Gravitation, Mathematical Physics and Field Theory Revista Mexicana de Física **69** 050703 1–9 SEPTEMBER-OCTOBER 2023

Generalized equations and their solutions in the (1/2,0)+(0,1/2) representations of the Lorentz group

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Received 28 January 2023; accepted 19 April 2023

We present explicit examples of generalizations in relativistic quantum mechanics. First of all, we discuss the generalized spin-1/2 equations for neutrinos. They have been obtained by means of the Gersten-Sakurai method for derivations of arbitrary-spin relativistic equations. Possible physical consequences are discussed. Next, it is easy to check that both Dirac algebraic equations $Det(\hat{p}-m)=0$ and $Det(\hat{p}+m)=0$ for $u-$ and $v-$ 4-spinors have solutions with $p_0=\pm E_p=\pm\sqrt{\mathbf{p}^2+m^2}$. The same is true for higher-spin equations. Meanwhile, every book considers the equality $p_0=E_p$ for both $u-$ and $v-$ spinors of the $(1/2,0)\oplus(0,1/2)$ representation, thus applying the Dirac-Feynman-Stueckelberg procedure for elimination of the negative-energy solutions. The recent Ziino works (and, independently, the articles of several others) show that the Fock space can be doubled. We re-consider this possibility on the quantum field level for both $S=1/2$ and higher spin particles. The third example is: we postulate the non-commutativity of 4-momenta, and we derive the mass splitting in the Dirac equation. The applications are discussed.

Keywords: Relativistic quantum mechanics; spin-1/2; Dirac-Feynman-Stueckelberg.

DOI: https://doi.org/10.31349/RevMexFis.69.050703

1. Introduction

The problems of neutrino experiments and the problem of mass splitting of fermions have long history. We trust that they should be solved on the fundamental level of basic definitions of relativistic quantum equations.

In 1928, P.A.M. Dirac derived the so-called Dirac equation to explain the behavior of the relativistic motion of an electron and it was the first successful combination of quantum theory with the theory of relativity. It allows us to describe particles with spin $1/2$, Ref. [1]. However, one problem arose with the Dirac equation. It predicts solutions with negative energy and to solve this problem, Dirac introduced an hypothesis known as hole theory. In this hypothesis, the vacuum is a quantum state where all the electron eigenstates with negative energy are occupied. This description is called the Dirac sea. Before the experimental discovery of the positron in 1932, it was originally conceived as a hole in the Dirac sea and allowed to introduce the term antimatter in a natural way, Refs. [2, 3]. Beside these important results, a problematic fact existed for several years. The distribution of the kinetic energy of beta particles showed a continuous spectrum, however this result was not in concordance with the conservation of energy law, because if beta decay were electron emission as was thought at the time, then the emitted electron should have been emitted at a specific energy [4]. Additionally, an amount of momentum was missed and the angular momentum was not conserved. In a famous letter written in 1930, W. Pauli as a desperate remedy to save these conservation principles, proposed the existence of a new electrically neutral particle with spin $1/2$ (obeying in consequence the Dirac equation) and a tiny mass that should be emitted together with the electron. The energy of the emitted electron was part of the discussions of the seventh Solvay conference (1933). The existence of the neutrino proved the conservation of energy, but it was also determined that its mass could be zero. The same year, E. Fermi used the neutrino in his theory of the beta decay in an article submitted to *Nature* which was rejected because it contained speculations too remote from reality to be of interest to the reader. It was published in Italian by *Nuovo Cimento* and a few months later in German by *Zeitschrift für Physik* [4]. However the problem continued, how can a particle be detected without electric charge and whose mass could be zero? That is, how can a particle be detected if it basically does not interact?

The answer came from B. Pontecorvo in 1946. Because of the beta decay, a neutron would be transformed into one proton, therefore a chemical element would be transformed to a second one whose atomic number would be greater by one unit (an additional proton). Thus, he proposed to put a huge container full of chlorine close to a nuclear reactor. The chlorine (N=17) would be transformed to argon-37, enough inert as to get a chemical reaction, but slightly radioactive (its half-life is 34 days), making it possible to detect a neutrino [5]. The experiment was unsuccessful because, unknown at that time, the nuclear reactors produce antineutrinos, instead of neutrinos. This experiment led F. Reines and C. Cowan to a new one in 1953 using the inverse beta decay (IBD). They used a detector full of water with cadmium chloride ($CdCl_2$) surrounded with a scintillator material (material that gives off flashes of light in response to gamma rays) connected to photomultiplier tubes, to detect the flashes. Because of the IBD, a positron (discovered in 1932) and a neutron would be created. The positron would be annihilated with some electron in the media creating photons. On the other hand, the neutron produced by the interaction of the neutrino with the hydrogen proton would slow down gradually until it is captured by one of the cadmium nuclei. It would remain in an excited state, then it decays emitting photons. Both radiations could be detected with a few microseconds of difference. Enough time to measure both signals, creating an unequivocal proof of the presence of the neutrino [4]. However, the signal was also detected even if the reactor was switched off, but the number of signals increased if the reactor was on. This effect could be a consequence of the cosmic rays. Thus, they moved the experiment two years later to the Savannah River site because it had better shielding against cosmic rays. This shield was 11 m from the reactor and 12 m underground. They found almost four times more signals with the reactor on than off. After 25 years, the neutrino was finally captured and detected! [6]. In 1939, H. Bethe proposed a process for the energy production in the stars through two cycles. One of them, the proton-proton chain reaction, would produce neutrinos. In 1968, R. Davis and J.N. Bahcall detected solar neutrinos in the Homestake experiment using a similar idea as the Reines-Cowan experiment. However, even when the experiment was successful in detecting and counting solar neutrinos, the number detected was almost one third of the theoretical prediction. This became the *solar neutrino problem*.

Additionally, in 1935, H. Yukawa proposed the existence of a particle with a mass approximately 300-400 times the electron mass that should be responsible for the strong interaction between protons and neutrons inside the atomic nuclei. Searching for this particle, in 1937 was detected a particle with a mass 200 higher than the electron mass, negative electric charge but it did not interact with atomic nuclei. In consequence it was not the proposed by Yukawa and was called muon (μ) [4]. In that time, it was believed that muon was an excited and heavier electron and it was found that part of its energy decayed into an electron, therefore the difference of both energies should be emitted as electromagnetic radiation. However, the number of detected electrons was seriously lower than should have shown up. Fermi and his doctoral student J. Steinberger, thought that two neutrinos should also be produced. In 1948 [4], Steinberger experimentally confirmed this hypothesis, however, why did these two emitted neutrinos not annihilate creating electromagnetic radiation? That possibility exists if both neutrinos were the same, then it should be possible to observe the disintegration of muons into electrons and photons, a process not observed. Does there exist any fundamental property prohibiting this decay? The answer will come in 1953, when the leptonic number was introduced, therefore there would be two neutrinos: the electron neutrino and the muon neutrino [6].

In 1959, Pontecorvo listed 21 possible reactions involving neutrinos [7]. He observed that few reactions cannot happen except if the electron neutrino and the muon neutrino were the same. To solve this problem, he proposed that the neutrinos can vary between their states, that is, that neutrinos may transform into other kinds of neutrinos. This phenomenon is known as *neutrino oscillation* and it is actually an open research area. There have been several attempts to describe the neutrino oscillations. In Ref. [10] it is proposed a description for the neutrino oscillations using fields of Dirac neutrinos with definite masses, whereas in Ref. [11], it is proposed using massive Majorana neutrino fields (A Majorana field is described by a real-valued wave equation, in consequence a fermion is also its own antiparticle. All the fermions in the Standard Model are described as Dirac fermions and none as Majorana fermions).

Our paper deals with some examples that pursue generalizations of the Dirac equation in the context of relativistic quantum mechanics. The first example treats a generalized spin-1/2 equation; the second one, deals with a couple of algebraic equations, $Det(\hat{p} - m) = 0$ and $Det(\hat{p} + m) = 0$ for two 4-spinors. Finally, the authors try to implement non-commutativity of 4-momenta and derive a mass splitting in the Dirac equation.

2. Generalized neutrino equations

A. Gersten [12] proposed a method for derivations of massless equations of arbitrary-spin particles. In fact, his method is related to the van der Waerden-Sakurai [13] procedure for the derivation of the massive Dirac equation. In the present paper we first apply this procedure to the spin-1/2 fields. As a result one obtains equations which generalize the well-known Weyl equations. However, these equations are known for a long time [14]. Raspini [15, 16] analized them in detail.

Let us look at the equation (4) of the Gersten paper [12a] for the two-component spinor field function:

$$(E^2 - c^2\vec{\mathbf{p}}^{\,2})I^{(2)}\psi = \left[EI^{(2)} - c\vec{\mathbf{p}}\cdot\vec{\boldsymbol{\sigma}}\right] \times \left[EI^{(2)} + c\vec{\mathbf{p}}\cdot\vec{\boldsymbol{\sigma}}\right]\psi = 0\,. \qquad (1)$$

Actually, this equation is the massless limit of the Klein-Gordon equation for spin-1/2 in the Sakurai book [13]. In the massive case one should substitute m^2c^4 into the right-hand side of Eq. (1). However, instead of equation (3.25) of [13] one can define the two-component 'right' field function

$$\phi_R = \frac{1}{m_1 c}\left(i\hbar\frac{\partial}{\partial x_0} - i\hbar\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\right)\psi, \quad \phi_L = \psi \qquad (2)$$

with the different mass parameter m_1. In such a way we come to the system of the first-order differential equations:

$$\left(i\hbar\frac{\partial}{\partial x_0} + i\hbar\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\right)\phi_R = \frac{m_2^2 c}{m_1}\phi_L\,, \quad (3)$$

$$\left(i\hbar\frac{\partial}{\partial x_0} - i\hbar\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\right)\phi_L = m_1 c\phi_R\,. \quad (4)$$

It can be re-written in the 4-component form:

$$\begin{pmatrix} i\hbar(\partial/\partial x_0) & i\hbar\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} \\ -i\hbar\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} & -i\hbar(\partial/\partial x_0)\end{pmatrix}\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}$$
$$=\frac{c}{2}\begin{pmatrix}(m_2^2/m_1+m_1) & (-m_2^2/m_1+m_1)\\ (-m_2^2/m_1+m_1) & (m_2^2/m_1+m_1)\end{pmatrix}\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}, \quad (5)$$

for the function $\Psi = column(\psi_A \quad \psi_B) = column(\phi_R + \phi_L \quad \phi_R - \phi_L)$. The generalized Eq. (5) can be written in the covariant form:

$$\left[i\gamma^\mu\partial_\mu - \frac{m_2^2 c}{m_1\hbar}\frac{(1-\gamma^5)}{2} - \frac{m_1 c}{\hbar}\frac{(1+\gamma^5)}{2}\right]\Psi = 0\,. \quad (6)$$

The standard representation of γ^μ matrices has been used here. You may compare this framework with the spin-1 case [17].

If $m_1 = m_2$ we can recover the standard Dirac equation. As noted in [14b] this procedure can be viewed as the simple change of the representation of γ^μ matrices. However, this is true unless $m_2 \neq 0$ only. Otherwise, the entries in the transformation matrix become to be singular. However, one can either repeat a similar procedure (the modified Sakurai procedure) starting from the *massless* Eq. (4) of [12a] or put $m_2 = 0$ in Eq. (6). It is necessary to stress that the term *'massless'* is used in the sense that $p_\mu p^\mu = 0$. The *massless equation* is:

$$\left[i\gamma^\mu\partial_\mu - \frac{m_1 c}{\hbar}\frac{(1+\gamma^5)}{2}\right]\Psi = 0\,. \quad (7)$$

Then, we may have different physical consequences following from (7) comparing with those which follow from the Weyl equation. The mathematical reason of such a possibility of different massless limits is that the corresponding change of representation of γ^μ matrices involves mass parameters m_1 and m_2 themselves.

It is interesting to note that we can also repeat this procedure for other definition (or for even more general definitions):

$$\phi_L = \frac{1}{m_3 c}(i\hbar\frac{\partial}{\partial x_0} + i\hbar\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\psi, \quad \phi_R = \psi\,. \quad (8)$$

This is due to the fact that the parity properties of the two-component spinor are undefined in the two-component equation. The resulting equation is

$$\left[i\gamma^\mu\partial_\mu - \frac{m_4^2 c}{m_3\hbar}\frac{(1+\gamma^5)}{2} - \frac{m_3 c}{\hbar}\frac{(1-\gamma^5)}{2}\right]\tilde\Psi = 0\,, \quad (9)$$

which gives us yet another equation in the massless limit ($m_4 \to 0$):

$$\left[i\gamma^\mu\partial_\mu - \frac{m_3 c}{\hbar}\frac{(1-\gamma^5)}{2}\right]\tilde\Psi = 0\,, \quad (10)$$

differing in the sign at the γ_5 term.

The above procedure can be generalized to *any* Lorentz group representations, *i. e.*, to any spins. The physical content of the generalized $S = 1/2$ *massless* equations is not the same as that of the Weyl equation. The excellent discussion can be found in Ref. [14]. The theory does *not* have chiral invariance. Those authors call the additional parameters as the measures of the degree of chirality. Apart of this, Tokuoka introduced the concept of the gauge transformations for the 4-spinor fields. He also found some strange properties of the anti-commutation relations (see §3 in [14a]). And finally, the Eqs. (7,10) describe *four* states, two of which answer for the positive energy $p_0 = |\mathbf{p}|$, and two others answer for the negative energy $p_0 = -|\mathbf{p}|$.

We just want to add the following remarks to the discussion. The operator of the *chiral-helicity* $\hat\eta = (\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})$ (in the spinorial representation) used in [14b] does *not* commute, *e.g.*, with the Hamiltonian of the Eq. (7):

$$[\mathcal{H}, \boldsymbol{\alpha}\cdot\hat{\mathbf{p}}]_- = 2\frac{m_1 c}{\hbar}\frac{1-\gamma^5}{2}(\boldsymbol{\gamma}\cdot\hat{\mathbf{p}})\,. \quad (11)$$

For the eigenstates of the *chiral-helicity* the system of corresponding equations can be read ($\eta =\uparrow,\downarrow$):

$$i\gamma^\mu\partial_\mu\Psi_\eta - \frac{m_1 c}{\hbar}\frac{1+\gamma^5}{2}\Psi_{-\eta} = 0\,. \quad (12)$$

The conjugated eigenstates of the Hamiltonian $|\Psi_\uparrow + \Psi_\downarrow >$ and $|\Psi_\uparrow - \Psi_\downarrow >$ are connected, in fact, by γ^5 transformation $\Psi \to \gamma^5\Psi \sim (\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})\Psi$. However, the γ^5 transformation is related to the PT ($t \to -t$ only) transformation, which, in its turn, can be interpreted as $E \to -E$, if one accepts the Stueckelberg idea about antiparticles. We associate $|\Psi_\uparrow + \Psi_\downarrow >$ with the positive-energy eigenvalue of the Hamiltonian $p_0 = |\mathbf{p}|$ and $|\Psi_\uparrow - \Psi_\downarrow >$, with the negative-energy eigenvalue of the Hamiltonian ($p_0 = -|\mathbf{p}|$). Thus, the free chiral-helicity massless eigenstates may oscillate one to another with the frequency $\omega = E/\hbar$ (as the massive chiral-helicity eigenstates, see [18a] for details). Moreover, a special kind of interaction which is not symmetric with respect to the chiral-helicity states (for instance, if the left chiral-helicity eigenstates interact with the matter only) may induce changes in the oscillation frequency, like in the Wolfenstein (MSW) formalism.

The conclusion may be similar to that which was achieved before: the dynamical properties of the massless particles (*e.g.*, neutrinos and photons) may differ from those defined by the well-known Weyl and Maxwell equations [18–20].

3. Negative energies in the Dirac Equation

Usually, everybody uses the following definition of the field operator [21] in the pseudo-Euclidean metrics:

$$\Psi(x) = \frac{1}{(2\pi)^3}\sum_h \int \frac{d^3\mathbf{p}}{2E_p}[u_h(\mathbf{p})a_h(\mathbf{p})e^{-ip\cdot x} + v_h(\mathbf{p})b_h^\dagger(\mathbf{p})]e^{+ip\cdot x}]\,, \tag{13}$$

as given *ab initio*. After actions of the Dirac operator at $\exp(\mp ip_\mu x^\mu)$ the 4-spinors ($u-$ and $v-$) satisfy the momentum-space equations: $(\hat{p}-m)u_h(p) = 0$ and $(\hat{p}+m)v_h(p) = 0$, respectively; the h is the polarization index. However, it is easy to prove from the characteristic equations $Det(\hat{p}\mp m) = (p_0^2 - \mathbf{p}^2 - m^2)^2 = 0$ that the solutions should satisfy the energy-momentum relation $p_0 = \pm E_p = \pm\sqrt{\mathbf{p}^2+m^2}$ in both cases.

Let us remind the general scheme of construction of the field operator, which has been presented in Ref. [22]. In the case of the $(1/2,0)\oplus(0,1/2)$ representation we have:

$$\begin{aligned}\Psi(x) &= \frac{1}{(2\pi)^3}\int d^4p\,\delta(p^2-m^2)e^{-ip\cdot x}\Psi(p) = \frac{1}{(2\pi)^3}\sum_h\int d^4p\,\delta(p_0^2-E_p^2)e^{-ip\cdot x}u_h(p_0,\mathbf{p})a_h(p_0,\mathbf{p})\\
&= \frac{1}{(2\pi)^3}\int\frac{d^4p}{2E_p}[\delta(p_0-E_p)+\delta(p_0+E_p)][\theta(p_0)+\theta(-p_0)]e^{-ip\cdot x}\sum_h u_h(p)a_h(p)\\
&= \frac{1}{(2\pi)^3}\sum_h\int\frac{d^4p}{2E_p}[\delta(p_0-E_p)+\delta(p_0+E_p)]\left[\theta(p_0)u_h(p)a_h(p)e^{-ip\cdot x}+\theta(p_0)u_h(-p)a_h(-p)e^{+ip\cdot x}\right]\\
&= \frac{1}{(2\pi)^3}\sum_h\int\frac{d^3\mathbf{p}}{2E_p}\theta(p_0)[u_h(p)a_h(p)|_{p_0=E_p}e^{-i(E_pt-\mathbf{p}\cdot\mathbf{x})}+\ u_h(-p)a_h(-p)|_{p_0=E_p}e^{+i(E_pt-\mathbf{p}\cdot\mathbf{x})}].\end{aligned} \tag{14}$$

During the calculations above we had to represent $1 = \theta(p_0)+\theta(-p_0)$ in order to get positive- and negative-frequency parts. Moreover, during these calculations we did not yet assumed, which equation this field operator (namely, the $u-$ spinor) does satisfy, with negative- or positive- mass?

In general we should transform $u_h(-p)$ to the $v(p)$. The procedure is the following one [23]. In the Dirac case we should assume the following relation in the field operator:

$$\sum_h v_h(p)b_h^\dagger(p) = \sum_h u_h(-p)a_h(-p)\,. \tag{15}$$

We know that [3]

$$\bar{u}_{(\mu)}(p)u_{(\lambda)}(p) = +m\delta_{\mu\lambda}\,, \tag{16}$$

$$\bar{u}_{(\mu)}(p)u_{(\lambda)}(-p) = 0\,, \tag{17}$$

$$\bar{v}_{(\mu)}(p)v_{(\lambda)}(p) = -m\delta_{\mu\lambda}\,, \tag{18}$$

$$\bar{v}_{(\mu)}(p)u_{(\lambda)}(p) = 0\,, \tag{19}$$

but we need $\Lambda_{(\mu)(\lambda)}(p) = \bar{v}_{(\mu)}(p)u_{(\lambda)}(-p)$. By direct calculations, we find

$$-mb_{(\mu)}^\dagger(p) = \sum_\lambda \Lambda_{(\mu)(\lambda)}(p)a_{(\lambda)}(-p)\,. \tag{20}$$

Hence, $\Lambda_{(\mu)(\lambda)} = -im(\boldsymbol{\sigma}\cdot\mathbf{n})_{(\mu)(\lambda)}$, $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$, and

$$b_{(\mu)}^\dagger(p) = i\sum_\lambda(\boldsymbol{\sigma}\cdot\mathbf{n})_{(\mu)(\lambda)}a_{(\lambda)}(-p)\,. \tag{21}$$

Multiplying (15) by $\bar{u}_{(\mu)}(-p)$ we obtain

$$a_{(\mu)}(-p) = -i\sum_\lambda(\boldsymbol{\sigma}\cdot\mathbf{n})_{(\mu)(\lambda)}b_{(\lambda)}^\dagger(p)\,. \tag{22}$$

The equations are self-consistent. In the $(1,0)\oplus(0,1)$ representation the similar procedure leads to somewhat different situation:

$$a_{(\mu)}(p) = [1-2(\mathbf{S}\cdot\mathbf{n})^2]_{(\mu)(\lambda)}a_{(\lambda)}(-p)\,. \tag{23}$$

This signifies that in order to construct the Sankaranarayanan-Good field operator, it satisfies $[\gamma_{\mu\nu}\partial_\mu\partial_\nu - ([i\partial/\partial t]/E)m^2]\Psi(x) = 0$, we need additional postulates. For instance, one can try to construct the left- and the right-hand side of the field operator separately each other [24].

First of all to mention, we have, in fact, $u_h(E_p,\mathbf{p})$ and $u_h(-E_p,\mathbf{p})$, and $v_h(E_p,\mathbf{p})$ and $v_h(-E_p,\mathbf{p})$, originally, which may satisfy the equations:

$$\left[E_p(\pm\gamma^0) - \boldsymbol{\gamma}\cdot\mathbf{p} - m\right]u_h(\pm E_p,\mathbf{p}) = 0\,. \tag{24}$$

Due to the properties $U^\dagger\gamma^0U = -\gamma^0$, $U^\dagger\gamma^iU = +\gamma^i$ with the unitary matrix

$$U = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \gamma^0\gamma^5$$

in the Weyl basis, we have

$$[E_p\gamma^0 - \boldsymbol{\gamma}\cdot\mathbf{p} - m]U^\dagger u_h(-E_p,\mathbf{p}) = 0\,. \tag{25}$$

The properties of the $U-$ matrix are opposite to those of $P^\dagger\gamma^0 P = +\gamma^0$, $P^\dagger\gamma^i P = -\gamma^i$ with the usual $P = \gamma^0$, thus giving[i]

$$[-E_p\gamma^0 + \gamma\cdot\mathbf{p} - m]Pu_h(-E_p,\mathbf{p}) \\ = -\left[\hat{p}+m\right]\tilde{v}_?(E_p,\mathbf{p}) = 0.$$

The relations of the $v-$ spinors of the positive energy to $u-$ spinors of the negative energy are frequently forgotten, $\tilde{v}_?(E_p,\mathbf{p}) = \gamma^0 u_h(-E_p,\mathbf{p})$. Thus, unless the unitary transformations do not change the physical content, we have that the negative-energy spinors $\gamma^5\gamma^0 u^-$ [see (25)] satisfy the accustomed "positive-energy" Dirac equation. We should then expect the same physical content. Their explicit forms $\gamma^5\gamma^0 u^-$ are different from the textbook "positive-energy" Dirac spinors. They are the following ones:

$$\tilde{u}(p) = \frac{N}{\sqrt{2m(-E_p+m)}}\begin{pmatrix}-p^+ + m\\ -p_r\\ p^- - m\\ -p_r\end{pmatrix}, \qquad (26)$$

$$\tilde{\tilde{u}}(p) = \frac{N}{\sqrt{2m(-E_p+m)}}\begin{pmatrix}-p_l\\ -p^- + m\\ -p_l\\ p^+ - m\end{pmatrix}. \qquad (27)$$

$E_p = \sqrt{\mathbf{p}^2+m^2} > 0$, $p_0 = \pm E_p$, $p^\pm = E \pm p_z$, $p_{r,l} = p_x \pm i p_y$. Their normalization is to $(-2N^2)$. Next, $\tilde{v}(p) = \gamma^0 u^-$. They are not equal to $v_h(p) = \gamma^5 u_h(p)$. Obviously, they also do not have well-known forms of the usual $v-$ spinors in the Weyl basis, differing by phase factors and in the signs at the mass terms.

One can again prove that the matrix

$$P = e^{i\theta}\gamma^0 = e^{i\theta}\begin{pmatrix}0 & 1_{2\times2}\\ 1_{2\times2} & 0\end{pmatrix} \qquad (28)$$

can be used in the parity operator as well as in the original Weyl basis. However, if we would take the phase factor to be zero we obtain that while $u_h(p)$ have the eigenvalue $+1$ of the parity, but $(R = (\mathbf{x}\to-\mathbf{x}, \mathbf{p}\to-\mathbf{p}))$

$$PR\tilde{u}(p) = PR\gamma^5\gamma^0 u(-E_p,\mathbf{p}) = -\tilde{u}(p), \qquad (29)$$

$$PR\tilde{\tilde{u}}(p) = PR\gamma^5\gamma^0 u(-E_p,\mathbf{p}) = -\tilde{\tilde{u}}(p). \qquad (30)$$

In the case of choosing the phase factor $\theta = \pi$ we recover usual parity properties. We again confirmed that the relative (particle-antiparticle) intrinsic parity has physical significance only.

Similar formulations have been presented in Refs. [25], and [26]. The group-theoretical basis for such doubling has been given in the papers by Gelfand, Tsetlin and Sokolik [27], who first presented the theory in the 2-dimensional representation of the inversion group in 1956. M. Markov wrote *two* Dirac equations with the opposite signs at the mass term [25] long ago:

$$\left[i\gamma^\mu\partial_\mu - m\right]\Psi_1(x) = 0, \qquad (31)$$

$$\left[i\gamma^\mu\partial_\mu + m\right]\Psi_2(x) = 0. \qquad (32)$$

In fact, he studied all properties of this relativistic quantum model (while he did not know yet the quantum field theory in 1937). Next, he added and subtracted these equations:

$$i\gamma^\mu\partial_\mu\varphi(x) - m\chi(x) = 0, \qquad (33)$$

$$i\gamma^\mu\partial_\mu\chi(x) - m\varphi(x) = 0. \qquad (34)$$

Thus, φ and χ solutions can be presented as some superpositions of the Dirac 4-spinors $u-$ and $v-$. These equations, of course, can be identified with the equations for the Majorana-like $\lambda-$ and $\rho-$, which we presented in Ref. [18]:

$$i\gamma^\mu\partial_\mu\lambda^S(x) - m\rho^A(x) = 0, \qquad (35)$$

$$i\gamma^\mu\partial_\mu\rho^A(x) - m\lambda^S(x) = 0, \qquad (36)$$

$$i\gamma^\mu\partial_\mu\lambda^A(x) + m\rho^S(x) = 0, \qquad (37)$$

$$i\gamma^\mu\partial_\mu\rho^S(x) + m\lambda^A(x) = 0. \qquad (38)$$

Neither of them can be regarded as the Dirac equation. However, they can be written in the 8-component form as follows:

$$\left[i\Gamma^\mu\partial_\mu - m\right]\Psi_{(+)}(x) = 0, \qquad (39)$$

$$\left[i\Gamma^\mu\partial_\mu + m\right]\Psi_{(-)}(x) = 0, \qquad (40)$$

with

$$\Psi_{(+)}(x) = \begin{pmatrix}\rho^A(x)\\ \lambda^S(x)\end{pmatrix}, \quad \Psi_{(-)}(x) = \begin{pmatrix}\rho^S(x)\\ \lambda^A(x)\end{pmatrix}, \\ \Gamma^\mu = \begin{pmatrix}0 & \gamma^\mu\\ \gamma^\mu & 0\end{pmatrix}. \qquad (41)$$

It is easy to find the corresponding projection operators, and the Feynman-Stueckelberg propagator.

In the previous papers we explained that the connection with the Dirac spinors has been found [18, 28]. For instance,

$$\begin{pmatrix}\lambda^S_\uparrow(\mathbf{p})\\ \lambda^S_\downarrow(\mathbf{p})\\ \lambda^A_\uparrow(\mathbf{p})\\ \lambda^A_\downarrow(\mathbf{p})\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & i & -1 & i\\ -i & 1 & -i & -1\\ 1 & -i & -1 & -i\\ i & 1 & i & -1\end{pmatrix}\begin{pmatrix}u_{+1/2}(\mathbf{p})\\ u_{-1/2}(\mathbf{p})\\ v_{+1/2}(\mathbf{p})\\ v_{-1/2}(\mathbf{p})\end{pmatrix}, \qquad (42)$$

provided that the 4-spinors have the same physical dimension. Thus, we can see that the two 4-spinor systems are connected by the unitary transformations, and this represents itself the rotation of the spin-parity basis. However, it is usually assumed that the $\lambda-$ and $\rho-$ spinors describe the neutral particles, meanwhile $u-$ and $v-$ spinors describe the charged particles. Kirchbach [28] found the amplitudes for neutrinoless double beta decay $(00\nu\beta)$ in this scheme. It is obvious from (42) that there are some additional terms comparing with the standard calculations of those amplitudes. As Markov wrote himself, he was expecting "new physics" from these equations.

Barut and Ziino [26] proposed yet another model. They considered γ^5 operator as the operator of the charge conjugation. Thus, the charge-conjugated Dirac equation has the different sign comparing with the ordinary formulation:

$$[i\gamma^\mu\partial_\mu + m]\Psi^c_{BZ} = 0\,, \tag{43}$$

and the so-defined charge conjugation applies to the whole system, fermion + electromagnetic field, $e \to -e$ in the covariant derivative. The superpositions of the Ψ_{BZ} and Ψ^c_{BZ} also give us the "doubled Dirac equation", as the equations for $\lambda-$ and $\rho-$ spinors. The concept of the doubling of the Fock space has been developed in the Ziino works (cf. [27,29]) in the framework of the quantum field theory. In their case the self/anti-self charge conjugate states are simultaneously the eigenstates of the chirality. Next, it is interesting to note that we have for the Majorana-like field operators ($a_\eta(\mathbf{p}) = b_\eta(\mathbf{p})$):

$$\left[\nu^{^{ML}}(x^\mu) + \mathcal{C}\nu^{^{ML\,\dagger}}(x^\mu)\right]/2 = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_p}\sum_\eta \left[\begin{pmatrix} i\Theta\phi^{*\,\eta}_{_L}(p^\mu)\\ 0\end{pmatrix} a_\eta(p^\mu)e^{-ip\cdot x} + \begin{pmatrix} 0\\ \phi^\eta_{_L}(p^\mu)\end{pmatrix} a^\dagger_\eta(p^\mu)e^{ip\cdot x}\right], \tag{44}$$

$$\left[\nu^{^{ML}}(x^\mu) - \mathcal{C}\nu^{^{ML\,\dagger}}(x^\mu)\right]/2 = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_p}\sum_\eta \left[\begin{pmatrix} 0\\ \phi^\eta_{_L}(p^\mu)\end{pmatrix} a_\eta(p^\mu)e^{-ip\cdot x} + \begin{pmatrix} -i\Theta\phi^{*\,\eta}_{_L}(p^\mu)\\ 0\end{pmatrix} a^\dagger_\eta(p^\mu)e^{ip\cdot x}\right], \tag{45}$$

which, thus, naturally lead to the Ziino-Barut scheme of massive chiral fields, Ref. [26].

Finally, we would like to mention that, in general, in the Weyl basis the $\gamma-$ matrices are *not* Hermitian, $\gamma^{\mu^\dagger} = \gamma^0\gamma^\mu\gamma^0$. So, $\gamma^{i^\dagger} = -\gamma^i$, $i = 1, 2, 3$, the pseudo-Hermitian matrix. The energy-momentum operator $i\partial_\mu$ is obviously Hermitian. So, the question, if the eigenvalues of the Dirac operator $i\gamma^\mu\partial_\mu$ (the mass, in fact) would be always real? The question of the complete system of the eigenvectors of the *non*-Hermitian operator deserve careful consideration.

The main points of this Section are: there are "negative-energy solutions" in that is previously considered as "positive-energy solutions" of relativistic wave equations, and vice versa. Their explicit forms have been presented in the case of spin-1/2. Next, the relations to the previous works have been found. For instance, the doubling of the Fock space and the corresponding solutions of the Dirac equation obtained additional mathematical bases. Similar conclusion can be deduced for the higher-spin equations.

4. Non-commutativity in the Dirac Equation

The non-commutativity [30] exibits interesting peculiarities in the Dirac case. We analized Sakurai-van der Waerden method of derivations of the Dirac (and higher-spins too) equation [31]:

$$(EI^{(4)} + \boldsymbol{\alpha}\cdot\mathbf{p} + m\beta)$$
$$\times\,(EI^{(4)} - \boldsymbol{\alpha}\cdot\mathbf{p} - m\beta)\Psi_{(4)} = 0\,. \tag{46}$$

As in the original Dirac work, we have

$$\beta^2 = 1\,,\quad \alpha^i\beta + \beta\alpha^i = 0\,,\quad \alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij}\,. \tag{47}$$

For instance, their explicite forms can be chosen

$$\alpha^i = \begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix},\quad \beta = \begin{pmatrix} 0 & 1_{2\times 2}\\ 1_{2\times 2} & 0\end{pmatrix}, \tag{48}$$

where σ^i are the ordinary Pauli 2×2 matrices. Obviously, the inverse operators of the Dirac operators of both the positive- and negative- masses exist in the non-commutative case.

We also postulate the non-commutativity relations for the components of 4-momenta: $[E, \mathbf{p}^i]_- = \Theta^{0i} = \theta^i$. Therefore the Eq. (46) will *not* lead to the well-known equation $E^2 - \mathbf{p}^2 = m^2$. Instead, we have

$$\{E^2 - E(\boldsymbol{\alpha}\cdot\mathbf{p}) + (\boldsymbol{\alpha}\cdot\mathbf{p})E - \mathbf{p}^2$$
$$- m^2 - i(\boldsymbol{\sigma}\otimes I_{(2)})[\mathbf{p}\times\mathbf{p}]\}\Psi_{(4)} = 0\,. \tag{49}$$

For the sake of simplicity, we may assume the last term to be zero.[ii] Thus, we come to

$$\left\{E^2 - \mathbf{p}^2 - m^2 - (\boldsymbol{\alpha}\cdot\boldsymbol{\theta})\right\}\Psi_{(4)} = 0\,. \tag{50}$$

Let us apply the unitary transformation. It is known [18, 34] that one can

$$U_1(\sigma\cdot\mathbf{a})U_1^{-1} = \sigma_3|\mathbf{a}|. \quad [34] \tag{51}$$

Some relations for the components $\mathbf{a}$ should be assumed. Moreover, in our case $\boldsymbol{\theta}$ should not depend on E and $\mathbf{p}$. Otherwise, we must take the non-commutativity $[E, \mathbf{p}^i]_-$ into account again. For $\boldsymbol{\alpha}$ matrices we re-write (51) to

$$\mathcal{U}_1(\boldsymbol{\alpha}\cdot\boldsymbol{\theta})\mathcal{U}_1^{-1} = |\boldsymbol{\theta}|\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} = \alpha_3|\boldsymbol{\theta}|\,. \tag{52}$$

The explicit form of the U_1 matrix is ($a_{r,l} = a_1 \pm i a_2$):

$$
\begin{aligned}
U_1 &= \frac{1}{\sqrt{2a(a+a_3)}} \begin{pmatrix} a+a_3 & a_l \\ -a_r & a+a_3 \end{pmatrix} \\
&= \frac{1}{\sqrt{2a(a+a_3)}} [a + a_3 + i\sigma_2 a_1 - i\sigma_1 a_2] \,, \\
\mathcal{U}_1 &= \begin{pmatrix} U_1 & 0 \\ 0 & U_1 \end{pmatrix} . \qquad (53)
\end{aligned}
$$

Let us apply the second unitary transformation:

$$
\begin{aligned}
\mathcal{U}_2 \alpha_3 \mathcal{U}_2^\dagger &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \alpha_3 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \qquad (54)
\end{aligned}
$$

The final equation is

$$
[E^2 - \mathbf{p}^2 - m^2 - \gamma^5_{chiral} |\boldsymbol{\theta}|] \Psi'_{(4)} = 0 \,. \qquad (55)
$$

In the physical sense this implies the mass splitting for a Dirac particle over the non-commutative space, $m_{1,2} = \pm\sqrt{m^2 \pm \theta}$. This procedure may be attractive for explanation of the mass creation and the mass splitting for fermions.

5. The conclusions

Currently, there exist additional alternative models of neutrino masses due to the amount of ambiguities that arise in the experiments. For instance, the Liquid Scintillator Neutrino Detector (LSND) results are not compatible with the oscillation parameters, creating a conflict with the expected results of only three neutrino flavors. These results are tested by other experiments, like MiniBooNE, MicroBooNE or KARMEN. There are additional controversies. For example, if neutrinos are an important fraction of the cosmological density, then they should be heavier than the splittings registered by the atmospheric and solar oscillation frequencies. All these results have also imply different models for the neutrino masses. A new kind of neutrino: the sterile neutrino has been proposed, which could interact via gravity and not by the other fundamental interactions of the Standard Model.

All these different results demand some theory that can describe them successfully. In the paper we presented some examples that generalize the Dirac equation in relativistic quantum mechanics. The conclusion of the second Section is: the dynamical properties of the massless particles (*e. g.*, neutrinos and photons) may differ from those defined by the well-known Weyl and Maxwell equations [18–20]. The main points of the third Section are: there are "negative-energy solutions" in that is previously considered as the "positive-energy solutions" of relativistic wave equations, and vice versa. Their explicit forms have been presented in the case of the spin 1/2. The relations to the previous works have been found. For instance, the doubling of the Fock space and the corresponding solutions of the Dirac equation obtained additional mathematical bases. The non-commutativity in the Dirac equation may impliy the mass splitting, $m_{1,2} = \pm\sqrt{m^2 \pm \theta}$.

Acknowledgments

This paper was also presented at the 31st ICGTMP, June 19-25, 2016, CBPF, Rio de Janeiro, Brasil and at the XI SILAFAE. Antigua Guatemala, Nov. 14-18, 2016. we appreciate useful discussions with colleagues at the recent Conferences.

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i. The $\tilde{v}$ is still the solution of the Dirac equation. And it is related to the standard-basis 4-spinors. We use tildes because we do not yet know their polarization properties.

ii. In general we can continue with this term. However, the calculations become immense and troubleome. This omitted term may be related to the magnetic field as shown by Feynman-Dyson [32] and Dvoeglazov [33].

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