Cosmological-static metric correspondence and Kruskal type solutions from symmetry transformations

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We develop a formalism which provides a new view for the transformation of spherically symmetric metrics, regarding cosmological and Kruskal type metrics. Our analysis begins with some general relevant dynamically metrics in cosmology, and prove that they all can be transformed to a unique static form. We extend the formalism to obtain generalized Kruskal type coordinates in cosmology and black hole theory. This extended formalism provides a novel mechanism to obtain suitable coordinate charts associated with spherically symmetric metrics. In particular, we obtain explicitly new Kruskal type coordinates for extremal Reissner-Nordström and Schwarzschild-de-Sitter metrics, as well for an extension of the de-Sitter metric.

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1. Introduction

The most illustrative analytical solutions to the field equations in general relativity, are those where the space has spherical symmetry. For instance, the Schwarzschild metric for black holes and the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, that describes the behavior of the Universe at cosmological distances, have spherical symmetry (see Refs. [1, 2] and also [3] for generalized models with extra dimensions). However, it is well known that the description of a non-trivial space-time cannot be complete with just one coordinate chart, and usually one has to consider several patches, where in each patch a distinct set of coordinates is valid [2, 4]. In view of this, it can be useful to define new coordinates that cover more parts of the manifold. Then one sees that a transformation of space-time coordinates can fulfill two purposes: to reveal explicit symmetries of the space-time, as well as to extend the description of the space-time to regions that cannot be considered in the original setup. This allows, in cosmological and black hole models, to extend the description beyond the event horizons appearing in both cases [5, 6]. In this article, we explore the relations between two possible forms of the metric: one where the coordinates associated with the space-time appears as dynamical, and other where the coordinates takes a static form.

The relation between dynamical type and static forms of the metric has been a topic of great interest in the literature [7–12]. For instance, the well known association of de-Sitter space with the FLRW metric has been used in a deeper analysis in general relativity [7, 8]. However, interpretational problems between a static and a non-static representation of the same underlying space have been subject of debate [13]. In this work, one of the main ideas is to find, at a level of coordinate transformations, a link between spherically symmetric spaces (relevant to black hole theory) and cosmology. But more generally, we develop a general formalism based in coordinate transformation, that establishes a correspondence between static/non-static metrics. In particular, starting from time-dependent metrics we find the corresponding static metrics which turn out to be unique solutions.

We argue that our method can be extended to obtain Kruskal type coordinates in a number of scenarios in black-hole physics and cosmology. Specifically, assuming a general metric in the form

$$dS^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2,$$

we transform it to coordinates where the metric takes a conformally flat form, at hypersurfaces with $d\Omega^2 = 0$. We show that this leads to different possible mappings, including the Kruskal type transformations. From there, we discuss the resulting transformation for several spherically symmetric metrics, such as Schwarzschild, Reissner-Nordström, extremal Reissner-Norsdtröm, de-Sitter and Schwarzschild-de-Sitter. A relevant aspect of our approach is that we obtain three novel Kruskal transformations that highlights interesting features of Reissner-Norsdtröm and Schwarzschild-de-Sitter spaces, as well as a type of space described by a Generalized-de-Sitter metric. For all cases, the appropriate selection of integration constants assures two things: first, that the coordinates singularities can be removed; and second, that the different regions -for instance, interior and exterior of a black hole- can be distinguished in the Kruskal representation. We argue that this may shed some light on the underlying symmetries of a more general Kruskal formalism.

The rest of this work is divided as follows: in Sec. 2 we deal with the FLRW metric transformed to a 'static' type metric. We show the way this leads to a Friedmann equation with cosmological constant and zero matter density. The result is that for all spherical, hyperbolic and plane geometries, all converge to the same de-Sitter type metric. For sake of completeness, we also solve for the scale parameter $a = a(T)$. In Sec. 3 we review a further generalization and find that the previous result of a Friedmann equation for vacuum is unique, as well as the general form for the FLRW metric. In Sec. 4 we show the way this procedure can be applied in general to spherically symmetric metrics; we find that there exists several possibilities for the solutions. One of this possibilities leads to Kruskal type coordinates, and in Sec. 5 we review some particular solutions for different static metrics. Finally, in Sec. 6 we make some final remarks.

2. From FLRW cosmology to static metrics

Consider the gravitational field equations with cosmological constant Λ:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R - 2\Lambda) = 8\pi G T_{\mu\nu}. \tag{1}$$

By assuming that the space is maximally symmetric with commoving coordinates (T, ρ, θ, ϕ) describing a spherically symmetric space, one can solve (1) for the metric in the form

$$dS^2_{(1)} = -dT^2 + a^2(T)\left(\frac{d\rho^2}{1-k\rho^2} + \rho^2 d\Omega^2\right), \tag{2}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the solid angle line element. Also, k can take the values 1, 0 or -1 denoting space-like slices at constant T corresponding to spherically, flat and hyperbolic topologies, respectively. Furthermore $a(T)$ is the scale factor, whose evolution is obtained by assuming that the energy-momentum tensor takes the form

$$T^{\mu\nu} = (\rho_f + p_f)u^\mu u^\nu + p_f g^{\mu\nu}. \tag{3}$$

Here u^μ is the four velocity, while ρ_f and p_f are the density of energy and pressure describing a perfect fluid. From there, one obtains the Friedmann equations

$$\frac{\dot{a}^2 + k}{a^2} = \frac{8\pi G}{3}\rho_f + \frac{\Lambda}{3}, \tag{4}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_f + 3p_f) + \frac{\Lambda}{3}. \tag{5}$$

Now consider the transformation from $dS^2_{(1)} = g_{\mu\nu}dx^\mu dx^\nu$ given in (2), to the static spherical symmetric form $dS^2_{(2)} = \gamma_{\alpha\beta}dx'^\alpha dx'^\beta$, namely

$$dS^2_{(2)} = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2. \tag{6}$$

The solid angle is the same for both cases in such a way that the angular terms in (2) and (6) imply

$$r = a\rho. \tag{7}$$

From now on, we will denote partial derivatives respect to T with an overdot, while prime will mean partial derivatives respect to ρ, such as $\dot{t} = \partial t/\partial T$ and $t' = \partial t/\partial\rho$. We start with the tensor transformation

$$g_{\mu\nu} = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu}\gamma_{\alpha\beta}. \tag{8}$$

For g_{00}, g_{11} and g_{01}, after rearranging some terms, this leads to

$$\dot{t}^2 = \frac{1}{f^2}(f + \rho^2\dot{a}^2), \tag{9}$$

$$t'^2 = \frac{a^2}{f^2}\left(1 - \frac{f}{1-k\rho^2}\right), \tag{10}$$

and

$$f^4\dot{t}^2 t'^2 = \rho^2 a^2\dot{a}^2, \tag{11}$$

respectively. Substituting (9) and (10) into (11) one obtains

$$f = 1 - \rho^2(\dot{a}^2 + k), \tag{12}$$

where f is considered to be a function of ρ and T.

By inserting (12) into (9), one obtains

$$\dot{t}^2 = \frac{1-k\rho^2}{f^2}, \tag{13}$$

while combining Eqs. (10) and (12) one finds that

$$t'^2 = \frac{\rho^2 a^2\dot{a}^2}{f^2(1-k\rho^2)}. \tag{14}$$

Now we use the fact that $f' = -2\rho(\dot{a}^2 + k)$ and $\dot{f} = -2\rho^2\dot{a}\ddot{a}$ to take the partial derivatives of (13) respect to ρ and of (14) respect to T, in order to obtain $\partial^2 t/\partial T\partial\rho$ in both cases. After equating and performing some simplifications, the next relation appears:

$$\left[(\dot{a}^2 + k) - a\ddot{a}\right]\left[1 + \rho^2(\dot{a}^2 - k)\right] = 0. \tag{15}$$

In general, the second factor is nonzero, since then it would imply that a is a function of ρ and this is incongruent with $a = a(T)$; equivalently, the second factor equal to zero would imply, by (16), that f can be put as a function of ρ only. It follows that only

$$\frac{\ddot{a}}{a} = \frac{\dot{a}^2 + k}{a^2}, \tag{16}$$

holds. By noticing that $d(\dot{a}^2 + k)/dt = 2\dot{a}\ddot{a}$, one can see that this equation is equivalent to

$$\frac{d(\dot{a}^2 + k)}{dt} = 2\frac{\dot{a}}{a}(\dot{a}^2 + k). \tag{17}$$

As this can be expressed as a total derivative of logarithms, this leads to

$$\frac{\dot{a}^2 + k}{a^2} = \Gamma, \tag{18}$$

where Γ is an integration constant. Then, by (16) we have also the relation $\ddot{a} = \Gamma a$.

Observe that (18) reduces to the first Friedmann equation (4) for vacuum ($\rho_f = 0$) with cosmological constant $\Lambda = 3\Gamma$. This identification is validated by the comparison of $\ddot{a} = \Gamma a$ with the second Friedmann equation in vacuum, namely $\ddot{a}/a = \Lambda/3$ in (5). It is interesting to note that the Friedmann equation emerges from a symmetry transformation, without invoking any dynamic equation such as the Einstein field equations.

Going back to (18), rewriting it as $\dot{a} = \sqrt{\Gamma a^2 - k}$, we can solve for $k = 0$, 1 and -1. For the moment we take into account the cases $\Gamma \neq 0$. It turns out that the equality would describe Minkowski space, as we shall see below.

For $k = 0$ ($\Gamma > 0$), the result is $a = e^{\sqrt{\Gamma}T}$. For the closed topology where $k = 1$ (here also $\Gamma > 0$ is forced), $a(T)$ becomes $a = (1/\sqrt{\Gamma}) \cosh \sqrt{\Gamma}T$, where we have chosen $T = 0$ as the comoving time when $a = 1/\sqrt{\Gamma}$. Meanwhile, with $k = -1$, Γ can be either positive or negative. For $\Gamma > 0$, the solution is $a = (1/\sqrt{\Gamma}) \sinh \sqrt{\Gamma}T$. In this case, we have chosen the origin of time in such a way that $a = 0$ when $T = 0$. For $k = -1$ and $\Gamma < 0$, the solution corresponds to $a = (1/\sqrt{|\Gamma|}) \sin \sqrt{|\Gamma|}T$.

Concerning the function f, we remember from (7) that $a = r/\rho$, that together with (12) and (17) imply that

$$f = 1 - \Gamma r^2. \tag{19}$$

Summarizing this section, for $\Gamma \neq 0$ we have the following solutions:

Curvature	Metric
$k{=}0, \Gamma > 0$	$dS^2{=} - dT^2 + e^{2\sqrt{\Gamma}T}\left(d\rho^2 + \rho^2 d\Omega^2\right)$
$k{=}1, \Gamma > 0$	$dS^2{=} - dT^2 + \frac{\cosh^2(\sqrt{\Gamma}T)}{\Gamma}\left(\frac{d\rho^2}{1-\rho^2} + \rho^2 d\Omega^2\right)$
$k{=}-1, \Gamma > 0$	$dS^2{=} - dT^2 + \frac{\sinh^2(\sqrt{\Gamma}T)}{\Gamma}\left(\frac{d\rho^2}{1+\rho^2} + \rho^2 d\Omega^2\right)$
$k{=}-1, \Gamma < 0$	$dS^2{=} - dT^2 + \frac{\sin^2\left(\sqrt{\lvert\Gamma\rvert}T\right)}{\lvert\Gamma\rvert}\left(\frac{d\rho^2}{1+\rho^2} + \rho^2 d\Omega^2\right)$

The first solution is the usual de-Sitter space, while the second and third ones are the two types of Lanczos universe. The fourth solution is the only allowed solution with $\Gamma < 0$, and it corresponds to anti-de-Sitter space [12, 13].

Finally, choosing $\Gamma = 0$ in Eq. (18) implies that the scale parameter obeys the equation $\dot{a}^2 + k = 0$, and the corresponding solutions for g_{11} in Eq. (1) are $g_{11} = 1$ for $k = 0$ and $g_{11} = t^2/(1 + r^2)$ for $k = -1$; this last solution is referred as Milne model. Both get mapped (in a trivial way) to Minkowski space. These two solutions, as well as the ones listed in Eq. (20), corresponds to the six possible transformations to the static form given in Eq. (2) (see Refs. [12, 13]). The whole set of solutions share the corresponding static form of the metric

$$dS^2_{(2)} = -\left(1 - \Gamma r^2\right) dt^2 + \frac{dr^2}{1 - \Gamma r^2} + r^2 d\Omega^2. \tag{21}$$

As we shall see in the next section, this static form will be preserved even when generalizing the line element given in Eq. (2).

3. A further generalization.

Now, let us consider a more general form of the metric, but still assuming commoving time and radial symmetry. In this sense, the ansatz now reads as

$$dS^2_{(3)} = -dT^2 + a^2(T)\left(b^2(\rho)d\rho^2 + \rho^2 d\Omega^2\right). \tag{22}$$

If this metric is transformed to (6), then the relation (7), $r = a\rho$, is satisfied again. Even more, transformations (9) and (11) hold again. However, instead of (10) we have

$$t'^2 = \frac{a^2}{f^2}(1 - b^2 f). \tag{23}$$

Substituting (9) and (23) into (11) leads after simplification to

$$f = \frac{1}{b^2} - \rho^2 \dot{a}^2. \tag{24}$$

Insertion in Eq. (9) and (23) leads to the succinct expressions $\dot{t} = 1/bf$ and $t' = \rho a\dot{a}b/f$. As in the previous section, we derive this relations with respect to ρ and with respect to T, respectively. By using $\dot{f} = -2\rho^2\dot{a}\ddot{a}$ and $f' = -2(b'b^{-3} + \rho\dot{a}^2)$, and equating $\partial\dot{t}/\partial\rho$ with $\partial t'/\partial T$, we see that after some algebra the next relation holds:

$$b' = \rho b^3\left(a\ddot{a} - \dot{a}^2\right). \tag{25}$$

Since $a = a(T)$ and $b = b(\rho)$, (25) implies that

$$a\ddot{a} - \dot{a}^2 = \frac{b'}{\rho b^3} = \kappa, \tag{26}$$

where κ is a constant. Note that the values of κ can be identified with those of k (1, 0 or -1), by rescaling adequately the parameter $a(T)$. With this identification, the previous relation for $a(T)$ is just equation (16) [1, 2]. Furthermore, integration of $b'b^{-3} = \kappa\rho$ leads to

$$\frac{1}{b^2} = B - \kappa\rho^2, \tag{27}$$

with B another integration constant. Assuming local flatness at slices with T constant, B can be set equal to 1. Hence, even by considering a more general metric in our formalism, namely $dS^2_{(3)}$ in Eq. (22), a transformation to the static form given by (6) restricts the metric to the form given by (2), and with same solutions listed at the end of Sec. 2.

4. Extending the formalism to include Kruskal type coordinates

It turns out that the same formalism can lead to Kruskal type coordinates. Let us assume that the metric can take a form that is conformally flat in the space-time slices with $d\theta = d\phi = 0$:

$$dS^2_{(4)} = N^2(T,\rho)(-dT^2 + d\rho^2) + \varphi^2(T,\rho)d\Omega^2. \quad (28)$$

The transformation (8) of the metric components given in Eq. (28) to the static form (6), $dS^2_{(2)} = -f dt^2 + f^{-1}dr^2 + r^2 d\Omega^2$, leads to

$$(\partial_t T)^2 = (\partial_t \rho)^2 + f N^{-2}, \quad (29)$$

$$(\partial_r T)^2 = (\partial_r \rho)^2 - f^{-1} N^{-2}, \quad (30)$$

and

$$\partial_t T \partial_r T = \partial_t \rho \partial_r \rho. \quad (31)$$

Substitution of (29) and (30) into (31) gives, after clearing, the next relation:

$$N^{-2} = (\partial_r \rho)^2 f - (\partial_t \rho)^2 f^{-1}. \quad (32)$$

This allows to simplify (29) and (30) as

$$\partial_t T = f \partial_r \rho \quad (33)$$

and

$$\partial_r T = f^{-1} \partial_t \rho, \quad (34)$$

respectively. The form of these two expressions suggests to take $T(t,r) = \Theta(t)\Phi(r)$ and $\rho(t,r) = \xi(t)\chi(r)$. Then those two relations leads to

$$\frac{1}{\xi}\frac{d\Theta}{dt} = \frac{f}{\Phi}\frac{d\chi}{dr} = \alpha, \quad (35)$$

and

$$\frac{1}{\Theta}\frac{d\xi}{dt} = \frac{f}{\chi}\frac{d\Phi}{dr} = \beta, \quad (36)$$

where α and β are constants. Both expressions imply various relations. First, from (35) we derive $d\Theta/dt = \alpha\xi$ with respect to t and use (36). It results in

$$\frac{d^2\Theta}{dt^2} - \alpha\beta\Theta = 0. \quad (37)$$

On the other hand, dividing (36) by (35) we have that $\alpha\Phi d\Phi = \beta\chi d\chi$, giving the function relation

$$\alpha\Phi^2 - \beta\chi^2 = \sigma, \quad (38)$$

with σ another integration constant. There are several relevant possibilities for the product $\alpha\beta$ in Eq. (37):

Case $\alpha\beta = 0$. Assuming that $\alpha = 0$, then (35) implies that Θ is a constant, in such a way that $\partial_t T = 0$; this in turn leads to $\partial_r \rho = 0$ due to (33), and also that χ is constant. By (36), rescaling and shifting the origin of time, ξ can be set equal to t. Then, setting $\Theta = 1$ and $\chi = \beta^{-1}$, from the same relation (36), it results $dT = d\Phi = f^{-1}dr$, while $d\rho = dt$. From (30) and (34) one learns that $N^2 = -f/(\partial_t\rho)^2 = -f$. Inserting all this into the form of the metric (28) we obtain the same metric given in Eq. (6). A similar argument holds for the case $\beta = 0$. Thus, with $\alpha\beta = 0$ the transformation maps onto itself and N^2 is proportional to f, a reminiscence of what occurs with the use of tortoise coordinates, where $dr^* = (1 - r_s/r)^{-1}dr$ in such a way that $-(1 - r_s/r)\,dt^2 + (1 - r_s/r)^{-1}dr^2$ transforms into $-(1 - r_s/r)\left(dt^2 + dr^{*2}\right)$, where r_s is the Schwarzschild radius [1,2].

Case $\alpha\beta < 0$

By (37) we have that $\Theta \propto \sin(\sqrt{|\alpha\beta|}t + \phi_0)$ and consequently -by (36)- that $\xi \propto \cos(\sqrt{|\alpha\beta|}t + \phi_0)$. We can fix the phase angle to zero in such a way that $T = 0$ coincides with $t = 0$. Also, without loss of generality we take $\alpha > 0$ and $\beta < 0$, that implies that $\sigma > 0$ in Eq. (38). Then the solutions are given by

$$\Theta(t) = B_1 \sin\left(\sqrt{-\alpha\beta}t\right), \quad (39)$$

and

$$\xi(t) = B_2 \cos\left(\sqrt{-\alpha\beta}t\right), \quad (40)$$

where the relation $B_2 = \sqrt{-\beta/\alpha}B_1$ holds in such a way that $d\Theta/dt = \alpha\xi$ in Eq. (36), is still satisfied.

From (36) we have that $\chi = (f/\beta)d\Phi/dr$, that together with (38) leads to

$$\frac{d\Phi}{dr} = \frac{\sqrt{-\beta}\sqrt{\sigma - \alpha\Phi^2}}{f}, \quad (41)$$

which can be integrated, yielding

$$\Phi = \sqrt{\frac{\sigma}{\alpha}} \sin\left(\sqrt{-\alpha\beta}\int\frac{dr}{f}\right). \quad (42)$$

Inserting this result in (38), we have that

$$\chi = \sqrt{\frac{\sigma}{-\beta}} \cos\left(\sqrt{-\alpha\beta}\int\frac{dr}{f}\right). \quad (43)$$

Remembering that $T(t,r) = \Theta(t)\Phi(r)$ and $\rho(t,r) = \xi(t)\chi(r)$, in this case we have

$$T(t,r) = \sin\left(\sqrt{-\alpha\beta}\int\frac{dr}{f}\right)\sin\left(\sqrt{-\alpha\beta}t\right), \quad (44)$$

and

$$\rho(t,r) = \cos\left(\sqrt{-\alpha\beta}\int\frac{dr}{f}\right)\cos\left(\sqrt{-\alpha\beta}t\right). \quad (45)$$

Here we have set $B_1\sqrt{\sigma/\alpha} = 1$ (justified by rescaling coordinates). By using Eq. (45) in Eq. (32) for different f in the metric, the factor N^2 appearing in (30) can be obtained. The result is

$$N^2 = \frac{f}{-\alpha\beta\left[\cos\left(2\sqrt{-\alpha\beta}t\right) - \cos\left(2\sqrt{-\alpha\beta}\int\frac{dr}{f}\right)\right]}. \quad (46)$$

Case $\alpha\beta > 0$

Equation (37) leads to $\Theta(t) = C_1 \sinh(\sqrt{\alpha\beta}t + \phi_0)$. In order to get $t = 0$ when $T = 0$, we choose $\phi_0 = 0$ and then we have the solutions

$$\Theta(t) = C_1 \sinh\left(\sqrt{\alpha\beta}t\right), \tag{47}$$

and

$$\xi(t) = C_2 \cosh\left(\sqrt{\alpha\beta}t\right), \tag{48}$$

where $C_2 = \sqrt{\beta/\alpha}C_1$. Clearly, this two relations lead to $\alpha\xi^2 - \beta\Theta^2 = \beta C_1^2$, obtained also by using (35) and (36).

Now take into account that (38) implies $\chi = \pm\sqrt{\alpha/\beta}\Phi$, with $\sigma = 0$. Inserting this in (35), we obtain

$$\frac{1}{\Phi}\frac{d\Phi}{dr} = \pm\sqrt{\alpha\beta}f^{-1}. \tag{49}$$

By integrating this expression, we see that

$$\Phi(r) = Ae^{\pm\sqrt{\alpha\beta}\int \frac{dr}{f}}, \tag{50}$$

and consequently (38) implies that

$$\chi(r) = \pm A\sqrt{\frac{\alpha}{\beta}}e^{\pm\sqrt{\alpha\beta}\int \frac{dr}{f}}, \tag{51}$$

with A constant. Thus, for $\alpha\beta > 0$, the coordinates $T(t,r) = \Theta(t)\Phi(r)$ and $\rho(t,r) = \xi(t)\chi(r)$ are

$$T(t,r) = \Phi(r)\sinh\left(\sqrt{\alpha\beta}t\right), \tag{52}$$

and

$$\rho(t,r) = \Phi(r)\cosh\left(\sqrt{\alpha\beta}t\right). \tag{53}$$

Here, $\Phi(r)$ is given in Eq. (50) and we set $A = C_1 = 1$. Also, we have omitted a possible minus sign in $\rho(t,r)$, since it just plays the role of an inversion of coordinates in the analysis of the regions considered.

The function $N(T,\rho)$ can be obtained by inserting Eqs. (33), (52) and (53) in Eq. (32). The result is:

$$N^2 = \frac{f\Phi^{-2}}{\alpha\beta}. \tag{54}$$

Observe that the function $f(r)$ determines all possible transformations, and the relations (52) and (53) determine Kruskal type coordinates for a given f. In the next section we obtain the explicit form for several cases of interest.

5. Kruskal type solutions

For simplicity, we define $\gamma = \pm\sqrt{\alpha\beta}$ and proceed to obtain the Kruskal type solution for different cases, by changing f in $dS^2_{(2)} = -fdt^2 + f^{-1}dr^2 + r^2d\Omega^2$.

5.1. Schwarzschild

In this case we have $f = 1 - r_s/r$, where $r_s = 2M$ is the Schwarzschild radius. This leads to $\int dr/\left(1 - [r_s/r]\right) = r + r_s \ln\left(r/r_s - 1\right)$, for $r > r_S$. Thus, substituting this result into (50), yields

$$\Phi_{\text{Schw}} = e^{\frac{r}{2r_s}}\sqrt{\frac{r}{r_s} - 1}, \tag{55}$$

where we have set $\gamma = 1/(2r_s)$. Then (52) and (53) become

$$T_{\text{Schw}} = e^{\frac{r}{2r_s}}\sqrt{\frac{r}{r_s} - 1}\sinh\frac{t}{2r_s}, \tag{56}$$

and

$$\rho_{\text{Schw}} = e^{\frac{r}{2r_s}}\sqrt{\frac{r}{r_s} - 1}\cosh\frac{t}{2r_s}, \tag{57}$$

respectively. Further, from (28) and (54) we find that the the metric is given by

$$dS^2_{\text{Schw}} = \frac{4r_s^3}{r}e^{-\frac{r}{r_s}}(-dT^2 + d\rho^2) + r^2d\Omega^2. \tag{58}$$

We recognize in Eqs. (56)-(58) the Kruskal transformation associated with the Schwarzschild metric [14, 15]. As usual, the relation $\rho^2_{\text{Schw}} - T^2_{\text{Schw}} = e^{r/r_s}\left(r/r_s - 1\right)$ is useful to verify the properties of this space-time, and in particular to extend the analysis to the region $r < r_S$ (See [1, 2] and also [16] and references therein for recent developments).

Moreover, it is worth mentioning that our formalism is more direct and general than the usually given in textbooks, since we just need to specify f and then solve for Φ in order to obtain the full set of coordinate transformations for T, ρ and $N(T,\rho)$.

5.2. Reissner-Nordström

For the electric charged static black hole we have $f = 1 - r_s/r + Q^2/r^2 = (r - r_+)(r - r_-)/r^2$, where $r_\pm = M \pm \sqrt{M^2 - Q^2}$. The solution to $\int f^{-1}dr$ (again region with $r > r_+$) is given by

$$\int \frac{dr}{f} = r + \frac{1}{2\sqrt{M^2 - Q^2}} \times \left[r_+^2 \ln\left(r - r_+\right) - r_-^2 \ln(r - r_-)\right] + \text{const}. \tag{59}$$

By setting the integration constant to

$$\frac{\left(r_-^2 \ln r_- - r_+^2 \ln r_+\right)}{\left(2\sqrt{M^2 - Q^2}\right)},$$

and following the same steps as in the Schwarzschild case, the Reissner-Nordström metric is transformed into the Kruskal form by means of

$$\Phi_{\text{R-N}}(r) = e^{\gamma r}\frac{\left(\frac{r}{r_+} - 1\right)^{\zeta r_+^2}}{\left(\frac{r}{r_-} - 1\right)^{\zeta r_-^2}},$$

where $\zeta = \gamma(r_+ - r_-)$. The corresponding coordinate transformations acquire the form

$$T_{\text{R-N}}(t,r) = e^{\gamma r}\frac{\left(\frac{r}{r_+}-1\right)^{\zeta r_+^2}}{\left(\dfrac{r}{r_-}-1\right)^{\zeta r_-^2}}\sinh\gamma t,$$
$$\rho_{\text{R-N}}(t,r) = e^{\gamma r}\frac{\left(\frac{r}{r_+}-1\right)^{\zeta r_+^2}}{\left(\dfrac{r}{r_-}-1\right)^{\zeta r_-^2}}\cosh\gamma t. \quad (60)$$

Then the Kruskal type solution for the charged static black hole is

$$dS^2_{\text{R-N}} = \frac{1}{\gamma^2 r^2}e^{-2\gamma r}\frac{\left(\frac{r}{r_+}-1\right)^{2\zeta r_-^2+1}}{\left(\frac{r}{r_-}-1\right)^{2\zeta r_+^2-1}} \times\left(-dT^2+d\rho^2\right)+r^2d\Omega^2. \quad (61)$$

Thus, the different regions of this space can be visualized from

$$\rho^2_{\text{R-N}} - T^2_{\text{R-N}} = e^{2\gamma r}\frac{\left(\frac{r}{r_+}-1\right)^{2\zeta r_+^2}}{\left(\frac{r}{r_-}-1\right)^{2\zeta r_-^2}}.$$

(For comparison set $\gamma = (r_+ - r_-)/r_+^2$ and see for instance Refs. [17] and [18]).

5.3. Reissner-Nordström (Extremal)

The Reissner-Nordström extremal metric is of theoretical interest in several contexts. This solution is obtained from $r_\pm = M \pm \sqrt{M^2-Q^2}$ when $M^2 = Q^2$, which means that $r_\pm = M$. In such a case the metric takes the form $dS^2 = -(1-M/r)^2dt^2 + (1-M/r)^{-2}dr^2 + r^2d\Omega^2$. So, our task is to determine the integral $\int f^{-1}dr = \int r^2(r-M)^{-2}dr$. We obtain $\int f^{-1}dr = r - M^2(r-M) + 2M\ln(r-M) + \text{const.}$ and hence one may write the radial function Φ as $\Phi_{\text{Ex}}(r) = (r/r_+ - 1)\exp\{(r/r_+-\varphi_1)(r/r_+-\varphi_2)/[2(r/r_+-1)]\}$, where $\varphi_1 = (1+\sqrt{5})/2$ and $\varphi_2 = (1-\sqrt{5})/2$. Also, we have set $\gamma = (4M)^{-1}$ and the integration constant equal to $-2M\ln M$. With this at hand, the coordinates (in the patch where $r > r_+$) are given by

$$T_{\text{Ex}}(t,r) = e^{\frac{\left(\frac{r}{r_+}-\varphi_1\right)\left(\frac{r}{r_+}-\varphi_2\right)}{4\left(\frac{r}{r_+}-1\right)}}\sqrt{\frac{r}{r_+}-1}\sinh\frac{t}{4r_+}, \quad (1)$$
$$\rho_{\text{Ex}}(t,r) = e^{\frac{\left(\frac{r}{r_+}-\varphi_1\right)\left(\frac{r}{r_+}-\varphi_2\right)}{4\left(\frac{r}{r_+}-1\right)}}\sqrt{\frac{r}{r_+}-1}\cosh\frac{t}{4r_+}. \quad (62)$$

Now the metric would be specified by using $N^2 = f\Phi^{-2}\gamma^{-2}$, that yields

$$dS^2_{\text{Ex}} = \frac{16r_+^4}{r^2}\left(\frac{r}{r_+}-1\right)e^{\frac{-\left(\frac{r}{r_+}-\varphi_1\right)\left(\frac{r}{r_+}-\varphi_2\right)}{\left(\frac{r}{r_+}-1\right)}} \times(-dT^2+d\rho^2)+r^2d\Omega^2. \quad (63)$$

Also,

$$\rho^2_{\text{Ex}} - T^2_{\text{Ex}} = (r/r_+-1)\exp\left\{\frac{(r/r_+-\varphi_1)(r/r_+-\varphi_2)}{[2(r/r_+-1)]}\right\}.$$

It is interesting to observe that the golden ratios φ_1 and φ_2 emerge in this extremal case (see Refs. [19, 20] and references therein).

5.4. De-Sitter space

In this case, for (17) we have $f = 1 - \Gamma r^2$, where $\Gamma > 0$ leads to de-Sitter space. Now we have $\int f^{-1}dr = \Gamma^{-1/2}\text{arctanh}(\sqrt{\Gamma}r) + \text{const.}$ This leads to the radial function $\Phi_{\text{dS}}(r) = \sqrt{(1-\sqrt{\Gamma}r)/(1+\sqrt{\Gamma}r)} = (1-\sqrt{\Gamma}r)/\sqrt{1-\Gamma r^2}$ for $r < 1/\sqrt{\Gamma}$. We have chosen $\gamma = -\sqrt{\Gamma}$ and the integration constant equal to zero. The Kruskal coordinates are then

$$T_{\text{dS}}(t,r) = \frac{1-\sqrt{\Gamma}r}{\sqrt{1-\Gamma r^2}}\sinh\left(\sqrt{\Gamma}t\right),$$
$$\rho_{\text{dS}}(t,r) = \frac{1-\sqrt{\Gamma}r}{\sqrt{1-\Gamma r^2}}\cosh\left(\sqrt{\Gamma}t\right). \quad (64)$$

Thus, in this case the metric takes the simple form

$$dS^2_{\text{dS}} = \frac{1}{\Gamma}\left(1-\sqrt{\Gamma}r\right)^2(-dT^2+d\rho^2)+r^2d\Omega^2. \quad (65)$$

The coordinates (64) yield $\rho^2_{\text{dS}} - T^2_{\text{dS}} = \left(1-\sqrt{\Gamma}r\right)/\left(1+\sqrt{\Gamma}r\right)$. Note that the negative sign chosen in $\gamma = -\sqrt{\Gamma}$ makes sense: with this selection the region $0 \le r \le 1/\sqrt{\Gamma}$ would be described, in the (T,ρ) system, by hyperbolas starting from $\rho^2_{\text{dS}} - T^2_{\text{dS}} = 1$ until reaching the asymptotes $T = \pm\rho$. Observe that this is valid in the two quadrants where $\rho > 0$. On the other hand, $\gamma > 0$ would result in unbounded hyperbolas, since then $\rho^2_{\text{dS}} - T^2_{\text{dS}} \to \infty$ when $r \to 1/\sqrt{\Gamma}$. Compare this solution with Refs. [21–23].

5.5. Schwarzschild-de-Sitter metric

As we mentioned before, the solution for black hole with cosmological constant comes from taking the radial function as $f = 1 - r_s/r - \Gamma r^2$. The three roots of the cubic equation $r - r_s - \Gamma r^3 = 0$, namely (λ_1, λ_2, λ_3), are real and distinct if $M < 1/3\sqrt{3\Gamma}$ (see [24] and references therein). In this case we set $\lambda_1 > \lambda_2 > \lambda_3$, in such a way that the first root corresponds to the cosmological horizon $r = \lambda_1$, the second root to the black hole event horizon $r = \lambda_2$, and the third one is a negative (unphysical) root. It is worthwhile to mention that the two horizons at λ_1 and λ_2 coincide when $M = \left(3\sqrt{3\Gamma}\right)^{-1}$. Then

$\int f^{-1}dr = \ln\left[(r-\lambda_1)^{\sigma_1}(r-\lambda_2)^{\sigma_2}(r-\lambda_3)^{\sigma_3}\right] + const.$, where $\sigma_i = \lambda_i/(1-3\Gamma\lambda_i^2)$, where $i = 1, 2, 3$. By (50) we have

$$\Phi_{\text{S-dS}}(r) = \left(\frac{r}{\lambda_1}-1\right)^{\gamma\sigma_1}\left(\frac{r}{\lambda_2}-1\right)^{\gamma\sigma_2}\left(1-\frac{r}{\lambda_3}\right)^{\gamma\sigma_3}. \tag{66}$$

Here, we have chosen the integration constant equal to $-\sigma_1\ln\lambda_1 - \sigma_2\ln\lambda_2 - \sigma_3\ln(-\lambda_3)$. From (52) and (53), it results

$$\begin{aligned} T_{\text{S-dS}}(t,r) &= \left(\frac{r}{\lambda_1}-1\right)^{\gamma\sigma_1}\left(\frac{r}{\lambda_2}-1\right)^{\gamma\sigma_2} \\ &\times\left(1-\frac{r}{\lambda_3}\right)^{\gamma\sigma_3}\sinh\gamma t, \\ \rho_{\text{S-dS}}(t,r) &= \left(\frac{r}{\lambda_1}-1\right)^{\gamma\sigma_1}\left(\frac{r}{\lambda_2}-1\right)^{\gamma\sigma_2} \\ &\times\left(1-\frac{r}{\lambda_3}\right)^{\gamma\sigma_3}\cosh\gamma t, \end{aligned} \tag{67}$$

where we have taken into account that $\lambda_3 < 0$.

From (28) and (54) we find

$$\begin{aligned} dS^2_{\text{S-dS}} &= \frac{\Gamma\lambda_1\lambda_2\lambda_3}{\gamma^2 r}\left(\frac{r}{\lambda_1}-1\right)^{1-2\gamma\sigma_1} \\ &\times\left(\frac{r}{\lambda_2}-1\right)^{1-\gamma\sigma_2}\left(1-\frac{r}{\lambda_3}\right)^{1-2\gamma\sigma_3} \\ &\times(-dT^2+d\rho^2)+r^2(T,\rho)d\Omega^2. \end{aligned} \tag{68}$$

5.6. A generalized de-Sitter metric

In Ref. [25] a generalization of de-Sitter space is considered, where the metric is of the type (6), with $f = 1-h^2r^2+q^4r^4$. Here two cosmological horizons arise, given by $r_\pm^2 = (h^2 \pm \sqrt{h^4-4q^4})/(2q^4)$. In order to calculate $\Phi(r)$, we perform the integral

$$\begin{aligned} \int\frac{dr}{f} &= \frac{1}{q^4(r_+^2-r_-^2)} \\ &\times\left(\frac{1}{r_+}\tanh^{-1}\left[\frac{r}{r_+}\right]-\frac{1}{r_-}\tan^{-1}\left[\frac{r}{r_-}\right]\right). \end{aligned} \tag{69}$$

By choosing $\gamma = -q^4(r_+^2-r_-^2)r_+$, we have from (50) that

$$\Phi_{\text{GdS}}(r) = \sqrt{\frac{r_+-r}{r_++r}}\,e^{-\frac{r_+}{r_-}\tan^{-1}\left(\frac{r}{r_-}\right)}, \tag{70}$$

for the region with $r < r_-$, where "GdS" stands for *Generalized-de-Sitter space*. The corresponding Kruskal type coordinates are given by

$$\begin{aligned} T_{\text{GdS}}(t,r) &= \sqrt{\frac{r_+-r}{r_++r}}\,e^{-\frac{r_+}{r_-}\tan^{-1}\left(\frac{r}{r_-}\right)}\sinh(\gamma t), \\ \rho_{\text{GdS}}(t,r) &= \sqrt{\frac{r_+-r}{r_++r}}\,e^{-\frac{r_+}{r_-}\tan^{-1}\left(\frac{r}{r_-}\right)}\cosh(\gamma t). \end{aligned} \tag{71}$$

Also, the relation

$$\rho^2_{\text{GdS}} - T^2_{\text{GdS}} = \frac{r_+-r}{r_++r}e^{-\frac{2r_+}{r_-}\tan^{-1}\left(\frac{r}{r_-}\right)}$$

holds, confirming that the qualitative behavior of the space in these coordinates is very similar to that of the de-Sitter space analyzed before, in the patch where $r < r_-$.

6. Final remarks

In this work we have analyzed the relationship between some spherical symmetric metrics for two cases: cosmological FLRW and Kruskal type metrics. In the first case, we have shown that, by imposing that the FLRW metric to be transformed into the form

$$dS^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega^2, \tag{72}$$

leads to $f = 1-\Gamma r^2$ with solutions summarized at the end of Sec. 2. In fact, the only possibilities resulting from this symmetry transformation are the spaces known as: de-Sitter, anti-de-Sitter, Lanczos, Milne, and Minkowski. It is remarkable that as a by-product of the symmetry transformation, the Friedmann equation with cosmological constant emerges.

Next we moved in the reverse order: starting from the general metric (72), we applied a transformation to obtain a metric which is conformally flat in hypersurfaces with $d\theta = d\phi = 0$. This led to two non-trivial possibilities, one in which the coordinates are proportional to sines and cosines; and a second solution in terms of hyperbolic trigonometric functions, that resembles the Kruskal solution for Schwarzschild space.

In Sec. 5 we used the method to explicitly obtain the coordinate (T,ρ) for several well known spaces: Schwarzschild, Reissner-Nordström, extremal Reissner-Norsdtröm, de-Sitter and Schwarzschild-de-Sitter. Here, the analysis was not exhaustive, in the sense that the main purpose was to show how the method of two-metric transformation correctly works. For instance, we only solved for the exterior regions in the case of black holes, and for the region inside the cosmological horizon in the de-Sitter case. Meanwhile, we found that choosing properly the integration constant γ in $\Phi(r) = A\exp\left(\gamma\int[dr/f]\right)$, given in Eq. (50), one preserves desirable properties of the Kruskal extended space. Specifically, in the extremal Reissner-Norsdström case we have chosen γ in such a way that the interior and the exterior regions of the black hole get uniquely represented in the Kruskal (T,ρ)-coordinates framework. Furthermore, in the de-Sitter case we favored a negative sign in γ, since this ensures that in the limit $r \to 1/\sqrt{\Gamma}$, the hyperbolas representing the space do not open to infinity. It is worthwhile to remark that we have obtained three novel Kruskal transformations: for the Reissner-Norsdtröm (where interestingly the *golden ratio* arises), Schwarzschild-de-Sitter space and the mentioned as Generalized-de-Sitter space.

Further interesting prospects of our work could emerge as follows. Clearly, one can find other Kruskal type solutions by considering other functions $f(r)$ in the metric (2) (for instance the space analyzed in Ref. [26]). In this sense, the formulation of Secs. 4 and 5 complements other works that consider the properties of static metrics in which $g_{11} = -1/g_{00}$ [27–30].

Another interesting aspect is that the formulation of this two-metric transformation can be generalized in a straightforward manner to higher dimensions. First, notice that the angular term $r^2 d\Omega^2$ is a passive term in all the development. Then one may readily generalize it to higher dimensions. Of course, in this case one must modify also the function $f(r)$. For instance, in the Schwarzschild type metric in arbitrary D-dimensions one has

$$f = 1 - \frac{r_s}{r^{D-3}}. \tag{73}$$

There is at least one possible scenario in which such a generalization may have important and interesting consequences, namely black holes associated with parallelizable spheres. As it is known, the only parallelizable spheres are S^1, S^3 and S^7, which corresponds to the existence of normed division algebras: real numbers, complex numbers, quaternions and octonions, respectively [31, 32]. In this way, from the point of view of parallelizable spheres, the event horizon of black holes associated with the spheres S^1, S^3 and S^7 seems even more interesting that the traditional S^2-event/horizon.

Also, for further research it may be interesting to consider the connection between the transformations corresponding to negative and positive values of $\alpha\beta$ in Sec. 4. Since for $\alpha\beta < 0$ the coordinate transformations are related to the trigonometric functions sine and cosine, while for $\alpha\beta > 0$ corresponds to hyperbolic trigonometric functions, one may expect a connection between these two scenarios. This may be analogue to the transformation between spheres and hyperbolas. In fact, one finds such example in complex variable, where the mapping $f = b\left(2a^2 z^{-2} - 1\right)$ transforms the complex variable $z' = u + iv$ to $z = x + iy$ connecting the circumference $u^2 + v^2 = b^2$ with the hyperbola $x^2 - y^2 = a^2$. Moreover, we argue that this transformation admits a conformal mapping interpretation.

Appendix

A. Coordinate transformations from metric with scale parameter to its static form

For sake of completeness, we derive the explicit relation between the systems (t, r) and (T, ρ) listed in (16). First, we combine (8) and (10) to obtain

$$\frac{\partial t}{\partial \rho} = \frac{\rho a \dot{a}}{\sqrt{1 - k\rho^2}[1 - \rho^2(\dot{a}^2 + k)]}. \tag{A.1}$$

Thus, the idea is to integrate (A.1) for the different cases.

Case $k = 0$

Considering $a = e^{\sqrt{\Gamma}T}$ [see (16) for $k = 0$], (A.1) yields $\partial t/\partial\rho = \partial/\partial\rho\left[-\left(2\sqrt{\Gamma}\right)^{-1}\ln(1 - \Gamma e^{2\sqrt{\Gamma}T}\rho^2)\right]$. Hence, we can write

$$t = -\frac{1}{2\sqrt{\Gamma}}\ln(1 - \Gamma e^{2\sqrt{\Gamma}T}\rho^2) + g(T). \tag{A.2}$$

Note that inserting (8) and (19) in (9) yields $\partial t/\partial T = 1/\left(1 - \Gamma e^{2\sqrt{\Gamma}T}\rho^2\right)$. Comparing with $\partial t/\partial T$ obtained from (A.2), results in $\dot{g} = 1$, and then

$$t = T - \frac{1}{2\sqrt{\Gamma}}\ln\left(1 - \Gamma e^{2\sqrt{\Gamma}T}\rho^2\right), \tag{A.3}$$

where we have chosen $g(T) = T$. This result, with $r = e^{\sqrt{\Gamma}T}\rho$, leads to [10, 13]

$$T = \frac{1}{2\sqrt{\Gamma}}\ln\left(1 - \Gamma r^2\right) + t, \tag{A.4}$$

and

$$\rho = \frac{r}{\sqrt{1 - \Gamma r^2}}e^{-\sqrt{\Gamma}t}. \tag{A.5}$$

Case $k = 1$

Now we have that $a = \cosh(\sqrt{\Gamma}T)/\sqrt{\Gamma}$. Then (A.1) implies

$$\frac{\partial t}{\partial \rho} = \frac{\rho\sinh(\sqrt{\Gamma}T)\cosh(\sqrt{\Gamma}T)}{\sqrt{\Gamma}\sqrt{1 - \rho^2}[1 - \rho^2\cosh^2(\sqrt{\Gamma}T)]}. \tag{A.6}$$

Integration of this expression yields

$$t = \frac{1}{\sqrt{\Gamma}}\tanh^{-1}\left[\frac{\sqrt{1 - \rho^2}}{\tanh(\sqrt{\Gamma}T)}\right] + g(T). \tag{A.7}$$

Now, by using a and f in Eq. (9) gives $\partial t/\partial T = \sqrt{1 - \rho^2}/\left(1 - \rho^2\cosh^2(\sqrt{\Gamma}T)\right)$. Comparing with $\partial t/\partial T$ from (A.7) sets $\dot{g} = 0$. By choosing $g = 0$, (A.7) is

$$\tanh\left(\sqrt{\Gamma}t\right)\tanh\left(\sqrt{\Gamma}T\right) = \sqrt{1 - \rho^2}. \tag{A.8}$$

Meanwhile, as $a = \cosh(\sqrt{\Gamma}T)/\sqrt{\Gamma}$, from (3) we have

$$r = \frac{\rho\cosh(\sqrt{\Gamma}T)}{\sqrt{\Gamma}}. \tag{A.9}$$

Note that (A.8) must be used for $\tanh(\sqrt{\Gamma}T) > \sqrt{1 - \rho^2}$. But if $\tanh(\sqrt{\Gamma}T) < \sqrt{1 - \rho^2}$, one needs to use the transformation given in Refs. [11, 13].

Case $k=-1$**,** $\Gamma>0$

In this case $a=\sinh(\sqrt{\Gamma}T)/\sqrt{\Gamma}$, and (A.1) leads to

$$\frac{\partial t}{\partial \rho}=\frac{\rho\sinh(\sqrt{\Gamma}T)\cosh(\sqrt{\Gamma}T)}{\sqrt{\Gamma}\sqrt{1+\rho^2}[1-\rho^2\sinh^2(\sqrt{\Gamma}T)]}. \tag{A.10}$$

Following the same steps as before, this implies

$$t=\frac{1}{\sqrt{\Gamma}}\tanh^{-1}\left[\sqrt{1+\rho^2}\tanh(\sqrt{\Gamma}T)\right]+g(T). \tag{A.11}$$

Again, we derive with respect to T, that sets $\dot{g}=0$ and then we obtain

$$\tanh\left(\sqrt{\Gamma}t\right)=\sqrt{1+\rho^2}\tanh\left(\sqrt{\Gamma}T\right), \tag{A.12}$$

where we chose $g=0$. Furthermore, we also have that

$$r=\frac{\rho\sinh(\sqrt{\Gamma}T)}{\sqrt{\Gamma}}. \tag{A.13}$$

Here, we have used the scale parameter and (3). We note again that (A.12) is valid whenever $\tanh\left(\sqrt{\Gamma}T\right)>1/\sqrt{1+\rho^2}$, while $\tanh\left(\sqrt{\Gamma}T\right)<1/\sqrt{1+\rho^2}$ would lead to the transformation $\tanh\left(\sqrt{\Gamma}t\right)\tanh\left(\sqrt{\Gamma}T\right)=1/\sqrt{1+\rho^2}$.

Case $k=-1$**,** $\Gamma<0$

Here, $a=\sin(\sqrt{|\Gamma|}T)/\sqrt{|\Gamma|}$. Then (A.1) yields

$$\frac{\partial t}{\partial \rho}=\frac{\rho\sin(\sqrt{|\Gamma|}T)\cos(\sqrt{|\Gamma|}T)}{\sqrt{|\Gamma|}\sqrt{1+\rho^2}[1+\rho^2\sin^2(\sqrt{|\Gamma|}T)]}. \tag{A.14}$$

We can integrate the previous equation respect to ρ to obtain

$$t=\frac{1}{\sqrt{|\Gamma|}}\tan^{-1}\left[\sqrt{1+\rho^2}\tan(\sqrt{|\Gamma|}T)\right]+g(T). \tag{A.15}$$

Following the same steps as before, the coordinate transformation is

$$\tan\left(\sqrt{\Gamma}t\right)=\sqrt{1+\rho^2}\tan\left(\sqrt{\Gamma}T\right). \tag{A.16}$$

Additionally, in this case the coordinate r is

$$r=\frac{\rho\sin(\sqrt{|\Gamma|}T)}{\sqrt{|\Gamma|}}, \tag{A.17}$$

which completes all possible cases.

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