2c^2c_1 + 8c^2c_1^2Q - c_2}{cc_1} + 12cc_1Q\sec h(\sqrt{Q}(x-ct))^2,$$

$$v_{13}(x,t) = \frac{1}{4}\frac{-4c^2c_1^2 + 8c_2c^2Qc_1^2 + c_2^2}{c^2c_1^2} - 6c_2Q\sec h(\sqrt{Q}(x-ct))^2.$$

Case 14. While $P = 1$, $Q = m^2 + 2$, then $F(\xi) = dn\xi cn\xi/sn\xi$, therefore

$$u_{14} = -\frac{1}{2}\frac{-2c^2c_1 + 8c^2c_1^2Q - c_2}{cc_1} - 12cc_1P\frac{dn^2\xi cn^2\xi}{sn^2\xi},$$

$$v_{14} = \frac{1}{4}\frac{-4c^2c_1^2 + 8c_2Qc^2c_1^2 + c_2^2}{c^2c_1^2} + 6c_2P\frac{dn^2\xi cn^2\xi}{sn^2\xi}.$$

In the limit case when $m \to 0, F(\xi) = \cos\xi/\sin\xi$, and one of the traveling wave solutions of Eq. (12) can be evaluated as

$$u_{14}(x,t) = -\frac{1}{2}\frac{-2c^2c_1 + 16c^2c_1^2 - c_2}{cc_1} + 12cc_1\frac{\cos^2(x-ct)}{\sin^2(x-ct)},$$

$$v_{14}(x,t) = \frac{1}{4}\frac{-4c^2c_1^2 + 16c_2c^2c_1^2 + c_2^2}{c^2c_1^2} + 6c_2\frac{\cos^2(x-ct)}{\sin^2(x-ct)}.$$

Case 15. When $P = -4/m$, $Q = 6m - m^2 - 1$, then $F(\xi) = mdn\xi cn\xi/(msn^2\xi + 1)$, hence

$$u_{15} = -\frac{1}{2}\frac{-2c^2c_1 + 8c^2c_1^2Q - c_2}{cc_1} - 12cc_1P\frac{m^2dn^2\xi cn^2\xi}{m^2(sn^2\xi + 1)^2},$$

$$v_{15} = \frac{1}{4}\frac{-4c^2c_1^2 + 8c_2Qc^2c_1^2 + c_2^2}{c^2c_1^2} + 6c_2P\frac{m^2dn^2\xi cn^2\xi}{m^2(sn^2\xi + 1)^2}.$$

As long as $m \to 1, F(\xi) = m\sec h\xi\sec h\xi/(m\tanh\xi\tanh\xi + 1)$, and we get one of the solitary wave solutions of Eq. (12) as

$$u_{15}(x,t) = -\frac{1}{2}\frac{-2c^2c_1 + 32c^2c_1^2 - c_2}{cc_1} + \frac{48cc_1\sec h^4(x-ct)}{(\tanh^2(x-ct)+1)^2},$$

$$v_{15}(x,t) = \frac{1}{4}\frac{-4c^2c_1^2 + 32c_2c^2c_1^2 + c_2^2}{c^2c_1^2} - \frac{24c_2\sec h^4(x-ct)}{(\tanh^2(x-ct)+1)^2}.$$

Case 16. Setting $P = 4/m$, $Q : -6m - m^2 - 1$, then $F(\xi) = mdn\xi cn\xi/(msn^2\xi - 1)$, so

$$u_{16} = -\frac{1}{2}\frac{-2c^2c_1 + 8c^2c_1^2Q - c_2}{cc_1} - 12cc_1P\frac{m^2dn^2\xi cn^2\xi}{m^2(sn^2\xi - 1)^2},$$

$$v_{16} = \frac{1}{4}\frac{-4c^2c_1^2 + 8c_2Qc^2c_1^2 + c_2^2}{c^2c_1^2} + 6c_2P\frac{m^2dn^2\xi cn^2\xi}{m^2(sn^2\xi - 1)^2}.$$

When $m \to 1, F(\xi) = m\sec h\xi\sec h\xi/(m\tanh\xi\tanh\xi - 1)$, and we obtain one of the solitary wave solutions of Eq. (12) as

$$u_{16}(x,t) = -\frac{1}{2}\frac{-2c^2c_1 - 64c^2c_1^2 - c_2}{cc_1} - \frac{48cc_1\sec h^4(x-ct)}{(\tanh^2(x-ct)-1)^2},$$

$$v_{16}(x,t) = \frac{1}{4}\frac{-4c^2c_1^2 - 64c_2c^2c_1^2 + c_2^2}{c^2c_1^2} + \frac{24c_2\sec h^4(x-ct)}{(\tanh^2(x-ct)-1)^2}.$$

Case 17. If we get $P = 1/4$, $Q = (1-2m^2)/2$, then $F(\xi) = sn\xi/(1 \pm cn\xi)$, hence

$$u_{17} = -\frac{1}{2}\frac{-2c^2c_1 + 8c^2c_1^2Q - c_2}{cc_1} - 12cc_1P\frac{sn^2\xi}{(1 \pm cn\xi)^2},$$

$$v_{17} = \frac{1}{4}\frac{-4c^2c_1^2 + 8c_2Qc^2c_1^2 + c_2^2}{c^2c_1^2} + 6c_2P\frac{sn^2\xi}{(1 \pm cn\xi)^2}.$$

If $m \to 1$, $F(\xi) = \tanh\xi/(1 \pm \sec h\xi)$, and one of the solitary wave solutions of Eq. (12) can be stated as

$$u_{17}(x,t) = -\frac{1}{2}\frac{-2c^2c_1 - 4c^2c_1^2 - c_2}{cc_1} - \frac{3cc_1\tanh^2(x-ct)}{(1 \pm \sec h(x-ct))^2},$$

$$v_{17}(x,t) = \frac{1}{4}\frac{-4c^2c_1^2 - 4c_2c^2c_1^2 + c_2^2}{c^2c_1^2} + \frac{3c_2\tanh^2(x-ct)}{2(1 \pm \sec h(x-ct))^2}.$$

Case 18. Supposing $P = 1/4, Q = (1-2m^2)/2$, then $F(\xi) = sn\xi/(1 \pm cn\xi)$, therefore

$$u_{18} = -\frac{1}{2}\frac{-2c^2c_1 + 8c^2c_1^2Q - c_2}{cc_1} - 12cc_1P\frac{sn^2\xi}{(1 \pm cn\xi)^2},$$

$$v_{18} = \frac{1}{4}\frac{-4c^2c_1^2 + 8c_2Qc^2c_1^2 + c_2^2}{c^2c_1^2} + 6c_2P\frac{sn^2\xi}{(1 \pm cn\xi)^2}.$$

For $m \to 0, F(\xi) = \sin\xi/(1 \pm \cos\xi)$, and the travelling wave solution of Eq. (12) can be obtained as

$$u_{18}(x,t) = -\frac{1}{2}\frac{-2c^2c_1 + 4c^2c_1^2 - c_2}{cc_1} - \frac{3cc_1\sin^2(x-ct)}{(1 \pm \cos(x-ct))^2},$$
$$v_{18}(x,t) = \frac{1}{4}\frac{-4c^2c_1^2 + 4c_2c^2c_1^2 + c_2^2}{c^2c_1^2} + \frac{3c_2\sin^2(x-ct)}{2(1 \pm \cos(x-ct))^2}.$$

Case 19. If we get $P = (1-m^2)/4$, $Q = (1+m^2)/2$, then $F(\xi) = cn\xi/(1 \pm sn\xi)$, hence

$$u_{19} = -\frac{1}{2}\frac{-2c^2c_1 + 8c^2c_1^2Q - c_2}{cc_1} - 12cc_1P\frac{cn^2\xi}{(1 \pm sn\xi)^2},$$
$$v_{19} = \frac{1}{4}\frac{-4c^2c_1^2 + 8c_2Qc^2c_1^2 + c_2^2}{c^2c_1^2} + 6c_2P\frac{cn^2\xi}{(1 \pm sn\xi)^2}.$$

In the limit case when $m \to 0$, $F(\xi) = \cos\xi/(1 \pm \sin\xi)$, and the travelling wave solutions of Eq.(12) can be attained as

$$u_{19}(x,t) = -\frac{1}{2}\frac{-2c^2c_1 + 4c^2c_1^2 - c_2}{cc_1} - \frac{3cc_1\cos^2(x-ct)}{(1 \pm \sin(x-ct))^2},$$
$$v_{19}(x,t) = \frac{1}{4}\frac{-4c^2c_1^2 + 4c_2c^2c_1^2 + c_2^2}{c^2c_1^2} + \frac{3c_2\cos^2(x-ct)}{2(1 \pm \sin(x-ct))^2}.$$

4. Discussions

The dynamical behaviour of constructed solutions shows the different soliton type solutions. We obtained some important soliton solutions and profiles of the solutions is as follows: Figure 1, shows the physical structure of single soliton with parameters, $c = c_1 = c_2 = 1$. Figure 2, exhibits the physical structure of shock wave soliton with parameters, $c = c_1 = c_2 = 1$. Figure 3, represents the physical structure of periodic wave solution with parameters, $c = c_1 = c_2 = 1$. Figure 4, shows the physical structure of shock wave solution with parameters, $c = c_1 = c_2 = 1$. Figure 5, shows the physical structure of periodic wave solution with parameters, $c = c_1 = c_2 = 1$. Comparing with the results in [1, 2], we obtained more comprehensive solutions. As our knowledge, the results have not been previously reported. We expect that the results will be used future studies. In future work, conservation laws, which have a very important role in physics, can be obtained by group invariant analysis method. Also complexton and interactive solutions can be considered by various methods.

5. Conclusion

In this article we considered the $(1+1)-$dimensional Boussinesq System which were encountered in real world application problems such as coastal and civil engineering, harbour and coastal design. Jacobi elliptic function method were applied to investigate the traveling wave solutions of the governing system. By means of this method we have constructed exact solutions for nineteen cases. These solutions including trigonometric and hyperbolic functions and original to our knowledge. The hyperbolic solutions (including solitary wave solution) and trigonometric-function solutions of Eq. (12) can be attained in the limited case when the modulus $m \to 1$ and $m \to 0$ respectively. All solutions were verified Maple package program by putting them back into the original equation. Taking the parameters with special values, we presented 3-D and contour graphs of the Jacobi elliptic function solutions of the underlying equation.(Figs.1-5) The algorithm is very applicative and influential to investigate many solutions, therefore it might be also applied to many other nonlinear differential equations in mathematical physics.

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