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The impact of deformed space-phase parameters into HAs and HLM systems with the improved Hulthén plus Hellmann potentials model in the presence of temperature-dependent confined Coulomb potential within the framework of DSE

A. Maireche
Laboratory of Physics and Material Chemistry, Physics Department, Sciences Faculty, University of M'sila- Algeria, e-mail:abdelmadjid.maireche@univ-msila.dz

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In this work, the improved Hulthén plus Hellmann potentials model in the presence of temperature-dependent confined Coulomb potential IHHPTd model is adopted as the quark-antiquark interaction potential for studying the mass spectra of heavy mesons in the three-dimensional nonrelativistic quantum mechanics noncommutative phase space (3DNRQm-NCSP) symmetries. In addition, we found another application for this potential through its description for hydrogen atoms He^{+}, Li^{+2} and Be^{+}. We solved the deformed Schrödinger equation analytically using the generalized Bopp's shift method and standard perturbation theory. The new energy eigenvalues $E_{nc-n}^{(u,d)hy}$ and (E_{n-g}^{hlm}, $E_{n-m}^{hlm}$and E_{n-l}^{hlm}) for hydrogen atoms and heavy mesons such as charmonium $c\overline{c}$ and bottomonium $c\bar{b}$ and corresponding deformed Hamiltonian operators $H_{hhp}^{nc}(r,\Theta,\overline{\theta},\lambda,\overline{\lambda},\sigma,\overline{\sigma},\epsilon)$ and $H_{hhp}^{nc}(r,\Theta,\overline{\theta},\lambda,\overline{\lambda},\sigma,\overline{\sigma},g_s)$ were obtained, respectively. The present results are applied for calculating the new mass of heavy mesons. Four special cases were considered when some of the improved potential parameters were set to zero, resulting into improved Hellmann potential, improved Yukawa potential, improved Coulomb potential, and improved Hulthén potential, in (3DNRQm-NCSP) symmetries. The limiting cases are analyzed for Θ, σ and $\chi \longrightarrow 0$ are compared with those of literature.

Keywords: Schrödinger equation; Hulthen plus Hellmann potentials model; noncommutative quantum mechanics; star product; generalized Bopp's shift method.

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1. Introduction

Although a century has passed since the Schrödinger equation (SE), it is still an effective tool for physicists and chemists. In fact, it is a nonrelativistic (NR) equation that has a field of validity at low energies. It applies to many quantum mechanics problems in general, like quantum information theory, thermodynamic and thermochemical studies of diatomic and polyatomic molecule systems, and it plays a very vital role in the study of quarkonium systems (QS), among other systems. The Cornell potential (CP) is considered typical in the study of QS. Ikhdair obtained bound states (BS) of a spinless particle placed in scalar and vector CP under the influence of external magnetic and Aharonov-Bohm flux fields using the wave function Ansatz method for any arbitrary *l*-state with principal n and magnetic m quantum numbers [1]. The solutions of SE with many potential models or combined potentials such as Hua potential, Möbius square plus Kratzer potential, Möbius square plus Mie type potential, and Morse potential have been solved by applying various techniques, including the asymptotic iteration method, Nikiforov-Uvarov method (NUM), Laplace transforms, supersymmetric quantum mechanics, and a series expansion, in addition to other methods. Inyang *et al.* obtained analytical solutions of the N-dimensional SE for the Varshni-Hulthén potential within the framework of the NUM by using the Greene-Aldrich approximation scheme (GAS) to the centrifugal barrier and obtained the numerical energy eigenvalues and the corresponding normalized eigenfunctions [2]. William *et al.* obtained BS solutions of the radial SE by the superposition of Hulthén and Hellmann potentials within the framework of the NUM for arbitrary *l*-state with the GAS for the centrifugal term, in addition to the corresponding normalized wave functions and computed the numerical energy eigenvalues of different quantum states [3]. Edet *et al.* obtained an approximate solution of the SE in arbitrary dimensions for the generalized shifted Hulthén potential model within NUM and computed the BS energy eigenvalues and obtained the corresponding eigenfunctions [4]. The authors of Ref. [5] studied the Hellmann potential in the presence of external magnetic and Aharonov-Bohm flux fields within the framework of the SE and obtained the energy equation and wave function of the system in closed form. Edet and *et al.* obtained the BS approximate solution of the SE for the q-deformed Hulthén plus generalized inverse quadratic Yukawa potential in D-dimensions with the help of the NUM and the corresponding eigenfunctions are expressed in Jacobi polynomials [6]. Edet *et al.* obtained the approximate analytical solutions of the relativistic Schrödinger equation (RSE) with Hellmann-Kratzer potential and calculated the energy eigenvalue and the corresponding wave function and compact form using the NUM [7]. Al-Jamel and Widyan studied the spin-averaged mass spectra of heavy quarkonia ($c\overline{c}$ and $b\overline{b}$) in a Coulomb plus quadratic potential using the

NRSE and obtained the energy eigenvalues and eigenfunctions in compact forms for any *l*-value using the NUM [8]. Abu-Shady calculated quarkonia meson $b\overline{c}$ and $c\overline{s}$ meson masses for the N-dimensional relativistic Schrödinger equation (ND-RSE) under CP plus harmonic oscillator potential and obtained the energy eigenvalues and the corresponding wave functions in the ND-space using the NUM [9]. The authors of Ref. [10] have proposed an NR model that includes both Hulthén $V(r) = -\frac{A_0 \exp(-\alpha r)}{1-\exp(-\alpha r)}$ and Hellmann $V(r) = -\frac{A_1}{r} + \frac{A_2 \exp(-\alpha r)}{r}$ potentials and obtained BS solutions of the RSE within the framework of the NUM for arbitrary *l*-state with the GAS for the centrifugal term. Very recently, Akpac *et al.* adopted a Hulthén plus Hellmann potentials temperature-dependent (TD) model (HHPTd, in short) as the quark-antiquark interaction potential for studying the mass spectra of heavy mesons; the authors made it to be TD by replacing the screening parameter with a Debye mass and solving the RSE analytically using the series expansion method and obtained the energy eigenvalues and calculating the mass spectra of heavy mesons such as harmonium $c\overline{c}$ and bottomonium $b\bar{b}$ [11]**:**

$$V_{hhp}(r) = -\frac{A_0 \exp(-\alpha r)}{1-\exp(-\alpha r)} - \frac{A_1}{r} + \frac{A_2 \exp(-\alpha r)}{r}$$
$$\Longrightarrow$$
$$V_{hhp}(r,T) = -\frac{A_0 \exp(-m_D(T)\, r)}{1-\exp(-\alpha r)} - \frac{A_1}{r} + \frac{A_2 \exp(-m_D(T)\, r)}{r}, \tag{1}$$

where A_1, A_2 and A_0 are the potential strength, α is the screening parameter that controls the shape of the potential and $m_D(T)$ is the Debye mass which is TD potential $V_{hhp}(r,T)$ TD is obtained from $V_{hhp}(r)$ by replacing the screening parameter α with Debye mass $m_D(T)$ that is TD and vanishes when the temperature is zero. In the same context regarding temperature, Inyang *et al.* adopted Hulthén plus Hellmann potentials, which rendered TD by replacing the screening parameter with a Debye mass, as the quark-antiquark interaction potential for studying the thermodynamic properties and the mass spectra of heavy mesons [12].

Motivation

Relativistic and NR quantum mechanics have achieved great successes in terms of the convergence of theoretical treatments with experimental measurements. However, some indications show that there are many problems that quantum mechanics known in the literature has not been able to solve, for example, the problem of non-renormalizable electroweak interactions, the problem of quantization of gravity, and addition the problems in string theory. The idea of noncommutativity resulting from properties of deformation of space-space (Heisenberg in 1930 is the first to suggest the idea and then it was developed by Snyder in 1947) was one of the major solutions to these problems. As a result of all these motivational data, it is logical to consider the topographical properties of the noncommutativity space-space and phase-phase have a clear effect on the various physical properties of relativistic and nonrelativistic quantum systems [13–18].

A lot of research has been devoted to study the properties of quarkonium in the noncommutative space phase (NCSP) in the framework of the two cases: the nonrelativistic NR and relativistic, based on the three fundamental equations related to the Yukawa potentials such as modified Klien-Gordon equation (KGE) with modified scalar-vector Yukawa potential [19] and the relativistic interactions in one-electron atoms with modified Yukawa potential for spin-1/2 particles [20]. Moreover, we have treated the nonrelativistic behavior of hydrogen-like and neutral atoms subjected to the generalized perturbed Yukawa potential with a centrifugal barrier [21]. We are studying the modified unequal mixture scalar vector Hulthén-Yukawa potentials model as a quark-antiquark interaction and neutral atoms with relativistic treatment using the approximation of the centrifugal term and Bopp's Shift method [22]. We have investigated the approximate solutions of DKG and DSE under the modified more general exponential screened Coulomb potential plus Yukawa potential in NCQM symmetries [23]. We have constructed a theoretical model of the DKG equation with generalized modified screened Coulomb plus inversely quadratic Yukawa potential in RNCQM symmetries [24]. We have obtained solutions of the KG equation for the modified central complex potential in the symmetries of noncommutative quantum mechanics [25]. We are studying spectra of heavy quarkonia with modified CP in the framework of modified SE [26]. We have obtained the new NR atomic energy spectrum of energy-dependent potential for heavy quarkonium in noncommutative spaces and phases symmetries [27]. In 2019, we constructed a new model for Heavy-Light Mesons (HLM) in the extended NR quark model under a new modified potential containing Cornell, Gaussian, and inverse square terms in the symmetries of NCQM [28]. We investigated a new asymptotic study to the 3D-RSE under modified quark-antiquark interaction potential [29]. We have calculated the new relativistic atomic mass spectra of quarks (u, d and s) for the extended modified CP at nano and Planck scales [30]. In addition, we have built a new model for HLM in the symmetries of the extended NR quark model [31]. Furthermore, we have investigated the NR-BS solution at finite temperature using the sum of a modified CP plus inverse quadratic potential in the framework of the DSE [32]. Motivated by the previous works in ordinary quantum mechanics and NCSP, we hope to investigate the Hulthén plus Hellmann potentials in the 3D-NR quantum mechanics noncommutative phase space 3DNRQM-NCSP symmetries to obtain new applications on the microscopic scale and contribute to the knowledge of elementary particles at the nanoscale. The NR energy levels under the improved Hulthén plus Hellmann potentials model temperature-dependent have not been obtained yet in the 3DNRQM-NCSP symmetries, we propose a new version of the improved Hulthén plus Hellmann potentials model temperature-dependent (IHHPTd model) $V_{nc}^{hhp}(r,T)$ and a corresponding Hamiltonian operator $H_{nc}^{hhp}(r,p,T)$ in the presence of TD confined Coulomb potential in the 3DNRQM-NCSP symmetries as follows:

$$\begin{cases} V_{nc}^{hhp}(r,T) = V_{hhp}(r,T) + \begin{pmatrix} \frac{l(l+1)}{r^4} - \beta_2 \\ -\frac{\beta_0}{2r^3} - \frac{\beta_1}{2r} \end{pmatrix} \mathbf{L\Theta} \\ H_{nc}^{hhp}(r,p,T) = H_{hhp}(r,p,T) + \frac{\mathbf{L}\overline{\theta}}{2\mu} + \left(l(l+1)r^{-4} - 2\beta_0 r^{-3} - 2\beta_1 r^{-1} + \beta_2\right)\mathbf{L\Theta} \end{cases}. \tag{2}$$

The two couplings $\mathbf{L\Theta}$ and $\mathbf{L}\overline{\theta}$ are equal to $L_x\Theta_{12} + L_y\Theta_{23} + L_z\Theta_{13}$ and $L_x\overline{\theta}_{12} + L_y\overline{\theta}_{23} + L_z\overline{\theta}_{13}$, respectively, and they appear automatically from the influence of noncommutativity NC effect space-space $\mathbf{\Theta}(\Theta_{12}, \Theta_{23}, \Theta_{13})$ on the potential $V_{hhp}(r,T)$ and NC effect phase-phase $\overline{\theta}\left(\overline{\theta}_{12}, \overline{\theta}_{23}, \overline{\theta}_{13}\right)$ on the kinetic term $\frac{p^2}{2\mu}$ with the angular momentum operator $\mathbf{L}$ (L_x, L_y, L_z), and $\Theta_{\tau\upsilon} = \theta_{\tau\upsilon/}/2$, ($-\beta_2$, β_1, β_0 and α_3) are the new potential parameters defined in Sec. 2, r is the distance between the two particles while $H_{hhp}(r,p,T)$ is the usual Hamiltonian operator of the HHPTd in NRQM symmetries. In the present work, we modify the HHPTd model $V_{nc}^{hhp}(r)$ by adding new terms ($l(l+1)r^{-4}\mathbf{L\Theta}$, $-\frac{\beta_0}{2}\ r^{-3}\mathbf{L\Theta}$, $-\frac{\beta_1}{2r}\mathbf{L\Theta}$ and ($\beta_2\mathbf{L\Theta} + \frac{\mathbf{L}\overline{\theta}}{2\mu}$)) due to the topological properties of the self-quantum influence of space-space and phase-phase including the effect of the centrifugal term $l(l+1)r^{-2}$ which appears in the first additive part. The study of quarkonia systems with the HHPTd model at finite temperature in 3DNRQM-NCSP symmetries is an essential tool for understanding the status of the matter formed in the heavy-ion collisions as in Ref. [33], on the other hand, it is a continuation of our previous efforts in this context as in Ref. [32] and which falls within the context of the investigation of the properties of the quarkonium system under the influence of the sum of modified Cornell plus inverse quadratic potential at finite temperature in 3DNRQM-NCSP symmetries. To the best of our knowledge, this new study of DSE with $V_{nc}^{hhp}(r,T)$ was not done before by any researcher. The structure of 3DNRQM-NCSP symmetries based on NC canonical commutations relations in (Schrödinger, Heisenberg and interactions) pictures (SP, HP and IP), respectively, as follows (throughout this article, the natural units $c = \hbar = 1$ will be applied) (see, *e.g.*, [34–41]):

$$\begin{aligned} &[x_\tau^{nc} \overset{*}{,} p_\upsilon^{nc}] = [x_\tau^{nc}(t) \overset{*}{,} p_\upsilon^{nc}(t)] = [x_{\tau I}^{nc}(t) \overset{*}{,} p_{\upsilon I}^{nc}(t)] = i\hbar_{eff}\delta_{\tau\upsilon}, \\ &[x_\tau^{nc} \overset{*}{,} x_\upsilon^{nc}] = [x_\tau^{nc}(t) \overset{*}{,} x_\upsilon^{nc}(t)] = [x_{\tau I}^{nc}(t) \overset{*}{,} x_{\upsilon I}^{nc}(t)] = i\theta_{\tau\upsilon}, \\ &[p_\tau^{nc} \overset{*}{,} p_\upsilon^{nc}] = [p_\tau^{nc}(t) \overset{*}{,} p_\upsilon^{nc}(t)] = [p_{\tau I}^{nc}(t) \overset{*}{,} p_{\upsilon I}^{nc}(t)] = i\overline{\theta}_{\tau\upsilon}, \end{aligned} \tag{3}$$

where $\hbar_{eff} = \hbar\left(1 + Tr\left(\theta\overline{\theta}/4\right)\right)$ and $\hbar$ are the effective Planck constant and the usual reduced Planck constant, respectively. However, the unified operators $\Upsilon_{\tau H}^{nc}(t) = (x_\tau^{nc} \vee p_\tau^{nc})(t)$ and $\Upsilon_{\tau I}^{nc}(t) = (x_{\tau I}^{nc} \vee p_{\tau I}^{nc})(t)$ in (HP and IP, respectively) depend on the corresponding operator $\Upsilon_{\tau S}^{nc} = x_\tau^{nc} \vee p_\tau^{nc}$ in SP with the following projection relations:

$$\begin{pmatrix} \Upsilon_{\tau H}(t) \\ \Upsilon_{\tau I}(t) \end{pmatrix} = \begin{pmatrix} \exp(i\frac{\mathbf{H}_{hhp}}{\hbar}T)\Upsilon_{\tau s}^{nc}\exp(-i\frac{\mathbf{H}_{hhp}}{\hbar}T) \\ \exp(i\frac{\mathbf{H}_o^{hhp}}{\hbar}T)\Upsilon_{\tau s}^{nc}\exp(-i\frac{\mathbf{H}_o^{hhp}}{\hbar}T) \end{pmatrix}, \tag{4.1}$$

$$\Longrightarrow$$

$$\begin{pmatrix} \Upsilon_{\tau H}^{nc}(t) \\ \Upsilon_{\tau I}^{nc}(t) \end{pmatrix} = \begin{pmatrix} \exp(i\frac{\mathbf{H}_{nc}^{hhp}}{\hbar_{eff}}T) * \Upsilon_{\tau s}^{nc} * \exp(-i\frac{\mathbf{H}_{nc}^{hhp}}{\hbar_{eff}}T) \\ \exp(i\frac{\mathbf{H}_{onc}^{hhp}}{\hbar_{eff}}T) * \Upsilon_{\tau s}^{nc} * \exp(-i\frac{\mathbf{H}_{onc}^{hhp}}{\hbar_{eff}}T) \end{pmatrix}. \tag{4.2}$$

Here $T = t - t_0$, $\Upsilon_{\tau s} = x_\tau \vee p_\tau$, $\Upsilon_{\tau H}(t) = (x_\tau \vee p_\tau)(t)$ and $\Upsilon_{\tau H}^{nc}(t) = (x_{\tau i}^{nc} \vee p_{\tau i}^{nc})(t)$ are the usual three representations of SP, HP and IP in NRQM, while the new dynamics of studied systems $\frac{d\Upsilon_{\tau H}^{nc}(t)}{dt}$describe from the new following motion equations in 3DNRQM-NCSP symmetries:

$$\frac{d\Upsilon_{\tau H}(t)}{dt} = -\frac{i}{\hbar}\left[\Upsilon_{\tau H}(t), \mathbf{H}_{hhp}\right] \quad \Longrightarrow \quad \frac{d\Upsilon_{\tau H}^{nc}(t)}{dt} = -\frac{i}{\hbar_{eff}}\left[\Upsilon_{\tau H}^{nc}(t) \overset{*}{,} \mathbf{H}_{nc}^{hhp}\right]. \tag{5}$$

Here $\mathbf{H}_{hhp}\left(\mathbf{H}_o^{hhp}\right)$and $\mathbf{H}_{nc}^{hhp}\left(\mathbf{H}_{onc}^{hhp}\right)$denote the ordinary and generalized quantum Hamiltonian (free) operators for the of IHHPTd model in the NRQM and 3DNRQM-NCSP, respectively. The two infinitesimal parameters $\left(\theta^{\tau\upsilon}, \overline{\theta}^{\tau\upsilon}\right) = \epsilon^{\tau\upsilon}\left(\theta, \overline{\theta}\right)$ (compared to the energy) are the elements of two antisymmetric real matrices with dimensions of $(\text{length})^2$ and $(\text{momentum})^2$, respectively while $\epsilon_{\mu\nu}$ just an antisymmetric number, for example $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = 0$. Furthermore, the star notation $*$, denote to the star product, which is generalized between two arbitrary functions $(fg)(x,p)$ of the form $(fg)(x^{nc}, p^{nc}) \equiv (f * g)(x,p)$ in 3D-NCSP symmetries (see, *e.g.*, [42–51]:

$$\left(f*g\right)\left(x,p\right)=\left(fg\right)\left(x,p\right)-\frac{i}{2}\left(\theta^{\tau\upsilon}\frac{\partial f}{\partial x^{\tau}}\frac{\partial g}{\partial x^{\upsilon}}+\overline{\theta}^{\tau\upsilon}\frac{\partial f}{\partial p^{\tau}}\frac{\partial g}{\partial p^{\upsilon}}\right)+O\left(\theta^{2},\overline{\theta}^{2}\right),\tag{6}$$

This allows the construction of two scales of space and phase cells with elementary volumes $\tau_{ncs}^{3}=\theta^{3/2}$ and $\tau_{ncp}^{3}=\overline{\theta}^{3/2}$respectively. On the other hand, Eq. (6) allows us to satisfy the postulated algebra in Eq. (3). The second and the third terms in the above equation are the effects of (space-space) and (phase-phase) noncommutativity properties, respectively. This paper aims to present approximate solutions of the deformed Schrödinger equation (DSE) with the improved Hulthén plus Hellmann potentials temperature-dependent (IHHPTd, in short) model in 3DNRQM-NCSP symmetries using the generalized Bopp's shift method (GBSM), in addition to the standard perturbation theory (SPT).

The organization of the present work is given as follows: in the next section, we briefly review the SE with the HHPTd model. We divided the third section into subsections, the first one reserved to the physical and mathematical model for the HHPTd model in 3D-NCSP by applying the GBSM, in the next subsection, we generate the new spin-orbit Hamiltonian operator for the hydrogenic atoms (HAs) and the HLM under the IHHPTd model and by applying SPT we find the corrected spectrum of n^{th} excited levels in the framework of the global group 3D-NCSP and then, we derive the modified magnetic and rotational spectra for the IHHPTd model, which produced with the effect of both perturbed Hamiltonians H_{z}^{hhp} and $H_{pert}^{hhp-rot}$ due to the topological properties of phase space. In the fourth section, we resume the global spectrum and corresponding NC Hamiltonian operator for the IHHPTd model and corresponding energy levels of the HAs (He^{+}, Li^{+2} and Be^{+}) and the HLM such as charmonium $c\overline{c}$ and bottomonium $b\overline{b}$ and we calculate the new mass spectra of ($c\overline{c}$, $b\overline{b}$) in 3D-NCSP symmetries, we will also treat some special cases that demonstrate the correctness of our results. Finally, the paper ends with concluding remarks in Sec. 5.

2. Background and preparation

2.1. Overview of the eigenfunctions and the energy eigenvalues for HHPTd in NRQM

In this section, we shall recall here the SE for the HHPTd model, which is an important short-range potential that behaves like a Coulomb potential for small values and decreases exponentially for large values presented in Eq. (1). For small values of $(-r/\beta_2)$ the HHPTd model takes the form [11,12]:

$$V_{hhp}\left(r,T\right)=-\frac{\beta_0}{r}+\beta_1 r-\beta_2 r^2+\beta_3,\tag{7}$$

with $\beta_0=A_2-A_1-(A_0/m_D\left(T\right))$, $\beta_1=(A_2m_D^2\left(T\right)/2)-A_0(m_D\left(T\right)/12)$, $\beta_2=(A_2m_D^3\left(T\right)/6)$ and $\beta_3=(A_0/2)-A_2m_D\left(T\right)$. If we consider a nonrelativistic virtual particle of reduced mass μ in a central potential like $V_{hhp}\left(r,T\right)$, the quantum evolution of this particle governed with the SE in the spherical coordinates (r,θ,φ) as follows:

$$H_{hhp}\left(p,x\right)\Psi\left(\overrightarrow{r}\right)=E_{nl}^{hhp}\Psi\left(\overrightarrow{r}\right)\Longrightarrow\frac{d^2R_{nl}\left(r\right)}{dr^2}+2\mu\left(E_{nl}^{hhp}-V_{eff}^{hhp}\left(r,T\right)\right)R_{nl}\left(r\right)=0\,.\tag{8}$$

The operator $H_{hhp}\left(p,x\right)$ is just the ordinary Hamiltonian operator in NRQM, $\Psi\left(r,\theta,\varphi,t\right)=(R_{nl}\left(r\right)/r)Y_l^m\left(\theta,\varphi\right)e^{-(i/\hbar)E_{nl}^{hhp}t}$, $Y_l^m\left(\theta,\varphi\right)$ are the spherical harmonic functions, $V_{eff}^{hhp}\left(r,T\right)=-(\beta_0/r)+\beta_1 r-\beta_2 r^2+\beta_3+(l[l+1]/r^2)$ is effective potential, E_{nl}^{hhp} are the eigenvalues of the Hulthén plus Hellmann potentials model in the presence of TD while n and l are the radial and orbital angular momentum quantum numbers. The wave function and the energy spectrum of Eq. (8) for HHPTd model (7) are respectively given by [11,12]:

$$\Psi_{nlm}\left(r,\theta,\varphi,t\right)=\frac{N_{nl}}{r}r^{\frac{\alpha}{2\sqrt{\epsilon_{nl}}}}\exp\left(-\sqrt{\epsilon_{nl}}r\right)L_n^{\left(\alpha/\sqrt{\epsilon_{nl}}\right)}\left(2\sqrt{\epsilon_{nl}}r\right)Y_l^m\left(\theta,\varphi\right)\exp\left(-\frac{i}{\hbar}E_{nl}^{hhp}t\right).\tag{9}$$

Here $\alpha=(6\mu\beta_1/\delta^2)+2\mu\beta_0-(16\mu\beta_2/\delta^3)$, $-\epsilon_{nl}=2\mu\left(E_{nl}-\beta_3\right)+(12\mu\beta_2/\delta^2)-(6\mu\beta_1/\delta)$ and $\delta=1/r_0$ while r_0 is a characteristic radius of the meson. The energy E_{nl}^{hhp} of the potential in Eq. (7) is given by [11,12]:

$$\begin{aligned}E_{nl}^{hhp}={}&A_0\left(\frac{1}{2}-\frac{m_D\left(T\right)}{4\delta}\right)+A_2m_D\left(T\right)-2A_0m_D^2\left(T\right)\left(\frac{3m_D\left(T\right)}{2\delta}-m_D^2\left(T\right)-1\right)-\frac{1}{8\mu}\\&\times\frac{2\mu\left[A_2-A_1+\frac{A_0}{m_D(T)}\right]+\frac{\mu m_D(T)}{\delta^2}\left[3A_2m_D\left(T\right)-\frac{A_0}{2}\right]-\frac{8\mu A_2m_D^3(T)}{3\delta^3}}{n+\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{\mu A_2m_D^3(T)}{\delta^3}\left(1-\frac{m_D(T)}{\delta}\right)-\frac{\mu A_0m_D(T)}{6\delta^3}}}\,.\end{aligned}\tag{10}$$

3. Solution of the DSE with the NR-IHHPTd model

3.1. Review of the concepts of GBSM

In this subsection, we devote this part to studying the nonrelativistic improved Hulthén plus Hellmann potentials model temperature-dependent in the presence of TD confined Coulomb potential $V_{nc}^{hhp}(r,T)$, in 3DNRQM-NCSP symmetries. To perform this task the physical form of the deformed Schrödinger equation DSE, it is necessary to replace the ordinary three-dimensional Hamiltonian operators $H_{hhp}(p,x)$, ordinary energy E_{nl}^{hhp} and corresponding complex wave function $\Psi(\overrightarrow{r})$ in the symmetries of NRQM by three-dimensional Hamiltonian operators $H_{nc}^{hhp}(p_{nc},x_{nc})$, new unknown values $E_{nc}^{hhp}$of energy and corresponding new complex wave function $\Psi(\overrightarrow{r_{nc}})$, respectively, in 3DNRQM-NCSP symmetries. In addition, we need to replace the ordinary product with the star product $(*)$, this allows us to construct the DSE in (NC-3D: RSP) symmetries as (see, *e.g.*, [52–56]):

$$H_{nc}^{hhp}(p_{nc},x_{nc})\Psi(\overrightarrow{r_{nc}})=E_{nc}^{hhp}\Psi(\overrightarrow{r_{nc}})\Rightarrow H_{hhp}(p,x)*\Psi(\overrightarrow{r})=E_{nl}^{hhp}\Psi(\overrightarrow{r})\ , \tag{11.1}$$

allowing us to obtain the modified radial part of the SE as follows:

$$\left(\frac{d^2}{dr^2}+2\mu\left[E_{nl}^{hhp}-V_{eff}^{hhp}\{r,T\}\right]\right)*R_{nl}(r)=0\,. \tag{11.2}$$

Among the possible paths to find the solutions to Eqs. (19) and (20), we make use of the Connex method or Seiberg and Witten map. It is known to specialists that the star product can be translated into the ordinary product known in the literature using what is called Bopp's shift method. Bopp was the first to consider pseudo-differential operators obtained from a symbol by the quantization rules $(x,p)\rightarrow(\widehat{x}=x-(i/2)\partial_p,\widehat{p}=p+(i/2)\partial_x)$ instead of the ordinary correspondence (x,p) $\rightarrow(\widehat{x}=x,\widehat{p}=p+(i/2)\partial_x)$, respectively. It is known to specialists that BSM has been applied effectively and has succeeded in simplifying the known four fundamental equations: the NRDSE [67–70], the relativistic deformed Klein-Gordon equation RDKGE [60–68], the relativistic deformed Dirac equation RDDE [20, 41, 69], and the deformed relativistic Duffin-Kemmer-Petiau equation (DRDKPE) [70] with the notion of star product to the NRSE, RKGE, RDE and RDKPE with the notion of ordinary product, respectively. Thus, BSM is based on reducing second-order linear differential equations of DNRSE, DRKGE, DRDE, and DRDKPE with star product to second-order linear differential equations of NRSE, RKGE, RDE, and RDKPE without star products with simultaneous translation in the 3D-NCSP. The CNCCRs with star product in Eqs. (3) become new CNCCRs without the notion of the star product as follows (see, *e.g.*, [59–61]):

$$\begin{cases}[x_\tau^{nc},p_\upsilon^{nc}]=[x_\tau^{nc}(t),p_\upsilon^{nc}(t)]=[x_{\tau I}^{nc}(t),p_{\upsilon I}^{nc}(t)]=i\hbar_{eff}\delta_{\tau\upsilon}\\ [x_\tau^{nc},x_\upsilon^{nc}]=[x_\tau^{nc}(t),x_\upsilon^{nc}(t)]=[x_{\tau I}^{nc}(t),x_{\upsilon I}^{nc}(t)]=i\theta_{\tau\upsilon}\\ [p_\tau^{nc},p_\upsilon^{nc}]=[p_\tau^{nc}(t),p_\upsilon^{nc}(t)]=[p_{\tau I}^{nc}(t),p_{\upsilon I}^{nc}(t)]=i\overline{\theta}_{\tau\upsilon}\end{cases}. \tag{12}$$

The generalized positions and momentum coordinates $(x_\tau^{nc},p_\tau^{nc})$ in 3DNRQM-NCSP depend on the corresponding usual generalized positions and momentum coordinates (x_τ,p_τ) in NRQM by the following, respectively (see, *e.g.*, [57–59]):

$$\begin{pmatrix}x_\tau\\p_\tau\end{pmatrix}\longrightarrow\begin{pmatrix}x_\tau^{nc}\\p_\tau^{nc}\end{pmatrix}=\begin{pmatrix}x_\tau-\sum\limits_{\upsilon=1}^{3}\frac{\theta_{\tau\upsilon}}{2}p_\upsilon\\ p_\tau+\sum\limits_{\upsilon=1}^{3}\frac{\overline{\theta}_{\tau\upsilon}}{2}x_\upsilon\end{pmatrix}. \tag{13}$$

The above equation allows us to obtain the two operators r_{nc}^2 and p_{nc}^2 in 3DNRQM-NCSP symmetries [52–55]:

$$\begin{pmatrix}r^2\\p^2\end{pmatrix}\longrightarrow\begin{pmatrix}r_{nc}^2\\p_{nc}^2\end{pmatrix}=\begin{pmatrix}r^2-\mathbf{L\Theta}\\p^2+\mathbf{L}\overline{\theta}\end{pmatrix}. \tag{14}$$

Thus, the reduced radial part of the SE (without star product) can be written as:

$$\left(\frac{d^2}{dr^2}+2\mu\left[E_{nl}-V_{eff}^{hhp}(r_{nc},T)\right]\right)R_{nl}(r)=0\,. \tag{15}$$

The Hamiltonian operator H_{nc}^{hhp} for the improved Hulthén plus Hellmann potentials model temperature-dependent can be expressed as:

$$H_{nc}^{hhp}=H\left(x_\tau^{nc}=x_\tau-\frac{\theta_{\tau\upsilon}}{2}p_\upsilon,p_\tau^{nc}=p_\tau+\frac{\overline{\theta}_{\tau\upsilon}}{2}x_\upsilon\right). \tag{16}$$

Now, we want to find the new effective potential of IHHPTd $V_{eff-nc}^{hhp}(r,T)$ in 3DNRQM-NCSP symmetries:

$$V_{eff}^{hhp}(r,T) \longrightarrow V_{eff-nc}^{hhp}(r) = V_{nc}^{hhp}(r_{nc}) + l(l+1)r_{nc}^{-2} \,. \tag{17.1}$$

The new IHHPTd in the presence of temperature-dependent confined Coulomb potential $V_{nc}^{hhp}(r,T)$ and the new centrifugal term $l(l+1)r_{nc}^{-2}$ in 3DNRQM-NCSP symmetries:

$$\begin{cases} V_{nc}^{hhp}(r_{nc},T) = -\frac{\beta_0}{r_{nc}} + \beta_1 r_{nc} - \beta_2 r_{nc}^2 + \beta_3 \\ l(l+1)r_{nc}^{-2} = l(l+1)r^2 + l(l+1)r^{-4}\mathbf{L\Theta} + O\left(\Theta^2\right) \end{cases} . \tag{17.2}$$

After straightforward calculations, we can obtain the important terms $-\beta_0/r_{nc}$, $\beta_1 r_{nc}$ and $-\beta_2 r^2$ which will be used to determine the IHHPTd in the presence of TD confined Coulomb potential $V_{nc}^{hhp}(r,T)$ in 3DNRQM-NCSP symmetries as:

$$\begin{pmatrix} -\frac{\beta_0}{r_{nc}} \\ \beta_1 r_{nc} \\ -\beta_2 r_{nc}^2 \end{pmatrix} = \begin{pmatrix} -\frac{\beta_0}{r} \\ \beta_1 r \\ -\beta_2 r^2 \end{pmatrix} - \begin{pmatrix} \frac{\beta_0}{2r^3} \\ \frac{\beta_1 \mathbf{L\Theta}}{2r} \\ -\beta_2 \end{pmatrix} \mathbf{L\Theta} + \mathbf{I}_{3\times 1} O\left(\Theta^2\right) . \tag{18}$$

By making the substitution above Eqs. (18) and (17.2) into Eq. (16) and (17.1), we find the global our working Hamiltonian operator H_{nc}^{hhp} and the new effective potential of IHHPTd $V_{eff-nc}^{hhp}(r)$ in 3DNRQM-NCSP symmetries:

$$\begin{cases} H_{nc}^{hhp} = H_{hhp} + (\frac{l(l+1)}{r^4} - \frac{\beta_0}{2r^3} - \frac{\beta_1}{2r} + \beta_2)\mathbf{L\Theta} + \frac{\mathbf{L}\overline{\theta}}{2\mu} + O\left(\Theta^2, \overline{\theta}^2\right) \\ V_{eff-nc}^{hhp}(r,T) = V_{eff}^{hhp}(r,T) + (\frac{l(l+1)}{r^4} - \frac{\beta_0}{2r^3} - \frac{\beta_1}{2r} + \beta_2)\mathbf{L\Theta} + O\left(\Theta^2\right) \end{cases} , \tag{19}$$

where the operator $H_{hhp}(p,x)$ is just the ordinary Hamiltonian operator in usual nonrelativistic quantum mechanics:

$$H_{hhp} = \frac{p^2}{2\mu} - \frac{\beta_0}{r} + \beta_1 r - \beta_2 r^2 + \beta_3 \,, \tag{20}$$

while the rest five terms are proportional with two infinitesimal parameters (Θ and $\overline{\theta}$) and then we can be considered as perturbations terms H_{pert}^{hhp} and $V_{pert}^{hhp}(r,T)$ in 3DNRQM-NCSP symmetries as:

$$\begin{cases} H_{pert}^{hhp} = \left(l(l+1)r^{-4} - 2\beta_0 r^{-3} - 2\beta_1 r^{-1} + \beta_2\right)\mathbf{L\Theta} + \frac{\mathbf{L}\overline{\theta}}{2\mu} + O\left(\overline{\theta}^2, \Theta^2\right) \\ V_{pert}^{hhp}(r,T) = \left(l(l+1)r^{-4} - 2\beta_0 r^{-3} - 2\beta_1 r^{-1} + \beta_2\right)\mathbf{L\Theta} + O\left(\Theta^2\right) \end{cases} . \tag{21}$$

It is clear that the operator $H_{hhp}(p,x)$ is just the Hamiltonian operator for HAs such as He^+, Li^{+2} and Be^+ and HLM in ordinary quantum mechanics while the generated part H_{pert}^{hhp} appears as a result of the deformation of the 3D-NCSP. In the following, we can disregard the second term in H_{pert}^{hhp} because we are interested in the corrections of first-order Θ and $\overline{\theta}$.

3.2. New spin-orbit Hamiltonian operator for HAs and HLM under the IHHPTd model

In this subsection, we want to derive the physical form of the induced perturbed Hamiltonian $H_{pert}^{hhp}(p_{nc}, x_{nc})$ due to space-phase noncommutativity effects. To achieve this goal, we replace $\mathbf{L\Theta}$ both and $\mathbf{L}\overline{\theta}$ with the useful physical forms ($\epsilon\Theta\mathbf{LS}$ or $g_s\Theta\mathbf{LS}$) and ($\epsilon\overline{\theta}\mathbf{LS}$ or $g_s\overline{\theta}\mathbf{LS}$), respectively (see, *e.g.*, [51–55]):

$$\mathbf{L\Theta} \mapsto \begin{pmatrix} \epsilon \\ g_s \end{pmatrix} \Theta\mathbf{LS} \text{ and } \mathbf{L}\overline{\theta} \mapsto \begin{pmatrix} \epsilon \\ g_s \end{pmatrix} \overline{\theta}\mathbf{LS} \,, \tag{22.1}$$

allowing us to construct the induced perturbed spin-orbit Hamiltonian operator as follows H_{so}^{hhp}:

$$H_{so}^{hhp} = \left[\left(l(l+1)r^{-4} - 2\beta_0 r^{-3} - 2\beta_1 r^{-1} + \beta_2\right)\Theta + \frac{\overline{\theta}}{2\mu}\right]\mathbf{LS} \begin{cases} \epsilon : \text{HAs} \\ g_s \text{: HLM} \end{cases} . \tag{22.2}$$

Here $\Theta = \left(\Theta_{12}^2 + \Theta_{23}^2 + \Theta_{13}^2\right)^{1/2}$, $\overline{\theta} = \left(\overline{\theta}_{12}^2 + \overline{\theta}_{23}^2 + \overline{\theta}_{13}^2\right)^{1/2}$, $\mathbf{LS}$ is just the scalar product $L_x S_x + L_y S_y + L_z S_z$, $\epsilon \approx 1/137$ is the atomic fine structure constant and, g_s is the strong coupling constant, and $\mathbf{S}$ denotes the spin of the hydrogenic atoms such as (He^+, Li^{+2} and Be^+) or the heavy-light mesons. Thus, the spin-orbit interactions H_{so}^{hhp} appear automatically as a result of the deformation of the space phase. Now, physically, we can rewrite the quantum spin-orbit $\mathbf{LS}$ coupling as follows:

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \Longrightarrow 2\mathbf{LS} = G^2 \text{ with } G^2 = \mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2 \,. \tag{23}$$

Here **J** is the total momentum of the hydrogenic atoms He^{+}, Li^{+2} and Be^{+} and HLM. Substituting this equation into Eq. (22) yields:

$$H_{so}^{hhp} = \epsilon\left(\left[l\{l+1\}r^{-4} - 2\beta_0 r^{-3} - 2\beta_1 r^{-1} + \beta_2\right]\Theta + \frac{\overline{\theta}}{2\mu}\right)G^2 \begin{cases} \epsilon: \text{HAs} \\ g_s\text{: HLM} \end{cases} . \tag{24}$$

Our recent study can apply in two principal cases: The first case considers $A_1 = Ze^2$, Z and e are the atomic number and the charge of the electron, the term $(-A_1/r\ )$ becomes an attractive Coulomb potential, thus, we can consider the Hamiltonian described hydrogenic atoms such as (He^{+}, Li^{+2} and Be^{+}) under the influence of external fields described by other terms $(-[A_0 \exp\left(-m_D\left(T\right)r\right)/1 - \exp\left(-\alpha r\right)] + [A_2 \exp\left(-m_D\left(T\right)r\right)/r])$ in ordinary quantum mechanics and its extension to 3DNRQM-NCSP, which allows us to get the eigenvalues j of the total angular momentum operator **J** from the interval $|l-1/2| \leq j \leq |l+1/2|$. Because the operator G^2 has two eigenvalues, we can obtain two values of energy, as follows

$$k\left(j,l,s\right) = j(j+1) - l(l+1) - s(s+1) = \begin{cases} k_{+}(j = l-1/2, l, s): & \text{Spin up} \\ k_{-}\left(j = l-1/2, l, s\right): & \text{Spin down} \end{cases} . \tag{25}$$

A second way of determining a diagonal matrix H_{so}^{hhp} of order (3×3) with diagonal elements $\left(H_{so}^{hhp}\right)_{11}$, $\left(H_{so}^{hhp}\right)_{22}$and $\left(H_{so}^{hhp}\right)_{33} = 0$ as:

$$\begin{aligned} \left(H_{so}^{hhp}\right)_{11} &= \epsilon k_{+}\left(\left[l\{l+1\}r^{-4} - 2\beta_0 r^{-3} - 2\beta_1 r^{-1} + \beta_2\right]\Theta + \frac{\overline{\theta}}{2\mu}\right) \quad \text{if} \quad j = l+1/2, \\ \left(H_{so}^{hhp}\right)_{22} &= \epsilon k_{-}\left(\left[l\{l+1\}r^{-4} - 2\beta_0 r^{-3} - 2\beta_1 r^{-1} + \beta_2\right]\Theta + \frac{\overline{\theta}}{2\mu}\right) \quad \text{if} \quad j = l-1/2\,. \end{aligned} \tag{26}$$

The non-null diagonal elements $\left(H_{so}^{hhp}\right)_{11}$and $\left(H_{so}^{hhp}\right)_{22}$of the perturbed Hamiltonian operator H_{pert}^{hhp} can be influenced by the energy values E_{nl} by creating three new values $\Delta E_{n-so}^{u-hhp} = \langle\Psi|\left(H_{so}^{hhp}\right)_{11}|\Psi\rangle$ and $\Delta E_{n-so}^{d-hhp} = \langle\Psi|\left(H_{so}^{hhp}\right)_{22}|\Psi\rangle$ corresponding the polarizations up $(j = l+1/2)$ and down $(l-1/2)$ that can expressed as:

$$\begin{cases} \Delta E_{n-so}^{u-hhp} = \epsilon k_{+}\langle\Psi|\left(\left[l\{l+1\}r^{-4} - 2\beta_0 r^{-3} - 2\beta_1 r^{-1} + \beta_2\right]\Theta + \frac{\overline{\theta}}{2\mu}\right)|\Psi\rangle \\ \Delta E_{n-so}^{d-hhp} = \epsilon k_{-}\langle\Psi|\left(\left[l\{l+1\}r^{-4} - 2\beta_0 r^{-3} - 2\beta_1 r^{-1} + \beta_2\right]\Theta + \frac{\overline{\theta}}{2\mu}\right)|\Psi\rangle \end{cases} . \tag{27}$$

The second case is for the heavy-light mesons (HLM), for example: scalar, vector, pseudoscalar, and pseudovector for (B, B_s , D and D_s) mesons, or the heavy quarkonioum systems, such as charmonioum $c\bar{c}$ and bottomonioum $b\bar{b}$, which quarks and antiquarks of the same system ($Q\overline{Q}$), the eigenvalues of the spin-orbit coupling operator **LS** are $k\left(j,l,s\right) = j(j+1) - l(l+1) - s(s+1)$ corresponding $j = l+1$ (spin great), $j = l$ (spin middle) and $j = l-1$ (spin little), respectively. Then, one can form a diagonal matrix for modified nonrelativistic quark-antiquark potential with diagonal element, and in 3DNRQM-NCSP symmetries:

$$\begin{cases} \left(H_{so}^{hhp}\right)_{11} = g_s k_1\left(\left[l(l+1)r^{-4} - 2\beta_0 r^{-3} - 2\beta_1 r^{-1} + \beta_2\right]\Theta + \frac{\overline{\theta}}{2\mu}\right) & \text{for } j = l+1 \\ \left(H_{so}^{hhp}\right)_{22} = g_s k_2\left(\left[l(l+1)r^{-4} - 2\beta_0 r^{-3} - 2\beta_1 r^{-1} + \beta_2\right]\Theta + \frac{\overline{\theta}}{2\mu}\right) & \text{for } j = l \\ \left(H_{so}^{hhp}\right)_{33} = g_s k_3\left(\left[l(l+1)r^{-4} - 2\beta_0 r^{-3} - 2\beta_1 r^{-1} + \beta_2\right]\Theta + \frac{\overline{\theta}}{2\mu}\right) & \text{for } j = l-1 \end{cases} . \tag{28}$$

Here $2\left(k_1, k_2, k_3\right) \equiv (l, -2, -2l-2)$. The non-null diagonal elements $\left(H_{so}^{hhp}\right)_{11}$, $\left(H_{so}^{hhp}\right)_{22}$ and $\left(H_{so}^{hhp}\right)_{33}$ of the perturbed Hamiltonian operator H_{pert}^{hhp} can be influenced to the energy values E_{nl} by creating three new values $\Delta E_{n-g}^{hhp} = \langle\Psi|\left(H_{so}^{hhp}\right)_{11}|\Psi\rangle$, $\Delta E_{n-m}^{hhp} = \langle\Psi|\left(H_{so}^{hhp}\right)_{22}|\Psi\rangle$ and $\Delta E_{n-l}^{hhp} = \langle\Psi|\left(H_{so}^{hhp}\right)_{33}|\Psi\rangle$ as:

$$\begin{cases} \Delta E_{n-g}^{hhp} = g_s k_1\langle\Psi|\left(\left[l(l+1)r^{-4} - 2\beta_0 r^{-3} - 2\beta_1 r^{-1} + \beta_2\right]\Theta + \frac{\overline{\theta}}{2\mu}\right)|\Psi\rangle \\ \Delta E_{n-m}^{hhp} = g_s k_2\langle\Psi|\left(\left[l(l+1)r^{-4} - 2\beta_0 r^{-3} - 2\beta_1 r^{-1} + \beta_2\right]\Theta + \frac{\overline{\theta}}{2\mu}\right)|\Psi\rangle \\ \Delta E_{n-l}^{hhp} = g_s k_3\langle\Psi|\left(\left[l(l+1)r^{-4} - 2\beta_0 r^{-3} - 2\beta_1 r^{-1} + \beta_2\right]\Theta + \frac{\overline{\theta}}{2\mu}\right)|\Psi\rangle \end{cases} . \tag{29}$$

After straightforward calculations, the radial functions $R_{nl}(r)$ satisfy the following differential equation in 3D-NCSP for hydrogenic atoms He^{+}, Li^{+2} and Be^{+} and HLM systems under the improved Hulthén plus Hellmann potentials model:

$$\frac{d^2 R_{nl}(r)}{dr^2} + 2\mu\left(E_{nl}^{hhp} - V_{eff-nc}^{hhp}(r,T)\right) R_{nl}(r) = 0 \tag{30}$$

with

$$V_{eff-nc}^{hhp}(r,T) = V_{eff}^{hhp}(r) + \left(\left\{\begin{array}{l} \epsilon \ \text{ for HAs} \\ g_s \text{ for HLM} \end{array}\right.\right)\left(\left[\ l\{l+1\}r^{-4} - 2\beta_0 r^{-3} - 2\beta_1 r^{-1} + \beta_2\ \right]\Theta + \frac{\overline{\theta}}{2\mu}\right)\mathbf{LS}\,, \tag{31}$$

here $V_{eff-nc}^{hhp}(r,T)$ is the new generalized effective potential in 3D-NCSP symmetries. We have seen previously that the induced spin-orbit H_{so}^{hhp} is infinitesimal compared to the principal Hamiltonian operator $H_{hhp}(p,x)$ in NRQM for HAs and HLM, such as charmonioum $c\overline{c}$ and bottomonioum $b\overline{b}$ under the improved Hulthén plus Hellmann potentials model, this allows us to apply standard perturbation theory to determine the nonrelativistic energy corrections E_{so}^{hhp} at the first order of two infinitesimal parameters Θ and $\overline{\theta}$ due to noncommutativity space-space and phase-phase properties.

3.3. BS Solution for the spin-orbit operator for HAs and HLM systems under the IHHPTd model

The Hulthén plus Hellmann potentials model is extended by including new radial terms $l(l+1)r^{-4}$, $\beta_0 r^{-3}$ and $\beta_1 r^{-1}$ to become an improved Hulthén plus Hellmann potentials model temperature-dependent in 3DNRQM-NCSP symmetries. The additive part H_{pert}^{hhp} (Eq. (19)) of the new Hamiltonian operator H_{nc}^{hhp} is also proportional to the infinitesimal vectors $\mathbf{\Theta}$ and $\overline{\theta}$. This allows us to consider the additive part H_{pert}^{hhp} as a perturbation potential compared with the main potential $H_{hhp}(p,x)$ (Eq. (20)) in the symmetries of 3DNRQM-NCSP, that is, the inequality $H_{pert}^{hhp} \ll H_{hhp}$ has to become satisfied. That is, all the physical justifications for applying the time-independent perturbation theory become satisfied. This allows us to give a complete prescription for determining the energy level of the generalized n^{th} excited states. Now, we use perturbation theory and in the case of relativistic 3DNRQM-NCSP, we find the expectation values of the radial terms $1/r^4$, $1/r^3$ and $1/r$ taking into account the wave function which we have seen previously in Eq. (9). Thus, we obtain

$$\langle nlm|\, r^{-4}\,|nlm\rangle = N_{nl}^2 \int_0^{+\infty} r^{\frac{\alpha}{\sqrt{\epsilon_{nl}}}-1-1} \exp\left(-2\sqrt{\epsilon_{nl}}r\right)\left(L_n^{\left(\alpha/\sqrt{\epsilon_{nl}}\right)}\left[2\sqrt{\epsilon_{nl}}r\right]\right)^2 dr, \tag{32.1}$$

$$\langle nlm|\, r^{-3}\,|nlm\rangle = N_{nl}^2 \int_0^{+\infty} r^{\frac{\alpha}{\sqrt{\epsilon_{nl}}}-1} \exp\left(-2\sqrt{\epsilon_{nl}}r\right)\left(L_n^{\left(\alpha/\sqrt{\epsilon_{nl}}\right)}\left[2\sqrt{\epsilon_{nl}}r\right]\right)^2 dr, \tag{32.2}$$

$$\langle nlm|\, r^{-1}\,|nlm\rangle = N_{nl}^2 \int_0^{+\infty} r^{\frac{\alpha}{\sqrt{\epsilon_{nl}}}+2-1} \exp\left(-2\sqrt{\epsilon_{nl}}r\right)\left(L_n^{\left(\alpha/\sqrt{\epsilon_{nl}}\right)}\left[2\sqrt{\epsilon_{nl}}r\right]\right)^2 dr\,. \tag{32.3}$$

where we have used the property of the spherical harmonics given by

$$\iint Y_l^m(\theta,\varphi)\, Y_{l'}^{m'}(\theta,\varphi)\, d\Omega = \delta_{ll'}\delta_{mm'}$$

with $d\Omega = \sin(\theta)\, d\theta d\varphi$. To ease the notation, we will provide useful abbreviations $\langle nlm|\, D\,|nlm\rangle \equiv \langle D\rangle_{(nlm)}$. Comparing Eqs. (32) with the integral of the form [71,72]:

$$\int_0^{+\infty} t^{\nu-1}\exp(-pt)\, L_m^{\lambda}(pt)\, L_n^{\beta}(pt)\, dt = \frac{p^{-\nu}\Gamma(\nu)\,\Gamma(n-\nu+\beta+1)\,\Gamma(m+\lambda+1)}{n!m!\Gamma(1-\nu+\beta)\,\Gamma(\lambda+1)}$$
$$\times\ {}_3F_2(-m,\nu,\nu-\beta;-n+\nu-\beta,\lambda+1;1) \tag{33}$$

Where $Re(\nu)\rangle 0$, $Re(p)\rangle 0$, $m \in N\ \Lambda\ n \in N$ and ${}_3F_2(-m,\nu,\nu-\beta;-n+\nu,\lambda+1;1)$ is obtained from the generalized hypergeometric function ${}_3F_2(\alpha_1,...,\alpha_p;\beta_1,...,\beta_q,z)$ and $\Gamma(\nu)$ being the Gamma function. After some manipulation, we can obtain the explicit results:

$$\langle nlm| r^{-4} |nlm\rangle = N_{nl}^2 \frac{(n+1)\left(n+\frac{\alpha}{\sqrt{\epsilon_{nl}}}\right)\left(2\sqrt{\epsilon_{nl}}\right)^{-\left(\frac{\alpha}{\sqrt{\epsilon_{nl}}}-1\right)}\Gamma\left(\frac{\alpha}{\sqrt{\epsilon_{nl}}}-1\right)\Gamma\left(n+\frac{\alpha}{\sqrt{\epsilon_{nl}}}\right)}{\frac{\alpha}{\sqrt{\epsilon_{nl}}} n!\Gamma\left(\frac{\alpha}{\sqrt{\epsilon_{nl}}}\right)} X_1 \,, \tag{34.1}$$

$$\langle nlm| r^{-3} |nlm\rangle = N_{nl}^2 \frac{\left(n+\frac{\alpha}{\sqrt{\epsilon_{nl}}}\right)\sqrt{\epsilon_{nl}}\left(2\sqrt{\epsilon_{nl}}\right)^{-\frac{\alpha}{\sqrt{\epsilon_{nl}}}}\Gamma\left(n+\frac{\alpha}{\sqrt{\epsilon_{nl}}}\right)}{n!\alpha} X_2 \,, \tag{34.2}$$

and

$$\langle nlm| r^{-1} |nlm\rangle = 0 \,, \tag{34.3}$$

with $X_1$and X_2 are equal ${}_3F_2\left(-n, [\alpha/\sqrt{\epsilon_{nl}}]-1, -1; -n-1, [\alpha/\sqrt{\epsilon_{nl}}]+1; 1\right)$ and ${}_3F_2(-n, [\alpha/\sqrt{\epsilon_{nl}}], 0; -n, [\alpha/\sqrt{\epsilon_{nl}}]$ $+1; 1)$, respectively. We have replaced $\Gamma(2)$ and $\Gamma(1)$ with a value of 1 in the denominators of both Eqs. (34.1) and (34.2), respectively, and $1/\Gamma(-1)$ with zero in Eq. (34.3). Furthermore, we have used the properties $\Gamma(n+1) = n\Gamma(n) = n!$. The main goal of this subsection is to determine the corrected energy spectrum $\Delta E_{n-so}^{u-hhp}(k_+, n, m_D(T), A_1, A_2, A_0, j, l, s) \equiv \Delta E_{n-so}^{u-hhp}$ and $\Delta E_{n-so}^{d-hhp}(k_-, n, m_D(T), A_1, A_2, A_0, j, l, s) \equiv \Delta E_{n-so}^{d-hhp}$ which come from $H_{so}^{hhp}(p_{nc}, x_{nc})$ corresponding to $j = l \pm 1/2$ at the first order of two parameters Θ and $\overline{\theta}$ for hydrogenic atoms He^+, Li^{+2} and Be^+ for (n, l) states by applying standard perturbation theory and through the structure constants which specified the dimensionality of the improved Hulthén plus Hellmann potentials model:

$$\begin{pmatrix} \Delta E_{n-so}^{u-hhp} \\ \Delta E_{n-so}^{d-hhp} \end{pmatrix} = \epsilon\left(F(n, m_D(T), A_1, A_2, A_0)\Theta + \frac{\overline{\theta}}{2\mu}\right)\begin{pmatrix} k_+ \text{for } j = l+1/2 \\ k_- \text{for } j = l-1/2 \end{pmatrix}, \tag{35}$$

with

$$F(n, m_D(T), A_1, A_2, A_0) = l(l+1)\langle r^{-4}\rangle_{(nlm)} - \frac{\beta_0}{2}\langle r^{-3}\rangle_{(nlm)} - \frac{\beta_1}{2}\langle r^{-1}\rangle_{(nlm)} + \beta_2 \,. \tag{36}$$

For the HLM, such as charmonioum $c\overline{c}$ and bottomonioum $b\overline{b}$, which quarks and antiquarks of the same system $(Q\overline{Q})$, the eigenvalues of the spin-orbit coupling, we obtain the following results, for the n^{th} excited states $\left(\Delta E_{n-g}^{hhp}, \Delta E_{n-m}^{hhp}, \Delta E_{n-l}^{hhp}\right)$, respectively:

$$\begin{pmatrix} \Delta E_{n-g}^{hhp} \\ \Delta E_{n-m}^{hhp} \\ \Delta E_{n-l}^{hhp} \end{pmatrix} = \begin{pmatrix} g_s k_1(l)\left(F(n, m_D(T), A_1, A_2, A_0)\Theta + \frac{\overline{\theta}}{2\mu}\right) \text{ if } j = l+1 \\ g_s k_2(l)\left(F(n, m_D(T), A_1, A_2, A_0)\Theta + \frac{\overline{\theta}}{2\mu}\right) \text{ if } j = l \\ g_s k_3(l)\left(F(n, m_D(T), A_1, A_2, A_0)\Theta + \frac{\overline{\theta}}{2\mu}\right) \text{ if } j = l-1 \end{pmatrix}. \tag{37}$$

4. BS solution for MZE for the IHHPTd model

In this subsection, having obtained the energy spectrum (ΔE_{n-so}^{u-hhp} and ΔE_{n-so}^{d-hhp}) from $H_{so}^{hhp}(p_{nc}, x_{nc})$ corresponding to $j = l+1/2$ and $j = l-1/2$ at the first order of two parameters Θ and $\overline{\theta}$ for the hydrogenic atoms for (n, l) states and the degenerated energy (ΔE_{n-g}^{hhp}, ΔE_{n-m}^{hhp}, ΔE_{n-l}^{hhp}) of the heavy quarkonioum systems, such as charmonioum $c\overline{c}$ and bottomonioum $b\overline{b}$. Now, it is possible to obtain the second self proper symmetry for the improved Hulthén plus Hellmann potentials model in the presence of temperature-dependent confined Coulomb potential. This physical phenomenon is induced automatically from the influence of an external uniform magnetic field $\aleph$, if we make the following two simultaneous transformations,

$$\begin{pmatrix} \Theta \\ \overline{\theta} \end{pmatrix} \longrightarrow \aleph \begin{pmatrix} \lambda \\ \overline{\lambda} \end{pmatrix}. \tag{38}$$

Here λ and $\overline{\lambda}$ are just two infinitesimal real proportional constants, for the purpose to simplify calculations without compromising the physics content we choose the magnetic field parallel to the z axis. Then, we make the replacement

$$\left(\left[l\{l+1\}r^{-4} - 2\beta_0 r^{-3} - 2\beta_1 r^{-1} + \beta_2\right]\mathbf{\Theta} + \frac{\overline{\theta}}{2\mu}\right)\mathbf{L} \tag{39}$$

$$\rightarrow \left(\left[l\{l+1\}r^{-4} - 2\beta_0 r^{-3} - 2\beta_1 r^{-1} + \beta_2\right]\lambda + \frac{\overline{\lambda}}{2\mu}\right)\aleph L_z \tag{40}$$

This allowed us to derive the modified magnetic Hamiltonian operator $H_z^{hhp}\left(r,\lambda,\overline{\lambda}\right)$ for the HAs He^{+}, Li^{+2} and Be^{+} and the HLM $c\overline{c}$ and $b\overline{b}$ under the improved Hulthén plus Hellmann potentials model in the presence of temperature-dependent confined Coulomb potential in global 3DNRQM-NCSP symmetries as:

$$H_z^{hhp}\left(r,\lambda,\overline{\lambda}\right)=\left(\left[l\{l+1\}r^{-4}-2\beta_0 r^{-3}-2\beta_1 r^{-1}+\beta_2\right]\lambda+\frac{\overline{\lambda}}{2\mu}\right)H_{\text{mod}}^{z}\epsilon \quad \text{for} \quad \text{HAs}\,, \tag{41}$$

and

$$H_z^{hhp}\left(r,\lambda,\overline{\lambda}\right)=\left(\left[l\{l+1\}r^{-4}-2\beta_0 r^{-3}-2\beta_1 r^{-1}+\beta_2\right]\lambda+\frac{\overline{\lambda}}{2\mu}\right)H_{\text{mod}}^{z}g_s \text{ for HLM}\,. \tag{42}$$

Here $H_{\text{mod}}^{z}=\aleph\mathbf{J}-H_z$ denotes the modified Zeeman effect (MZE) in nonrelativistic NCQM, while $H_z=-\overrightarrow{\aleph}\overrightarrow{\mathbf{S}}$ is just the usual Zeeman effect. To obtain the exact NC magnetic modifications of energy for the ground state, the first excited state and n^{th} excited states of the hydrogenic atoms He^{+}, Li^{+2} and Be^{+} and the heavy quarkonioum systems under the improved Hulthén plus Hellmann potentials model temperature-dependent $\Delta E_{n-mg}^{hy-hhp}\left(n,m_D\left(T\right),A_1,A_2,A_0\right)$ and $\Delta E_{n-mag}^{hlm-hhp}\left(n,m_D\left(T\right),A_1,A_2,A_0\right)$, we just replace ($k_{+}$ or $k\left(l\right)$)and $(\Theta,\overline{\theta})$ in the Eqs.(35) and Eqs.(37) by the following parameters m and $(\lambda,\overline{\lambda})\aleph$, respectively:

$$\begin{pmatrix}\Delta E_{n-mag}^{hy-hhp}\left(n,m_D\left(T\right),A_1,A_2,A_0\right)\\ E_{n-mag}^{hlm-hhp}\left(n,m_D\left(T\right),A_1,A_2,A_0\right)\end{pmatrix}=\aleph\left(F\left(n,m_D\left(T\right),A_1,A_2,A_0\right)\lambda+\frac{\overline{\lambda}}{2\mu}\right)m\begin{pmatrix}\epsilon\\ g_s\end{pmatrix}. \tag{43}$$

It is known that the discreet magnetic number m takes the possible values $\left(\overline{-l,+l}\right)$, which allows us to fix ($2l+1$) values. It should be noted that the results obtained $\left(\Delta E_{n-mag}^{hy-hhp},\Delta E_{n-mag}^{hlm-hhp}\right)$ in Eq. (43) can be found directly by applying the formula $\left\langle\Psi\right|H_z^{hhp}\left(r,\lambda,\overline{\lambda}\right)\left|\Psi\right\rangle$ that takes the following explicit relation:

$$\begin{aligned}\Delta E_{n-mag}^{hy-hhp}&=\epsilon N_{nl}^2\aleph m\int_0^{+\infty}r^{\frac{\alpha}{\sqrt{\epsilon_{nl}}}}\exp\left(-2\sqrt{\epsilon_{nl}}r\right)\left(L_n^{\left[\alpha/\sqrt{\epsilon_{nl}}\right]}\left[2\sqrt{\epsilon_{nl}}r\right]\right)^2\\&\times\left(\left[\frac{l(l+1)}{r^4}-\frac{\beta_0}{2r^3}-\frac{\beta_1}{2r}+\beta_2\right]\lambda+\frac{\overline{\lambda}}{2\mu}\right)dr\,,\end{aligned} \tag{44}$$

$$\begin{aligned}\Delta E_{n-mag}^{hlm-hhp}&=g_s N_{nl}^2\aleph m\int_0^{+\infty}r^{\frac{\alpha}{\sqrt{\epsilon_{nl}}}}\exp\left(-2\sqrt{\epsilon_{nl}}r\right)\left(L_n^{\left[\alpha/\sqrt{\epsilon_{nl}}\right]}\left[2\sqrt{\epsilon_{nl}}r\right]\right)^2\\&\times\left(\left[\frac{l(l+1)}{r^4}-\frac{\beta_0}{2r^3}-\frac{\beta_1}{2r}+\beta_2\right]\lambda+\frac{\overline{\lambda}}{2\mu}\right)dr\,.\end{aligned} \tag{45}$$

Now, for our purposes, we are interested in finding a new important symmetry for the improved Hulthén plus Hellmann potentials model at zero temperature in DSE symmetries. This physical phenomenon is induced automatically from the influence of a perturbed effective potential H_{pert}^{hhp} which we have seen in Eq. (21). We discover these important physical phenomena when our studied system consists of a non-interacting Fermi gas and it is formed from all the particles in their gaseous state (He^{+}, Li^{+2} and Be^{+}) undergoing rotation with angular velocity Ω if we make the following two simultaneous transformations to ensure that the previous calculations are not repeated:

$$\begin{pmatrix}\Theta\rightarrow\sigma\mathbf{\Omega}\\ \overline{\theta}\ \rightarrow\overline{\sigma}\mathbf{\Omega}\end{pmatrix}\Rightarrow\begin{pmatrix}\mathbf{L}\Theta\rightarrow\sigma\mathbf{L\Omega}\\ \mathbf{L}\overline{\theta}\rightarrow\overline{\sigma}\mathbf{L\Omega}\end{pmatrix}, \tag{46}$$

Here σ and $\overline{\sigma}$ are just infinitesimal real proportional constants. We can express the effective potential H_{pert}^{hhp} which induced the rotational movements of the hydrogenic atoms He^{+}, Li^{+2} and Be^{+} and the HLM ($c\overline{c}$ and $b\overline{b}$) as follows:

$$H_{pert}^{hhp}=\left(\left[\frac{l(l+1)}{r^4}-\frac{\beta_0}{2r^3}-\frac{\beta_1}{2r}+\beta_2\right]\sigma+\frac{\overline{\sigma}}{2\mu}\right)\mathbf{L\Omega}\begin{cases}\epsilon:\text{HAs}\\ g_s:\text{HLM}\end{cases}. \tag{47}$$

To simplify the calculations without compromising physical content, we choose the rotational velocity $\mathbf{\Omega}$ parallel to the (z) axis. Then,

$$S(\sigma,\overline{\sigma},r)\mathbf{L\Omega}=S(\sigma,\overline{\sigma},r)\Omega L_z\,, \tag{48}$$

with

$$S(\sigma,\overline{\sigma},r)=\left(\frac{l(l+1)}{r^4}-\frac{\beta_0}{2r^3}-\frac{\beta_1}{2r}+\beta_2\right)\sigma+\frac{\overline{\sigma}}{2\mu}\,. \tag{49}$$

All of this data allows for the discovery of the new corrected energy $\Delta E_{hhp}^{f-rot}\left(n,m_D\left(T\right),A_1,A_2,A_0,\sigma,\overline{\sigma},m\right)$ due to the perturbed Fermi gas effect $V_{pert}^{hhp-rot}\left(r\right)$ which is generated automatically by the influence of the improved Hulthén plus Hellmann potentials model temperature-dependent for the n^{th} excited state in DSE symmetries as follows:

$$\Delta E_{hhp}^{f-rot}\left(n,m_D\left(T\right),A_1,A_2,A_0,\sigma,\overline{\sigma},m\right)=\left(S\left(n,m_D\left(T\right),A_1,A_2,A_0\right)\sigma+\frac{\overline{\sigma}}{2\mu}\right)\quad\Omega m\left\{\begin{array}{l}\epsilon\text{: for HAs}\\ g_s\text{: for HLM}\end{array}\right.\,. \tag{50}$$

It is worth mentioning that the authors of Refs. [73] studied rotating isotropic and anisotropic harmonically confined ultra-cold Fermi gases in two and three-dimensional spaces at zero temperature, but in this study, the rotational term was added to the Hamiltonian operator, in contrast to our case, where this rotation term $S(\sigma,\overline{\sigma},r)\mathbf{L\Omega}$ automatically appears due to the large symmetries resulting from the deformation of space-phase.

5. Global results and discussion

In the previous subsections, we obtained the solution of the deformed Schrödinger equation for the improved Hulthén plus Hellmann potentials model, which is described by the Hamiltonian operator as given in Eq. (19) by using the GBSM and SPT. The energy eigenvalues are calculated with help of 3D-NCSP symmetries. The modified eigenenergies for the n-th excited states of the HAs (He^{+}, Li^{+2} and Be^{+}) under the improved Hulthén plus Hellmann potentials model temperature-dependent $E_{nc-n}^{(u,d)hy}(n,m_D\left(T\right),A_1,A_2,A_0,j,l,m,s)\equiv E_{nc-n}^{(u,d)hy}$ with spin-1/2 and the degenerated energy ($E_{n-g}^{hlm}(n,m_D\left(T\right),A_1,A_2,$ $A_0,j,l,m,s)\equiv E_{n-g}^{hlm}$, $E_{n-m}^{hlm}\left(n,m_D\left(T\right),A_1,A_2,A_0,j,l,m,s\right)\equiv E_{n-m}^{hlm}$, $E_{n-l}^{hlm}\left(n,m_D\left(T\right),A_1,A_2,A_0,j,l,m,s\right)\equiv$ E_{n-l}^{hlm}) of the heavy quarkonioum systems, such as charmonioum $c\overline{c}$ and bottomonioum $b\overline{b}$ are obtained in this paper based on our original results presented on the Eqs. (35), (37), (43) and (50), in addition to the ordinary energy for Hulthén plus Hellmann potentials model which presented in Eq. (10) take the form:

Case 1. For the HAs (He^{+}, Li^{+2} and Be^{+}), we have:

$$E_{nc-n}^{(u,d)hy}\left\{\begin{array}{l}E_{nc-n}^{u\_hy}\\ E_{nc-n}^{d\_hy}\end{array}\right.=E_{nl}^{hhp}+\left(F\left(n,m_D\left(T\right),A_1,A_2,A_0\right)\left(\aleph\lambda+\Omega\sigma\right)+\frac{\aleph\overline{\lambda}+\Omega\overline{\sigma}}{2\mu}\right)\epsilon m$$
$$+\epsilon N_{nl}^2\left(F\left(n,m_D\left(T\right),A_1,A_2,A_0\right)\Theta+\frac{\overline{\theta}}{2\mu}\right)\left\{\begin{array}{l}k_+\text{ for }j=l+1/2\\ k_-\text{ for }j=l-1/2\end{array}\right.\,. \tag{51}$$

For the HLM, such as charmonioum $c\overline{c}$ and bottomonioum $b\overline{b}$:

$$E_{n-g}^{hlm}=E_{nl}^{hhp}+g_s\left(F\left(n,m_D\left(T\right),A_1,A_2,A_0\right)\left(\aleph\lambda+\Omega\sigma\right)+\frac{\aleph\overline{\lambda}+\Omega\overline{\sigma}}{2\mu}\right)m$$
$$+g_s k_1\left(l\right)\left(F\left(n,m_D\left(T\right),A_1,A_2,A_0\right)\Theta+\frac{\overline{\theta}}{2\mu}\right)\text{ if }j=l+1\,, \tag{52.1}$$

$$E_{n-m}^{hlm}=E_{nl}^{hhp}+g_s\left(F\left(n,m_D\left(T\right),A_1,A_2,A_0\right)\left(\aleph\lambda+\Omega\sigma\right)+\frac{\aleph\overline{\lambda}+\Omega\overline{\sigma}}{2\mu}\right)m$$
$$+g_s k_2\left(l\right)\left(F\left(n,m_D\left(T\right),A_1,A_2,A_0\right)\Theta+\frac{\overline{\theta}}{2\mu}\right)\text{ if }j=l\,, \tag{52.2}$$

and

$$E_{n-l}^{hlm}=E_{nl}^{hhp}+g_s\left(F\left(n,m_D\left(T\right),A_1,A_2,A_0\right)\left(\aleph\lambda+\Omega\sigma\right)+\frac{\aleph\overline{\lambda}+\Omega\overline{\sigma}}{2\mu}\right)m$$
$$+g_s k_3\left(l\right)\left(F\left(n,m_D\left(T\right),A_1,A_2,A_0\right)\Theta+\frac{\overline{\theta}}{2\mu}\right)\text{ if }j=l-1\,. \tag{52.3}$$

Thus, the total energy $E_{nc-n}^{(u,d)hy}$ and $(E_{n-g}^{hlm}, E_{n-m}^{hlm}, E_{n-l}^{hlm})$ for the HAs and the HLM, respectively, under the improved Hulthén plus Hellmann potentials model temperature-dependent 3DNRQM-NCSP symmetries, is the sum of the ordinary part of energy E_{nl}^{hhp} and the three corrections of energy that are produced automatically with the effect of perturbed spin-orbit effect, MZE and perturbed Fermi rotational effect. This is one of the main objectives of our research. Finally, we end this section by introducing the important result of this work as:

Case 1. For the HAs:

$$\left(H_{hhp} + \epsilon\left[g(r)\lambda + \frac{\overline{\lambda}}{2\mu}\right] H_{\text{mod}}^{z} + \epsilon\left[g(r)\left(\sigma\mathbf{L\Omega} + \mathbf{\Theta LS}\right) + \frac{\overline{\overline{\theta}}\mathbf{LS} + \overline{\sigma}\mathbf{L\Omega}}{2\mu}\right]\right) \frac{R_{nl}(r)}{r} Y_l^m(\theta,\varphi)$$

$$= \left\{\begin{array}{l} E_{nc-n}^{u,hy} \text{ for } j = l+1/2 \\ E_{nc-n}^{-\beta_2,hy} \text{ for } j = l-1/2 \end{array}\right. \left(\frac{R_{nl}(r)}{r} Y_l^m[\theta,\varphi]\right), \tag{53}$$

with

$$g(r) = \frac{l(l+1)}{r^4} - \frac{\beta_0}{2r^3} - \frac{\beta_1}{2r} + \beta_2\,. \tag{54}$$

Case 1. For the HLM ($c\overline{c}$ and $b\overline{b}$), we have:

$$\left(H_{hhp} + g_s\left[g(r)\lambda + \frac{\overline{\lambda}}{2\mu}\right] H_{\text{mod}}^{z} + g_s\left[\; g(r)\left[\sigma\mathbf{L\Omega} + \mathbf{\Theta LS}\right] + \frac{\overline{\overline{\theta}}\mathbf{LS} + \overline{\sigma}\mathbf{L\Omega}}{2\mu}\;\right]\right) \frac{R_{nl}(r)}{r} Y_l^m(\theta,\varphi)$$

$$= \left\{\begin{array}{c} E_{n-g}^{hlm} \text{ if } j = l+1 \\ E_{n-m}^{hlm} \text{ if } j = l \\ E_{n-l}^{hlm} \text{ if } j = l-1 \end{array}\right. \left(\frac{R_{nl}(r)}{r} Y_l^m[\theta,\varphi]\right). \tag{55}$$

This is one of the main motivations for the topic of this work. It is clear, that the obtained eigenvalues of energies are real, which allows us to consider the NC diagonal Hamiltonian $H_{hhp}^{nc}\left(r,\Theta,\overline{\theta},\lambda,\overline{\lambda},\sigma,\overline{\sigma}\right)$ as a Hermitian operator. In addition, and regarding the previously obtained results (20), (24), (42) and (47), the global Hamiltonian operator, at first order in and with the improved Hulthén plus Hellmann potentials model temperature-dependent for hydrogenic atoms for (n,l) states takes the form as:

$$H_{hhp}^{nc}\left(r,\Theta,\overline{\theta},\lambda,\overline{\lambda},\sigma,\overline{\sigma}\right) = H_{hhp} + \left(g(r)\lambda + \frac{\overline{\lambda}}{2\mu}\right) H_{\text{mod}}^{z} \left\{\begin{array}{c} \epsilon \text{ :HAs} \\ g_s\text{: HLM} \end{array}\right.$$

$$+ \left(g(r)\left(\sigma\mathbf{L\Omega} + \mathbf{\Theta LS}\right) + \frac{\overline{\overline{\theta}}\mathbf{LS} + \overline{\sigma}\mathbf{L\Omega}}{2\mu}\right) \left\{\begin{array}{c} \epsilon \text{ :HAs} \\ g_s\text{: HLM} \end{array}\right. . \tag{56}$$

This is the equation for Has and the HLM, such as charmonioum $c\overline{c}$ and bottomonioum $b\overline{b}$ under the influence of the improved Hulthén plus Hellmann potentials temperature-dependent model interactions. It should be pointed out that this treatment considers only the first-order terms in either Θ or $\overline{\theta}$. Clearly, the first two parts of Eq. (55) presents the Hamiltonian operator in the ordinary QM for the Hulthén plus Hellmann potentials model, the third part is the MZ operators while the last part is the combined two effects correspond the spin-orbit and the rotational Fermi operator for the improved Hulthén plus Hellmann potentials model, which are induced automatically by the NC properties of space and phase. It is evident to consider the atomic quantum number m can take $(2l+1)$ values and we have also two values for $j = l+1/2$ and $j = l-1/2$ corresponding to up and down polarities for the HAs. For the HLM, such as charmonioum $c\overline{c}$ and bottomonioum $b\overline{b}$, we have also three values for $j = l \pm 1$ and $j = l$. Thus, every state in the ordinary NRQM symmetries of energy for the improved Hulthén plus Hellmann potentials model temperature-dependent will be $3(2l+1)$ a sub-state in 3DNRQM-NCSP symmetries. Thus, the total complete degeneracy of obtained energy level of the improved Hulthén plus Hellmann potentials model TD is obtained as a sum of all allowed values l. Total degeneracy is thus,

$$\underbrace{\underbrace{\sum_{l=1}^{n-1} 2(2l+1) = 2n^2}_{\text{Ordinary NRQM symmetries}} \rightarrow 2\left(\sum_{l=1}^{n-1} 2(2l+1)\right) = 4n^2}\text{: HAs}\,, \tag{57}$$

and

$$\underbrace{\sum_{l=1}^{n-1} 2(2l+1) = 2n^2}_{\text{Ordinary NRQM symmetries}} \rightarrow 3\underbrace{\left(\sum_{l=1}^{n-1} 2(2l+1)\right) = 6n^2}\;: HLM\,. \tag{58}$$

5.1. New mass spectra of HLM

This section is devoted to deriving the mass spectra of $Q\overline{Q}\,(Q=c,b)$ charmonium and bottomonium,in the improved Hulthén plus Hellmann potentials model. It is well known that the spin of charmonium and bottomonium equal two values (0 or 1), because it consists of quark and anti-quark. For spin-1, we have three values of j $(j_1=l+1,\, j_2=l,\, j_3=l-1)$, which allows us corresponding three values $(k_1,k_2,k_3)=(1/2)\,(l,-2,-2l-2)$ and thus, we obtain three values of energy:

$$E_{n-g}^{hlm}=E_{nl}^{hhp}+g_s\left(F\left(n,m_D\left(T\right),A_1,A_2,A_0\right)(\aleph\lambda+\Omega\sigma)+\frac{\aleph\overline{\lambda}+\Omega\overline{\sigma}}{2\mu}\right)m$$
$$+g_s l\left(F\left(n,m_D\left(T\right),A_1,A_2,A_0\right)\Theta+\frac{\overline{\theta}}{2\mu}\right)\text{ if }\ j=l+1, \tag{59.1}$$

$$E_{n-m}^{hlm}=E_{nl}^{hhp}+g_s\left(F\left(n,m_D\left(T\right),A_1,A_2,A_0\right)(\aleph\lambda+\Omega\sigma)+\frac{\aleph\overline{\lambda}+\Omega\overline{\sigma}}{2\mu}\right)m$$
$$-g_s\left(F\left(n,m_D\left(T\right),A_1,A_2,A_0\right)\Theta+\frac{\overline{\theta}}{2\mu}\right)\text{ if }\ j=l, \tag{59.2}$$

and

$$E_{n-l}^{hlm}=E_{nl}^{hhp}+g_s\left(F\left(n,m_D\left(T\right),A_1,A_2,A_0\right)(\aleph\lambda+\Omega\sigma)+\frac{\aleph\overline{\lambda}+\Omega\overline{\sigma}}{2\mu}\right)m$$
$$-g_s\left(l+1\right)\left(F\left(n,m_D\left(T\right),A_1,A_2,A_0\right)\Theta+\frac{\overline{\theta}}{2\mu}\right)\text{ if }\ j=l-1\,. \tag{59.3}$$

In the symmetries of usual NRQM, the mass spectra $Q\overline{Q}\,(Q=c,b)$ obtained by applying the following formula [74, 75]:

$$M^{hhp}=2m_Q+E_{nl}^{hhp}\,. \tag{60}$$

Here, m_Q are the bare quark masses. Thus, the modified mass $M_{nc}^{hhp}\,(s=1)$ with spin-1 of $Q\overline{Q}\,(Q=A_2,A_0)$ charmonium and bottomonium, becomes

$$M_{nc}^{hhp}\left(s=1\right)=2m_Q+\frac{1}{3}\left(E_{n-g}^{hlm}+E_{n-m}^{hlm}+E_{n-l}^{hlm}\right)\,. \tag{61}$$

The value $(1/3)\left(E_{n-g}^{hlm}+E_{n-m}^{hlm}+E_{n-l}^{hlm}\right)$ physically represents the non-polarized energy (energy independent of spin). After a simple calculation we obtain the δM:

$$\delta M\left(s=1\right)=g_s\left(F\left(n,m_D\left(T\right),A_1,A_2,A_0\right)(\aleph\lambda+\Omega\sigma)+\frac{\aleph\overline{\lambda}+\Omega\overline{\sigma}}{2\mu}\right)m$$
$$-\frac{2}{3}g_s\left(F\left(n,m_D\left(T\right),A_1,A_2,A_0\right)\Theta+\frac{\overline{\theta}}{2\mu}\right)\,, \tag{62}$$

with $\delta M\left(s=1\right)\equiv M_{nc}^{hhp}\left(s=1\right)-M^{hhp}$, M^{hhp} is the mass spectra of the heavy quarkonium system [11]:

$$M^{hhp}=2m+A_0\left(\frac{1}{2}-\frac{m_D\left(T\right)}{4\delta}\right)+A_2m_D\left(T\right)\left(\frac{3m_D\left(T\right)}{2\delta}-m_D^2\left(T\right)-1\right)$$
$$-\frac{1}{8\mu}\left(\frac{2\mu\left[A_2-A_1+\frac{A_0}{m_D(T)}\right]+\frac{\mu m_D(T)}{\delta^2}\left[3m_D\left(T\right)-\frac{A_0}{2}\right]-\frac{8\mu A_2m_D^3(T)}{3\delta^3}}{n+\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{\mu A_2m_D^2(T)}{\delta^3}\left(1-\frac{m_D(T)}{\delta}\right)-\frac{\mu A_0m_D(T)}{6\delta^3}}}\right)^2\,. \tag{63}$$

This is the noncommutativity contribution for the mass spectra of $Q\overline{Q}$ charmonium and bottomonium under an improved Hulthén plus Hellmann potentials model. For spin-0, we have only one value of $(\ j=l)$, allows us the values $k=0$ and thus, we obtain the energy:

$$E_{n-g}^{hlm}=E_{nl+}+g_s\left(F\left(n,m_D\left(T\right),A_1,A_2,A_0\right)(\aleph\lambda+\Omega\sigma)+\frac{\aleph\overline{\lambda}+\Omega\overline{\sigma}}{2\mu}\right)m. \tag{64}$$

Thus, the modified mass $M_{nc}^{hhp}\left(s=0\right)$ with spin-0 of $Q\overline{Q}$, becomes

$$\delta M\left(s=0\right)\equiv M_{nc}^{hhp}\left(s=0\right)-M^{hhp}=g_{s}\left(F\left(n,m_{D}\left(T\right),A_{1},A_{2},A_{0}\right)\left(\aleph\lambda+\Omega\sigma\right)+\frac{\aleph\overline{\lambda}+\Omega\overline{\sigma}}{2\mu}\right)m. \tag{65}$$

5.2. Special case

Considering that the studied IHHPTd potential, in the presence of a temperature-dependent confined Coulomb potential in our paper is composed of four important potentials in terms of physical and chemical applications, we will address four special cases.

First: When we set $A_0=A_1=0$, the potential $V_{hhp}\left(r,T\right)$ of the HHPTd model in Eq. (7) reduces to the Yukawa potential in the presence of a temperature dependence as follows:

$$V_{hhp}\left(r,T\right)=-\frac{\beta_0}{r}+\beta_1 r-\beta_2 r^2+\beta_3\longmapsto V_{yp}\left(r,T\right)=-\frac{\beta_0^{yp}}{r}+\beta_1^{yp}r-\beta_2^{yp}r^2+\beta_3^{yp}, \tag{66}$$

with $\beta_0^{yp}=A_2$, $\beta_1^{yp}=A_2 m_D^2\left(T\right)/2$, $\beta_2^{yp}=A_2 m_D^3\left(T\right)/6$ and $\beta_3^{yp}=-A_2 m_D\left(T\right)$. The new global perturbed Hamiltonian operator H_{pert}^{yp} and the new effective perturbed potential of the improved Yukawa potential $V_{pert}^{yp}\left(r,T\right)$ in 3DNRQM-NCSP symmetries:

$$H_{pert}^{yp}=(\frac{l(l+1)}{r^4}-\frac{\beta_0^{yp}}{2r^3}-\frac{\beta_1^{yp}}{2r}+\beta_2^{yp})\mathbf{L\Theta}+\frac{\mathbf{L}\overline{\theta}}{2\mu}+O\left(\Theta^2,\overline{\theta}^2\right), \tag{67}$$

and

$$V_{pert}^{yp}\left(r,T\right)=(\frac{l(l+1)}{r^4}-\frac{\beta_0^{yp}}{2r^3}-\frac{\beta_1^{yp}}{2r}+\beta_2^{yp})\mathbf{L\Theta}+O\left(\Theta^2\right). \tag{68}$$

The new energy of HAs, under the improved Yukawa potential $V_{pert}^{yp}\left(r,T\right)$ will be reduced to the following form:

$$\begin{aligned}E_{nc-n}^{(u,d)hy}&=E_{nl}^{yp}+\left(F^{yp}\left(n,m_D\left(T\right),A_2\right)\left(\aleph\lambda+\Omega\sigma\right)+\frac{\aleph\overline{\lambda}+\Omega\overline{\sigma}}{2\mu}\right)\epsilon m\\&+\epsilon N_{nl}^2\left(F^{yp}\left(n,m_D\left(T\right),A_2\right)\Theta+\frac{\overline{\theta}}{2\mu}\right)\left\{\begin{array}{l}k_+\text{ for }j=l+1/2\\k_-\text{ for }j=l-1/2\end{array}\right.\end{aligned} \tag{69}$$

with $F^{yp}\left(n,m_D\left(T\right),A_1,A_2\right)=l(l+1)\langle 1/r^4\rangle_{(nlm)}-(\beta_0^{yp}/2)\langle 1/r^3\rangle_{(nlm)}-(\beta_1^{yp}/2)\langle 1/r\rangle_{(nlm)}+\beta_2^{yp}$ and E_{nl}^{yp} is given by [11]:

$$E_{nl}^{yp}=A_2 m_D\left(T\right)-\frac{1}{8\mu}\frac{2\mu A_2+\frac{3\mu A_2 m_D^2(T)}{\delta^2}-\frac{8\mu A_2 m_D^3(T)}{3\delta^3}}{n+\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{\mu A_2 m_D^3(T)}{\delta^3}\left(1-\frac{m_D(T)}{\delta}\right)}}. \tag{70}$$

The new of energy of HLM ($c\overline{c}$ and $b\overline{b}$) for the improved Yukawa potential in the presence of a temperature dependence in the 3DNRQM-NCSP symmetries obtained from Eq. (52.1), (52.2) and (52.3) is

$$\begin{aligned}E_{n-g}^{hlm}&=E_{nl}^{yp}+g_s\left(F^{yp}\left(n,m_D\left(T\right),A_2\right)\left(\aleph\lambda+\Omega\sigma\right)+\frac{\aleph\overline{\lambda}+\Omega\overline{\sigma}}{2\mu}\right)m\\&+g_s k_1\left(l\right)\left(F^{yp}\left(n,m_D\left(T\right),A_2\right)\Theta+\frac{\overline{\theta}}{2\mu}\right)\text{ if }j=l+1,\end{aligned} \tag{71.1}$$

$$\begin{aligned}E_{n-m}^{hlm}&=E_{nl}^{yp}+g_s\left(F^{yp}\left(n,m_D\left(T\right),A_2\right)\left(\aleph\lambda+\Omega\sigma\right)+\frac{\aleph\overline{\lambda}+\Omega\overline{\sigma}}{2\mu}\right)m\\&+g_s k_2\left(l\right)\left(F^{yp}\left(n,m_D\left(T\right),A_2\right)\Theta+\frac{\overline{\theta}}{2\mu}\right)\text{ if }j=l,\end{aligned} \tag{71.2}$$

and

$$\begin{aligned}E_{n-l}^{hlm}&=E_{nl}^{yp}+g_s\left(F^{yp}\left(n,m_D\left(T\right),A_2\right)\left(\aleph\lambda+\Omega\sigma\right)+\frac{\aleph\overline{\lambda}+\Omega\overline{\sigma}}{2\mu}\right)m\\&+g_s k_3\left(l\right)\left(F^{yp}\left(n,m_D\left(T\right),A_2\right)\Theta+\frac{\overline{\theta}}{2\mu}\right)\text{ if }j=l-1.\end{aligned} \tag{71.3}$$

Second: When we set $A_1 = A_2 = 0$, the potential $V_{hhp}(r,T)$ of the HHPTd model in Eq. (7) reduces to the Hulthén potential in the presence of a temperature dependence as

$$V_{hhp}(r,T) = -\frac{\beta_0}{r} + \beta_1 r - \beta_2 r^2 + \beta_3 \longmapsto V_{hp}(r,T) = -\frac{\beta_0^{hp}}{r} + \beta_1^{hp} r + \beta_3^{hp}, \tag{72}$$

with $\beta_0^{hp} = -A_0/m_D(T)$, $\beta_1^{hp} = -A_0(m_D(T)/12)$, $\beta_2^{hp} = 0$ and $\beta_3^{hp} = A_0/2$. The new global perturbed Hamiltonian operator H_{pert}^{yp} and the new effective perturbed potential of the improved Hellmann potential $V_{pert}^{yp}(r,T)$ in 3DNRQM-NCSP symmetries:

$$\begin{cases} H_{pert}^{hp} = \left(\frac{l(l+1)}{r^4} - \frac{\beta_0^{hp}}{2r^3} - \frac{\beta_1^{hp}}{2r}\right)\mathbf{L\Theta} + \frac{\mathbf{L}\overline{\theta}}{2\mu} + O\left(\Theta^2, \overline{\theta}^2\right) \\ V_{pert}^{hp}(r,T) = \left(\frac{l(l+1)}{r^4} - \frac{\beta_0^{hp}}{2r^3} - \frac{\beta_1^{hp}}{2r}\right)\mathbf{L\Theta} + O\left(\Theta^2\right) \end{cases}. \tag{73}$$

The new energy of the HAs such as He^{+}, Li^{+2} and Be^{+},

$$\begin{aligned} E_{nc-n}^{(u,d)hy} &= E_{nl}^{hp} + \left(F^{hp}(n, m_D(T), A_2)(\aleph\lambda + \Omega\sigma) + \frac{\aleph\overline{\lambda} + \Omega\overline{\sigma}}{2\mu}\right)\epsilon m \\ &+ \epsilon N_{nl}^2 \left(F^{hp}(n, m_D(T), A_2)\Theta + \frac{\overline{\theta}}{2\mu}\right)\begin{cases} k_+ \text{ for } j = l + 1/2 \\ k_- \text{ for } j = l - 1/2 \end{cases}, \end{aligned} \tag{74}$$

with $F^{hp}(n, m_D(T), A_1, A_2) = l(l+1)\langle 1/r^4\rangle_{(nlm)} - (\beta_0^{hp}/2)\langle 1/r^3\rangle_{(nlm)} - (\beta_1^{hp}/2)\langle 1/r\rangle_{(nlm)}$ and E_{nl}^{hp} is given by [11]:

$$\begin{aligned} E_{nl}^{hp} &= A_0\left(\frac{1}{2} - \frac{m_D(T)}{4\delta}\right) - 2A_0 m_D^2(T)\left(\frac{3m_D(T)}{2\delta} - m_D^2(T) - 1\right) \\ &- \frac{1}{8\mu}\frac{2\mu\frac{A_0}{m_D(T)} - \frac{\mu m_D(T) A_0}{2\delta^2}}{n + \frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2\left(1 - \frac{m_D(T)}{\delta}\right)} - \frac{\mu A_0 m_D(T)}{6\delta^3}}. \end{aligned} \tag{75}$$

The new of energy of HLM ($c\overline{c}$ and $b\overline{b}$) for the improved Hulthén potential in the presence of a temperature dependence in the 3DNRQM-NCSP symmetries obtained from Eq. (52.1), (52.2) and (52.3) is

$$\begin{aligned} E_{n-g}^{hlm} &= E_{nl}^{hp} + g_s\left(F^{hp}(n, m_D(T), A_0)(\aleph\lambda + \Omega\sigma) + \frac{\aleph\overline{\lambda} + \Omega\overline{\sigma}}{2\mu}\right)m \\ &+ g_s k_1(l)\left(F^{hp}(n, m_D(T), A_0)\Theta + \frac{\overline{\theta}}{2\mu}\right) \text{ if } j = l + 1, \end{aligned} \tag{76.1}$$

$$\begin{aligned} E_{n-m}^{hlm} &= E_{nl}^{hp} + g_s\left(F^{hp}(n, m_D(T), A_0)(\aleph\lambda + \Omega\sigma) + \frac{\aleph\overline{\lambda} + \Omega\overline{\sigma}}{2\mu}\right)m \\ &+ g_s k_2(l)\left(F^{hp}(n, m_D(T), A_0)\Theta + \frac{\overline{\theta}}{2\mu}\right) \text{ if } j = l \end{aligned} \tag{76.2}$$

and

$$\begin{aligned} E_{n-l}^{hlm} &= E_{nl}^{hp} + g_s\left(F^{hp}(n, m_D(T), A_0)(\aleph\lambda + \Omega\sigma) + \frac{\aleph\overline{\lambda} + \Omega\overline{\sigma}}{2\mu}\right)m \\ &+ g_s k_3(l)\left(F^{hp}(n, m_D(T), A_0)\Theta + \frac{\overline{\theta}}{2\mu}\right) \text{ if } j = l - 1. \end{aligned} \tag{76.3}$$

Third: When we set $A_0 = 0$, the potential $V_{hhp}(r,T)$ of the HHPTd model in Eq. (7) reduces to the Hellmann potential in the presence of temperature dependence as

$$V_{hhp}(r,T) = -\frac{\beta_0}{r} + \beta_1 r - \beta_2 r^2 + \beta_3 \longmapsto V_{hp}(r,T) = -\frac{\beta_0^{hlp}}{r} + \beta_1^{hlp} r - \beta_2^{hlp} r^2 + \beta_3^{hlp}, \tag{77}$$

with $\beta_0^{hlp}=A_2-A_1$, $\beta_1^{hlp}=A_2m_D^2(T)/2$, $\beta_2^{hlp}=A_2m_D^3(T)/6$ and $\beta_3^{hlp}=-A_2m_D(T)$. The new global perturbed Hamiltonian operator H_{pert}^{yp} and the new effective perturbed potential of the improved Hellmann potential $V_{pert}^{hlp}(r,T)$ in 3DNRQM-NCSP symmetries:

$$\left\{\begin{array}{l} H_{pert}^{hlp}=\left(\frac{l(l+1)}{r^4}-\frac{\beta_0^{hlp}}{r}+\beta_1^{hlp}r-\beta_2^{hlp}r^2\right)\mathbf{L}\mathbf{\Theta}+\frac{\mathbf{L}\overline{\theta}}{2\mu}+O\left(\Theta^2,\overline{\theta}^2\right) \\ V_{pert}^{hlp}(r,T)=\left(\frac{l(l+1)}{r^4}-\frac{\beta_0^{hlp}}{r}+\beta_1^{hlp}r-\beta_2^{hlp}r^2\right)\mathbf{L}\mathbf{\Theta}+O\left(\Theta^2\right) \end{array}\right. . \tag{78}$$

The new energy of HAs,

$$E_{nc-n}^{(u,d)hy}=E_{nl}^{hlp}+\left(F^{hlp}(n,m_D(T),A_1,A_2)(\aleph\lambda+\Omega\sigma)+\frac{\aleph\overline{\lambda}+\Omega\overline{\sigma}}{2\mu}\right)\epsilon m$$
$$+\epsilon N_{nl}^2\left(F^{hlp}(n,m_D(T),A_1,A_2)\Theta+\frac{\overline{\theta}}{2\mu}\right)\left\{\begin{array}{l} k_+ \text{ for } j=l+1/2 \\ k_- \text{ for } j=l-1/2 \end{array}\right. \tag{79}$$

with $F^{hlp}(n,m_D(T),A_1,A_2)=l(l+1)\langle\frac{1}{r^4}\rangle_{(nlm)}-\frac{\beta_0^{hlp}}{2}\langle\frac{1}{r^3}\rangle_{(nlm)}-\frac{\beta_1^{hlp}}{2}\langle\frac{1}{r}\rangle_{(nlm)}+\beta_2^{hlp}$ and E_{nl}^{hlp} is given by [11]:

$$E_{nl}^{hlp}=A_2m_D(T)-\frac{1}{8\mu}\frac{2\mu[A_2-A_1]+\frac{3\mu m_D^2(T)}{\delta^2}-\frac{8\mu A_2m_D^3(T)}{3\delta^3}}{n+\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{\mu A_2m_D^3(T)}{\delta^3}\left(1-\frac{m_D(T)}{\delta}\right)}} \tag{80}$$

The new of energy of the HLM ($c\overline{c}$ and $b\overline{b}$) for the improved Hellmann potential in the presence of a temperature dependence in the 3DNRQM-NCSP symmetries obtained from Eq. (52.1), (52.2) and (52.3) is

$$E_{n-g}^{hlm}=E_{nl}^{hlp}+g_s\left(F^{hlp}(n,m_D(T),A_1,A_2)(\aleph\lambda+\Omega\sigma)+\frac{\aleph\overline{\lambda}+\Omega\overline{\sigma}}{2\mu}\right)m$$
$$+g_sk_1(l)\left(F^{hlp}(n,m_D(T),A_1,A_2)\Theta+\frac{\overline{\theta}}{2\mu}\right) \text{ if } j=l+1, \tag{81.1}$$

$$E_{n-m}^{hlm}=E_{nl}^{hlp}+g_s\left(F^{hlp}(n,m_D(T),A_1,A_2)(\aleph\lambda+\Omega\sigma)+\frac{\aleph\overline{\lambda}+\Omega\overline{\sigma}}{2\mu}\right)m$$
$$+g_sk_2(l)\left(F^{hlp}(n,m_D(T),A_1,A_2)\Theta+\frac{\overline{\theta}}{2\mu}\right) \text{ if } j=l, \tag{81.2}$$

and

$$E_{n-l}^{hlm}=E_{nl}^{hlp}+g_s\left(F^{hlp}(n,m_D(T),A_1,A_2)(\aleph\lambda+\Omega\sigma)+\frac{\aleph\overline{\lambda}+\Omega\overline{\sigma}}{2\mu}\right)m$$
$$+g_sk_3(l)\left(F^{hlp}(n,m_D(T),A_1,A_2)\Theta+\frac{\overline{\theta}}{2\mu}\right) \text{ if } j=l-1. \tag{81.3}$$

Forth: When we set $A_0=A_2=m_D(T)=0$, the potential $V_{hhp}(r,T)$ of the HHPTd model in Eq. (7) reduced to the Coulomb potential in the presence of a temperature dependence as

$$V_{hhp}(r,T)=-\frac{\beta_0}{r}+\beta_1r-\beta_2r^2+\beta_3\longmapsto V_{hp}(r,T)=-\frac{\beta_0^{cp}}{r}, \tag{82}$$

with $\beta_0^{cp}=-A_1$, $\beta_1^{cp}=0$, $\beta_2^{cp}=0$ and $\beta_3^{cp}=0$. The new global perturbed Hamiltonian operator H_{pert}^{yp} and the new effective perturbed potential of the improved Hellmann potential $V_{pert}^{hlp}(r,T)$ in 3DNRQM-NCSP symmetries:

$$\left\{\begin{array}{l} H_{pert}^{cp}=\left(\frac{l(l+1)}{r^4}-\frac{\beta_0^{cp}}{r}\right)\mathbf{L}\mathbf{\Theta}+\frac{\mathbf{L}\overline{\theta}}{2\mu}+O\left(\Theta^2,\overline{\theta}^2\right) \\ V_{pert}^{cp}(r,T)=\left(\frac{l(l+1)}{r^4}-\frac{\beta_0^{cp}}{r}\right)\mathbf{L}\mathbf{\Theta}+O\left(\Theta^2\right) \end{array}\right. . \tag{83}$$

The new energy of the HAs,

$$E_{nc-n}^{(u,d)hy}=E_{nl}^{cp}+\left(F^{cp}(n,m_D(T),A_1)(\aleph\lambda+\Omega\sigma)+\frac{\aleph\overline{\lambda}+\Omega\overline{\sigma}}{2\mu}\right)\epsilon m$$
$$+\epsilon N_{nl}^2\left(F^{cp}(n,m_D(T),A_1)\Theta+\frac{\overline{\theta}}{2\mu}\right)\left\{\begin{array}{l} k_+ \text{ for } j=l+1/2 \\ k_- \text{ for } j=l-1/2 \end{array}\right. , \tag{84}$$

with $F^{cp}\left(n, m_D\left(T\right), A_1\right) = l(l+1)\langle 1/r^4\rangle_{(nlm)} - (\beta_0^{cp}/2)\langle 1/r^3\rangle_{(nlm)}$ and E_{nl}^{cp} is given by [11]:

$$E_{nl}^{cp} = -\frac{1}{8\mu}\frac{-2\mu A_1}{n+\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2}}\,. \tag{85}$$

The new of energy of HLM ($c\overline{c}$ and $b\overline{b}$) for the improved Coulomb potential in the presence of a temperatura dependence in the 3DNRQM-NCSP symmetries obtained from Eq. (52.1), (52.2) and (52.3) is

$$E_{n-g}^{hlm} = E_{nl}^{hlp} + g_s\left(F^{cp}\left(n, m_D\left(T\right), A_1\right)\left(\aleph\lambda+\Omega\sigma\right) + \frac{\aleph\overline{\lambda}+\Omega\overline{\sigma}}{2\mu}\right)m$$
$$+ g_s k_1\left(l\right)\left(F^{cp}\left(n, m_D\left(T\right), A_1\right)\Theta + \frac{\overline{\theta}}{2\mu}\right) \text{ if } j = l+1\ , \tag{86.1}$$

$$E_{n-m}^{hlm} = E_{nl}^{hlp} + g_s\left(F^{cp}\left(n, m_D\left(T\right), A_1\right)\left(\aleph\lambda+\Omega\sigma\right) + \frac{\aleph\overline{\lambda}+\Omega\overline{\sigma}}{2\mu}\right)m$$
$$+ g_s k_2\left(l\right)\left(F^{cp}\left(n, m_D\left(T\right), A_1\right)\Theta + \frac{\overline{\theta}}{2\mu}\right) \text{ if } j = l\,, \tag{86.2}$$

and

$$E_{n-l}^{hlm} = E_{nl}^{hlp} + g_s\left(F^{cp}\left(n, m_D\left(T\right), A_1\right)\left(\aleph\lambda+\Omega\sigma\right) + \frac{\aleph\overline{\lambda}+\Omega\overline{\sigma}}{2\mu}\right)m$$
$$+ g_s k_3\left(l\right)\left(F^{cp}\left(n, m_D\left(T\right), A_1\right)\Theta + \frac{\overline{\theta}}{2\mu}\right) \text{ if } j = l-1 \tag{86.3}$$

We have obtained the solutions to the SchrÃ¶dinger equation, the most well-known nonrelativistic wave equation described without spin, but its extension in 3DNRQM-NCSP symmetries under the improved Hulthén plus Hellmann potentials model temperature-dependent in the presence of a temperature-dependent confined Coulomb potential has a physical behavior similar to the Dirac equation [76] for fermionic particles with spin-1/2, it can describe the dynamic state of a particle with spin-1/2 for HAs such as He^{+}, Li^{+2} and Be^{+}or similar to the relativistic Duffin-Kemmer equation [77–79] for mesons with spin-(0,1) for the heavy quarkonium systems $c\overline{c}$ and $b\overline{b}$, which can describe a dynamic state of a particle with spin one in the symmetries of RNCQM. The conventional nonrelativistic approach of a SE under the improved Hulthén plus Hellmann potentials model temperature-dependent in the presence of the temperature-dependent confined Coulomb potential involves solving the second-order KGE for spin-0 and the Proca equation for spin-1 [80]. Worthwhile it is better to mention that for the two simultaneous limits $(\Theta, \lambda, \sigma) \rightarrow (0,0,0)$ we recover the results of the Refs. [11,12]. It is possible to recover the results of commutative space when we consider $(\Theta\left(\lambda,\sigma\right), \overline{\theta}\left(\overline{\lambda},\overline{\sigma}\right))$ equal (0,0). In the limit $(\Theta, \overline{\theta}) \longrightarrow (0,0)$, we have:

$$\begin{cases} \lim\limits_{(\Theta,\overline{\theta})\longrightarrow(0,0)} E_{nc-n}^{(u,d)hy} = E_{nl} \\ \lim\limits_{(\Theta,\overline{\theta})\longrightarrow(0,0)} \left(E_{n-g}^{hlm}, E_{n-m}^{hlm}, E_{n-l}^{hlm}\right) = E_{nl} \end{cases}. \tag{87}$$

6. Conclusion

In this study, we adopted an inversely quadratic Yukawa potential plus Yukawa potential and Coulomb potential which we considered as Hulthén plus Hellmann potentials for the hydrogen atoms (He^{+}, Li^{+2} and Be^{+}) and quark-antiquark interaction. The potential was made to be temperature-dependent by replacing the screening parameter α with a Debye mass $m_D\left(T\right)$ which vanishes at $T \rightarrow 0$ in the presence of 3DNRQM-NCSP symmetries. The deformed Schrödinger equation is analytically solved using the generalized Bopp's shift method and standard perturbation theory. We obtained new approximate solutions of the eigenvalues $E_{nc-n}^{(u,d)hy}(n, m_D\left(T\right), A_1, A_2, A_0, j, l, m, s)$ for the hydrogenic atoms and ($E_{n-g}^{hlm}(n, m_D\left(T\right), A_1, A_2, A_0, j, l, m, s)$, $E_{n-m}^{hlm}(n, m_D\left(T\right), A_1, A_2, A_0, j, l, m, s)$, $E_{n-l}^{hlm}\left(n, m_D\left(T\right), A_1, A_2, A_0, j, l, m, s\right)$) for the heavy quarkonioum systems, such as charmonioum $c\overline{c}$ and bottomonioum $b\overline{b}$. The new energy values are sensitive to atomic quantum numbers (j, n, l, s and m), the noncommutativity parameter (Θ, σ, χ) due to the topological properties of the self-quantum influence of space-space and phase-phase, in NRNCQM_3D-RSP symmetries, in addition to the discreet atomic quantum numbers (n, l) and the parameters ($\mu, A_2,\ d$) of the IHH-PTd model that appeared in the literature. We have shown that new global Hamiltonian operator $H_{hhp}^{nc}\left(r, \Theta, \overline{\theta}, \lambda, \overline{\lambda}\right)$ in (NRNCQM_3D-RSP) symmetries is the sum of the main Hamiltonian operator of the Hulthén plus Hellmann potentials model temperature-dependent $H_{hhp}\left(p, x\right)$ and three perturbed operators; the first one is the modified spin-orbit interaction $H_{so}^{hhp}\left(r,\Theta,\overline{\theta}\right)$, the second is the modified Zeeman operator $H_{z}^{hhp}\left(r,\lambda,\overline{\lambda}\right)$ while the third operator $H_{pert}^{hhp-rot}$ is the perturbed Fermi Hamiltonian for the hydrogen atoms He^{+}, Li^{+2} and $Be^{+}$and the heavy quarkonioum systems. Consequently, the ordinary kinetic term $-\Delta/2\mu$ modified to the new form ($-\Delta/2\mu - \mathbf{L}\overline{\theta}/2\mu - \mathbf{L}\overline{\sigma}/2\mu - \mathbf{L}\overline{\lambda}/2\mu$) for the IHHPTd model in 3DNRQM-NCSP symmetries. Furthermore, we applied the present results to calculate heavy-meson masses such as charmonioum $c\overline{c}$ and bottomonioum $b\overline{b}$.

It has been shown that the DSE under an improved Hulthén plus Hellmann potentials model with temperature dependence presents a useful symmetry to solving the hydrogenic atoms He^{+},$Li^{2+}$and Be^{+} and the heavy quarkonioum systems, such as charmonioum $c\overline{c}$ and bottomonioum $b\overline{b}$. It should be noted that the results obtained in this research would be identical with corresponding results in ordinary quantum mechanics when the limits $(\Theta,\lambda,\sigma)\rightarrow(0,0,0)$ are applied simultaneously.

Consequently, the study of the analytical solution of the three-dimensional deformed Schrödinger equation for hydrogenic atoms and the heavy quarkonioum systems under the improved Hulthén plus Hellmann potentials temperature-dependent model in 3DNRQM-NCSP symmetries could provide valuable information in many physical fields, and opens a new big window for profound theoretical and experimental research. The generalized Bopp's shift method used in this paper is efficient and systematically gives physical and practical solutions to interesting problems, it provides logical and realistic solutions to physics problems that were considered very complex in the past, and can be used to obtain the solutions of other potentials of practical value and prospective importance.

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1. S.M. Ikhdair, *Relativistic Bound States of Spinless Particle by the Cornell Potential Model in External Fields*, Adv. High Energy Phys. **2013** (2013) 491648, https://doi.org/10.1155/2013/491648
2. E.P. Inyang, E.S. William and J.A. Obu, *Eigensolutions of the N-dimensional Schrödinger equation interacting with Varshni-Hulthén potential model*, Rev. Mex. Fis. **67** (2021) 193, https://doi.org/10.31349/RevMexFis.67.193
3. E.S. William, E.P. Inyang and E.A. Thompson, *Arbitrary l-solutions of the Schrödinger equation interacting with Hulthén-Hellmann potential model*, Rev. Mex. Fis. **66** (2021) 730, https://doi.org/10.31349/RevMexFis.66.730
4. C.O. Edet, P.O. Okoi, A.S. Yusuf and P.O. Amadi, *Bound state solutions of the generalized shifted Hulthén potential*,. Indian J Phys **95** (2021) 471, https://doi.org/10.1007/s12648-019-01650-0
5. C.O. Edet, P.O. Amadi, M.C. Onyeaju, U.S. Okorie, R. Sever, G.J. Rampho Hewa, Y. Abdullah, I.H. Salih, A.N. Ikot, *Thermal Properties and Magnetic Susceptibility of Hellmann Potential in Aharonov-Bohm (AB) Flux and Magnetic Fields at Zero and Finite Temperatures*, J Low Temp Phys **202** (2021) 83, https://doi.org/10.1007/s10909-020-02533-z
6. C.O. Edetaand P.O. Okoi, *Any l-state solutions of the Schrödinger equation for q-deformed Hulthén plus generalized inverse quadratic Yukawa potential in arbitrary dimensions*, Rev. Mex. Fis. **65** (2019) 333, https://doi.org/10.31349/RevMexFis.65.333
7. C.O. Edet, K.O. Okorie, H. Louis and N.A. Nzeata-Ibe, *Any l-state solutions of the Schrödinger equation interacting with Hellmann-Kratzer potential model*, Indian J Phys **94** (2020) 243, https://doi.org/10.1007/s12648-019-01467-x
8. A. Al-Jamel and H. Widyan, *Heavy Quarkonium Mass Spectra in A Coulomb Field Plus Quadratic Potential Using Nikiforov-Uvarov Method*, Applied Physics Research **4** (2012), http://dx.doi.org/10.5539/apr.v4n3p94
9. M. Abu-Shady, *Heavy Quarkonia and Mesons in the Cornell Potential with Harmonic Oscillator Potential in the N-dimensional Schrödinger Equation*, international Journal of Applied Mathematics and Theoretical Physics **2** (2016) 16, http://dx.doi.org/10.11648/j.ijamtp.20160202.11
10. E.S. William, E. P. Inyang and E. A. Thompson, Arbitrary *l*-solutions of the Schrödinger equation interacting with Hulthén-Hellmann potential model, Revista Mexicana de Física **66**(2020) 730, https://doi.org/10.31349/RevMexFis.66.730
11. I.O. Akpan, E.P. Inyang, E.P. Inyang, and E.S. William, Approximate solutions of the Schrödinger equation with Hulthén-Hellmann potentials for a quarkonium system, *Rev. Mex. Fis.* **67** (2021) 482, https://doi.org/10.31349/RevMexFis.67.482
12. E.P. Inyang, I.O. Akpan, J.E. Ntibi and E.S. William, *Masses and thermodynamic properties of a quarkonium system, Canadian Journal of Physics* **99** (2021) 1, https://doi.org/10.1139/cjp-2020-0578
13. R.J. Adler and D.I. Santigo, *On gravity and the uncertainty principale*, Mod. Phys. Lett. A **14** (1999) 1371, https://doi.org/10.1142/s0217732399001462.
14. F. Scardigli, *Generalized uncertainty principle in quantum gravity from micro-black hole gedanken experiment*, Phys.

Lett. B **452** (1999) 39, https://doi.org/10.1016/s0370-2693(99)00167-7

15. P. M. Ho and H. C. Kao, *Noncommutative Quantum Mechanics from Noncommutative Quantum Field Theory*, Phys. Rev. Lett. **88** (2002) 151602, https://doi.org/10.1103/physrevlett.88.151602
16. P. Gnatenkon, *Parameters of noncommutativity in Lie-algebraic noncommutative space*, Phys. Rev. D **99** (2019) 026009, https://doi.org/10.1103/physrevd.99.026009
17. A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and Matrix theory*, Journal of High Energy Physics **1998**(1998) 003, https://doi.org/10.1088/1126-6708/1998/02/003
18. O. Bertolami, J.G. Rosa, C.M.L. De Aragao, P. Castorina and D. Zappala, *Scaling of variables and the relation between noncommutative parameters in noncommutative quantum mechanics*, Mod. Phys. Lett. A **21** (2006) 795, https://doi.org/10.1142/s0217732306019840
19. A. Maireche, *A Model of Modified Klein-Gordon Equation with Modified Scalar-vector Yukawa Potential*, Afr. Rev Phys. **15** (2020) 1-11.
20. A. Maireche, *Investigations on the Relativistic Interactions in One-Electron Atoms with Modified Yukawa Potential for Spin 1/2 Particles*, International Frontier Science Letters **11** (2017) 29, https://doi.org/10.18052/www.scipress.com/IFSL.11.29
21. A. Maireche, *Nonrelativistic treatment of Hydrogen-like and neutral atoms subjected to the generalized perturbed Yukawa potential with centrifugal barrier in the symmetries of noncommutative Quantum mechanics*, International Journal of Geometric Methods in Modern Physics **17** (2020) 2050067, https://doi.org/10.1142/S021988782050067X
22. A. Maireche, *Modified Unequal Mixture Scalar Vector Hulthén-Yukawa Potentials Model as a quark-antiquark Interaction and Neutral Atoms via Relativistic Treatment Using the Approximation of the Centrifugal Term and Bopp's Shift Method*, Few-Body Syst. **61** (2020) 30. https://doi.org/10.1007/s00601-020-01559-z.
23. A. Maireche, *The Investigation of Approximate Solutions of Deformed Klein-Gordon and Schrödinger Equations Under Modified More General Exponential Screened Coulomb Potential Plus Yukawa Potential in NCQM Symmetries*, Few-Body Syst **62** (2021). https://doi.org/10.1007/s00601-021-01639-8
24. A. Maireche, *A Theoretical Model of Deformed Klein-Gordon Equation with Generalized Modified Screened Coulomb Plus Inversely Quadratic Yukawa Potential in RNCQM Symmetries*, Few-Body Syst. **62** (2021) 12, https://doi.org/10.1007/s00601-021-01596-2
25. A. Maireche, *Solutions of Klein-Gordon equation for the modified central complex potential in the symmetries of noncommutative quantum mechanics*, Sri Lankan Journal of Physics **22** (2021) 1, http://doi.org/10.4038/sljp.v22i1.8079
26. A. Maireche and Djenaoui Imane, *A New Nonrelativistic Investigation for Spectra of Heavy Quarkonia with Modified Cornell Potential: Noncommutative Three-Dimensional Space and Phase Space Solutions*, J. Nano- Electron. Phys. **8** (2016) 03024, https://doi.org/10.21272/jnep.8(3).03025
27. A. Maireche, *A New Nonrelativistic Atomic Energy Spectrum of Energy Dependent Potential for Heavy Quarkouniom in Noncommutative Spaces and Phases Symmetries*, J. Nano-Electron. Phys. **8** (2016) 02046, https://doi.org/10.21272/jnep.8(2).02046
28. A. Maireche, *A New Model for Describing Heavy-Light Mesons in The Extended Nonrelativistic Quark Model Under a New Modified Potential Containing Cornell, Gaussian and Inverse Square Terms in The Symmetries Of NCQM*, To Phys. J. **3** (2019) 197-215.
29. A. Maireche, *A New Asymptotic Study to the 3-Dimensional Radial Schrödinger Equation under Modified Quark-antiquark Interaction Potential*, J Nanosci Curr Res. **4**((2019) 1000131.
30. A. Maireche, *New Relativistic Atomic Mass Spectra of Quark (u, d and s) for Extended Modified Cornell Potential in Nano and Plank's Scales*, J. Nano-Electron. Phys. **8** (2016) 01020, https://doi.org/10.21272/JNEP.8(1).01020
31. A. Maireche, *Heay-light mesons in the symmetries of extended nonrelativistic quark model, Yanbu Journal of Engineering and Science* **17** (2019) 51,
32. A. Maireche, *A theoretical investigation of nonrelativistic bound state solution at finite temperature using the sum of modified Cornell plus inverse quadratic potential*, Sri Lankan Journal of Physics **21** (2020) 11, http://DOI.org/10.4038/sljp.v20i0
33. P. Faccioli, C. Lourenço, M. Araújo, V. Knünz, I. Krätschmerand J. Seixas, *Quarkonium production at the LHC: A data-driven analysis of remarkably simple experimental patterns*, Physics Letters B **773** (2017) 476, https://doi.org/10.1016/j.physletb.2017.09.00
34. A.A. Safaeia, H. Panahia, and H. Hassanabadi, *Analysis of nonrelativistic particles in noncommutative phase-space under new scalar and vector interaction terms*, Rev. Mex. Fis. **67** (2021) 84, https://doi.org/10.31349/RevMexFis.67.84.
35. J. Zhang, Fractional angular momentum in non-commutative spaces, Phys. Lett. B **584** (2004) 204, https://doi.org/10.1016/j.physletb.2004.01.049
36. A. Maireche, *A New Approach to the Approximate Analytic Solution of the Three-Dimensional Schrödinger Equation for hydrogenic and Neutral Atoms in the Generalized Hellmann Potential Model*, Ukr. J. Phys. **65** (2020) 987, https://doi.org/10.15407/ujpe65.11.987
37. A. Zachos, *Geometrical evaluation of star products*, Journal of Mathematical Physics **41** (2000) 5129, https://doi.org/10.1063/1.533395
38. E.M.C. Abreu, J.N. Ananias, A.C.R. Mendes, C. Neves, W. Oliveira and M.V. Marcial, *Lagrangian formulation for noncommutative nonlinear systems*, International Journal of Modern Physics A **27**(2012) 1250053. https://doi.org/10.1142/s0217751x12500534
39. J. Gamboa, M. Loewe and J.C. Rojasn, *Noncommutative quantum mechanics*, Phys. Rev. D **64** (2001) 067901, https://doi.org/10.1103/PhysRevD.64.067901.

40. E.F. Djemaï and H. Smail, *On Quantum Mechanics on Noncommutative Quantum Phase Space*, Commun. Theor. Phys. (Beijing, China) **41** (2004) 837, https://doi.org/10.1088/0253-6102/41/6/837

41. Y. Yi, L. Kang, W. Jian-Hua and C. Chi-Yi, *Spin-1/2 relativistic particle in a magnetic field in NC phase space*, Chinese Physics C **34** (2010) 543, https://doi.org/10.1088/1674-1137/34/5/005

42. S. Aghababaei and G. Rezaei, *Energy level splitting of a 2D hydrogen atom with Rashba coupling in non-commutative space, Commun*, Theor. Phys. **72** (2020) 125101, https://doi.org/10.1088/1572-9494/abb7cc

43. O. Bertolami and P. Leal, *Aspects of phase-space noncommutative quantum mechanics*, Phys. Lett. B **750** (2015) 6, https://doi.org/10.1016/j.physletb.2015.08.024

44. O. Bertolami, J.G. Rosa, C.M. L. de Aragao, P. Castorina and D. Zappala, *Noncommutative gravitational quantum well*, Phys. Rev. D **72** (2005) 025010, https://doi.org/10.1103/physrevd.72.025010

45. K. Li and J. Wang, *The topological AC effect on noncommutative phase space*, The European Physical Journal C **50** (2007) 1007, https://doi.org/10.1140/epjc/s10052-007-0256-0

46. J. Wang and K. Li, *The HMW effect in noncommutative quantum mechanics*, Journal of Physics A: Mathematical and Theoretical **40** (2007) 2197, https://doi.org/10.1088/1751-8113/40/9/021

47. A. Maireche, *New Bound State Energies for Spherical Quantum Dots in Presence of a Confining Potential Model at Nano and Plank's Scales,* NanoWorld J. **1** (2016) 122, https://doi.org/10.17756/nwj.2016-016.

48. M.A. De Andrade and C. Neves, *Noncommutative mapping from the symplectic formalism*, Journal of Mathematical Physics **59**(2018) 012105, https://doi.org/10.1063/1.4986964.

49. E.M.C. Abreu, C. Neves and W. Oliveira, *Noncommutative from the symplectic point of view*, International Journal of Modern Physics A **21** (2006) 5359, https://doi.org/10.1142/s0217751x06034094

50. K.P. Gnatenko and V.M. Tkachuk, *Upper bound on the momentum scale in noncommutative phase space of canonical type*, EPL (Euro physics Letters) **127** (2019) 20008, https://doi.org/10.1209/0295-5075/127/2000

51. K.P. Gnatenko and Tkachuk, *Composite system in rotationally invariant noncommutative phase space*, Int. J. Mod. Phys. A **33** (2018) 1850037. https://doi.org/10.1142/s0217751x18500379

52. A. Maireche, *A New Relativistic Study for Interactions in One-electron atoms (Spin Particles) with Modified Mie-type Potential*, J. Nano- Electron. Phys. **8** (2016) 04027, https://doi.org/10.21272/jnep.8(4(1)).04027

53. A. Maireche, *A Complete Analytical Solution of the Mie-Type Potentials in Non-commutative 3-Dimensional Spaces and Phases Symmetries,* Afr. Rev. Phys. **11** (2016) 111, http://lamp.ictp.it/index.php/aphysrev/article/view/1259/490.

54. A. Maireche, *Modified Schrödinger Equation with Modified QM Gravitational and Harmonic Oscillator Potentials in Symmetries of Non-commutative QM*, Afr. Rev. Phys. **14** (2019) 130, http://lamp.ictp.it/index.php/aphysrev/article/view/1717/613.

55. A. Maireche, *Bound state solutions of the modified Klein-Gordon and Schrödinger equations for arbitrary l-state with the modified Morse potential in the symmetries of noncommutative quantum mechanics*, Journal of Physical Studies **25** (2021) 1002, https://doi.org/10.30970/jps.25.1002

56. A. Maireche, *A theoretical study of the modified equal scalar and vector Manning-Rosen potential within the deformed Klein-Gordon and Schrödinger in relativistic noncommutative quantum mechanics and nonrelativistic noncommutative quantum mechanics symmetries*, Rev. Mex. Fis. **67** (2021) 050702 1, https://doi.org/10.31349/RevMexFis.67.050702

57. L. Gouba, *A comparative review of four formulations of noncommutative quantum mechanics*, Int. J. Mod. Phys. A **31** (2016) 1630025, https://doi.org/10.1142/s0217751x16300258

58. L. Mezincescu, Star operation in quantum mechanics, e-print (2000). arXiv:hep-th/0007046v2.

59. F. Bopp, La mécanique quantique est-elle une mécanique statistique classique particulière Ann. Inst. Henri Poincaré **15** (1956) 81.

60. A. Maireche, *The Relativistic and Nonrelativistic Solutions for the Modified Unequal Mixture of Scalar and Time-Like Vector Cornell Potentials in the Symmetries of Noncommutative Quantum Mechanics*, Jordan J. Phys. **14** (2021) 59, https://doi.org/10.47011/14.1.6

61. A. Maireche, *The Klein-Gordon equation with modified Coulomb plus inverse-square potential in the noncommutative three-dimensional space*, Mod. Phys. Lett. A **35** (2020) 052050015. https://doi.org/10.1142/s0217732320500157.

62. A. Maireche, *Bound-state Solutions of Klein-Gordon and Schrödinger Equations for Arbitrary l-state with Linear Combination of Hulthén and Kratzer Potentials*, Afr. Rev Phys. **15** (2020) 19, http://lamp.ictp.it/index.pIGCp/aphysrev/article/view/1779/620.

63. H. Motavalli and A.R. Akbarieh, *Klein-Gordon equation for the Coulomb potential in noncommutative space*, Mod. Phys. Lett. A **25** (2010) 2523, https://doi.org/10.1142/s0217732310033529

64. M. Darroodi, H. Mehraban and H. Hassanabadi, *The Klein Gordon equation with the Kratzer potential in the noncommutative space*, Mod. Phys. Lett. A **33** (2018) 1850203. https://doi.org/10.1142/s0217732318502036

65. A. Maireche, *Theoretical Investigation of the Modified Screened cosine Kratzer potential via Relativistic and Nonrelativistic treatment in the NCQM symmetries*, Lat. Am. J. Phys. Educ. **14** (2020) 3310-1.

66. A. Maireche, *A new theoretical study of the deformed unequal scalar and vector Hellmann plus modified Kratzer potentials within the deformed Klein–Gordon equation in RNCQM symmetries, Mod. Phys. Lett. A* **38** (2021) 2150232, https://doi.org/10.1142/S0217732321502321

67. A. Maireche, *The investigation of approximate solutions of Deformed Klein-Fock-Gordon and Schrödinger Equations under Modified Equal Scalar and Vector Manning-Rosen and Yukawa Potentials by using the Improved Approximation of the Centrifugal term and Bopp's shift Method in NCQM Symmetries*, Lat. Am. J. Phys. Educ. **15** (2021) 2310,

68. A. Maireche, Heavy quarkonium systems for the deformed unequal scalar and vector Coulomb-Hulthén potential within the deformed effective mass Klein-Gordon equation using the improved approximation of the centrifugal term and Bopp's shift method in RNCQM symmetries, *International Journal of Geometric Methods in Modern Physics* **18**(2021) 2150214, `https://doi.org/10.1142/S0219887821502145`

69. A. Maireche, *New Relativistic Bound States for Modified Pseudoharmonic Potential of Dirac Equation with Spin and Pseudo-Spin Symmetry in One-electron Atoms*, Afr. Rev. Phys. **12** (2018) 130.

70. A. Saidi and M.B. Sedra, *Spin-one (1 + 3) dimensional DKP equation with modified Kratzer potential in the non-commutative space*, Mod. Phys. Lett. A **35** (2019) 2050014, `https://doi.org/10.1142/s0217732320500145`

71. S. Gradshteyn and I. M. Ryzhik: Table of Integrals, Series and Products, 7th. ed.; Elsevier, edited by Alan Jeffrey (the University of Newcastle upon Tyne, England) and Daniel Zwillinger (Rensselaer Polytechnic Institute USA) 2007.

72. Wolfram Research: `https://functions.wolfram.com/`

73. K. Bencheikh, S. Medjedel and G. Vignale, *Current reversals in rapidly rotating ultracold Fermi gases*, Phys. Lett. A. **89** (2014). `https://doi.org/10.1103/physreva.89.063620`

74. R. Kumar and F. Chand, Asymptotic Study to the N-Dimensional Radial Schrödinger Equation for the Quark-Antiquark System. Communications in Theoretical Physics **59**(5) (2013) 528-532. `https://doi.org/10.1088/0253-6102/59/5/02`

75. S.M. Kuchin and N.V. Maksimenko, *Theoretical estimations of the spin-averaged mass spectra of heavy quarkonia and Bc mesons*, Universal Journal of Physics and Application **7** (2013) 295, `https://doi.org/10.13189/ujpa.2013.010310`

76. P.A. M. Dirac, *The quantum theory of the electron*. Proc. Roy. Soc. A Math. Phys. Eng. Sci. **117** (1928) 610, `https://doi.org/10.1098/rspa.1928.0023`

77. N. Kemmer, *Quantum theory of Einstein-Bose particles and nuclear interaction*, Proceedings of the Royal Society A **166** (1938) 127, `https://doi.org/10.1098/rspa.1938.0084`

78. R.J. Duffin, *On the characteristic matrices of covariant systems*, Phys. Rev. **54** (1938) 1114. `https://doi.org/10.1103/PhysRev.54.1114`.

79. G. Petiau, *University of Paris thesis, Published in Academie Royale de Médecine de Belgique*, Classe des Sciences, Memoires in 8 n, 16, 2, (1936).

80. M.D. Scadmn, Advanced Quantum Theory and Its Applications Through Feynman Diagrams (New York Springer) (1979).