Extraction of soliton solutions for the fractional Kaup-Boussinesq system: A comparative study

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This paper is based on finding soliton solutions to fractional Kaup-Boussinesq (FKB) system. The fractional derivatives such as β-derivative and truncated M-fractional derivative are used in this study. The unified approach, generalized projective riccati equations method (GPREM) and improved $\tan\left(\phi(\zeta)/2\right)$-expansion approaches are efficiently used for obtaining bright soliton, dark soliton, singular soliton, periodic soliton, dark-singular combo soliton and dark-bright combo soliton. The numerical simulations are also carried out by 3D and 2D, graphs of some of the obtained solutions to discuss the fractional effects.

Keywords: Integrating schemes; unified method; GPREM; improved $\tan\left(\phi(\zeta)/2\right)$-expansion; β-derivative; truncated M-fractional derivative.

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1. Introduction

In real world, majority of useful applications have nonlinear nature. Nonlinear partial differential equations (NLPDEs) have fundamental significance because these equations are used for mathematical modeling of various physical phenomena [1, 2]. In particular, to find the exact solution of NLPDEs which depend on time evolution known as nonlinear evolution equations (NLEEs) arising in different branches of science are a fascinating field of research [3–6]. The investigation of exact solitary wave solutions NLEEs is helpful to understand complex physical processes. Soliton theory has vast applications in the fields of biophysics, quantum mechanics, nonlinear optics, microbiology and engineering [7–10]. Solitons are solitary waves that travel with their original speed and shape even after nonlinear collision with other waves [11].

In recent decades, fractional NLPDEs have become an interesting research area. In the analysis of numerous processes in nonlinear sciences, fractional calculus has emerged as an effective and efficient mathematical gadget [12–14].

The fractional model provides more degrees of freedom. Moreover, FDEs own certain properties of a system other than that handled by the traditional integer-order equations. The solutions of FDEs contribute to innovative viewpoints in their dynamical investigation. Researchers have made many efforts to extract traveling wave solutions of FDEs [15–18].

This paper is devoted to finding soliton solutions for NLEE namely Kaup-Boussinesq (KB) system by considering β-fractional and M-truncated derivatives. KB model is adopted for the study of long and weakly nonlinear waves in shallow water. Other physical applications of the governing system include ion sound waves in plasma, vibrations and nonlinear lattice waves in a nonlinear series [19]. Many methods have been implemented to solve KB equations in literature [20]. This article concerns with the solutions of FKB equations using three robust and reliable integration schemes. Also, the comparative study permits to envision fractional behavior more precisely.

The unified method allows a researcher to find two kinds of traveling wave solutions, polynomial and rational form of functional solutions. Many authors have used this tool to recover the soliton solutions of NLPDEs [21]. GPREM has been utilized by many researchers and scientists [22–24] in recent years for obtaining new soliton solutions. Moreover, $\tan\left(\phi(\zeta)/2\right)$-expansion technique has been applied on NLPDEs [25] to obtain traveling wave solutions.

The manuscript includes twelve sections. Section 2 is allotted to the preliminaries. The suggested model is considered in Sec. 3. In Sec. 4, techniques descriptions of all three methods are provided. Section 5 covers the soliton extraction via unified method. Polynomial and rational function solutions are obtained in Secs. 6 and 7 respectively. Soliton solutions are developed through GPREM and $\tan\left(\phi(\zeta)/2\right)$-expansion method in Secs. 8 and 9. Section 10 and 11 show the comparison of the results and their discussion. In Sec. 12, concluding remarks are included.

2. Preliminaries

Some fundamental definitions of fractional calculus (FC) have been presented in this section. Riemann-Liouville definition and the Caputo definition being the most famous and commonly used definitions have some serious drawbacks. Few of them are stated below,

- Riemann-Liouville fractional derivative of an arbitrary constant is not zero.
- In Caputo definition the function is assumed to be differentiable.
- The product law and quotient law for derivatives of functions are not satisfied by these definitions.
- Chain rule and the index rule are also not satisfied by these definitions.

In order to overcome the above disadvantages of the existing definitions, a new form of conformable fractional derivative termed as β-derivative and truncated M-fractional derivative are applied in this paper.

2.1. β-derivative

The β-derivative ia also termed as another kind of conformable derivative. The β-derivative of differential function $g(x)$ can be defined, as [40]

$$ {}_0^B D_x^\alpha g(x) = \lim_{\varepsilon \to 0} \frac{g\left(x + \varepsilon\left(x + \frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}\right) - g(x)}{\varepsilon}, \quad 0 < \alpha < 1, $$

where α is taken as fractional parameter.

Theorem 1: β-derivative has the following properties [40],

$$ {}_0^B D_x^\alpha \left(ag(x) + bh(x)\right) = a\, {}_r^B D_x^\alpha g(x) + b\, {}_s^B D_x^\alpha h(x), \quad \forall\ a, b \in \Re $$

$$ {}_0^B D_x^\alpha \left(g(x) * h(x)\right) = h(x) {}_r^B D_x^\alpha g(x) + g(x) {}_r^B D_x^\alpha h(x), $$

$$ {}_0^B D_x^\alpha \left\{\frac{g(x)}{h(x)}\right\} = \frac{h(x) {}_r^B D_x^\alpha g(x) - g(x) {}_r^B D_x^\alpha h(x)}{h^2(x)}, $$

$$ {}_0^B D_x^\alpha c = 0, \quad \text{for } c \text{ any constant}, $$

where f and g are differential functions.

Truncated M-fractional derivative

It is defined as,

$$ {}_j D_M^{\alpha,\lambda} f(\phi) = \lim_{\epsilon \to 0} \frac{f\left(\phi +_j T_\lambda(\epsilon \phi^{-\alpha})\right) - f(\phi)}{\epsilon}, $$

for $\phi > 0$ and ${}_j T_\lambda(.),\ \lambda > 0$.

Theorem 2: Truncated M-fractional derivative has the following properties

$$ {}_j D_M^{\alpha,\lambda}(af + bg) = a_j D_M^{\alpha,\lambda}(f) + b_j D_M^{\alpha,\lambda}(g), \quad \forall a, b \in \Re $$

$$ {}_j D_M^{\alpha,\lambda}(\phi^z) = z\phi^{z-\alpha} \quad z \in R, $$

$$ {}_j D_M^{\alpha,\lambda}(fg) = f_j D_M^{\alpha,\lambda}(g) + g_j D_M^{\alpha,\lambda}(f), $$

$$ {}_j D_M^{\alpha,\lambda}\left(\frac{f}{g}\right) = \frac{f_j D_M^{\alpha,\lambda}(g) - g_j D_M^{\alpha,\lambda}(f)}{g^2}, $$

$$ {}_j D_M^{\alpha,\lambda}(f)(\phi) = \frac{\phi^{1-\alpha}}{\Gamma(\lambda+1)} \frac{df}{d\phi}, $$

$$ {}_j D_M^{\alpha,\lambda}(fog)(\phi) = f'(g(\phi))_j D_M^{\alpha,\lambda} g(\phi), $$

where f and g are differentiable functions of order α, $\alpha \in (0,1]$ and $\lambda > 0$.

3. Governing models

This paper investigates **Kaup-Boussinesq (KB) System** with β and M-fractional derivatives via three integrating techniques such as unified approach, GPREM and improved $\tan\left(\phi(\zeta)/2\right)$-technique for extracting new soliton solutions.

The motion of water wave is well described by the Kaup-Boussinesq system [37] given below

$$ u_t - v_{xxx} - 2(uv)_x = 0, \quad v_t + u_x - (v^2)_x = 0, \tag{1} $$

where $u(x,t)$ represents the height of the water surface above a horizontal bottom and $v(x,t)$ is the horizontal velocity. The governing model termed as Kaup-Boussinesq (KB) system because it has used Boussinesq scaling in its derivation, and Kaup [32] was the first who has investigated it. It has also been used by Broer [33]. The Proposed KB system also belongs to the family of long-waves models invented by Boussinesq, formed by [34, 35] and many others. In [36] solitary-wave solution of the KB system is obtained. In [38, 39], the authors have employed Adomian decomposition, homotopy methods and successive approximation methods for solving Kaup-Boussinesq system.

The fractional KB system using β-derivative has the following form

$$ {}^B D_t^\alpha u - {}^B D_x^{3\alpha} v - 2\, {}^B D_x^\alpha uv = 0, \quad {}^B D_t^\alpha v - {}^B D_x^\alpha u - {}^B D_x^\alpha v^2 = 0 \tag{2} $$

$D_t^\alpha = \partial^\alpha/\partial t^\alpha$ and $D_x^\alpha = \partial^\alpha/\partial x^\alpha$ represent β-fractional derivatives.

The fractional KB system using M-truncated derivative has the following form

$$ D_{M,t}^{\alpha,\lambda} u - D_{M,x}^{3\alpha,\lambda} v - 2D_{M,x}^{\alpha,\lambda} uv = 0, \quad D_{M,t}^{\alpha,\lambda} v - D_{M,x}^{\alpha,\lambda} u - D_{M,x}^{\alpha,\lambda} v^2 = 0, \tag{3} $$

where $D_{M,t}^{\alpha,\lambda}$ and $D_{M,x}^{\alpha,\lambda}$ represent M-derivatives.

4. Description of suggested methodologies

The PDE or fractional PDE is converted into ODE using traveling wave transformation. The unified method [29], GPREM [31] and $\tan\big(\phi(\zeta)/2\big)$-expansion [24] approaches are discussed in this section.

- **Technique I: Unified method**

  The unified method is an efficient analytical technique that extracts polynomial function solutions and rational solutions.

  Polynomial function solutions

  The converted ODE has a polynomial solution as

$$V(\zeta) = \sum_{i=0}^{n} a_i \phi^i(\zeta), \tag{4}$$

  where $\phi(\zeta)$ satisfying the ODE

$$(\phi'(\zeta))^d = \sum_{i=0}^{dk} b_i \phi^i(\zeta), \quad d = 1, 2, \tag{5}$$

  The constants a_i and b_i are to be found.

  Here n and k are determined using the balancing principle [29]. For $d = 1$, provides elementary solutions and for $d = 2$, elliptic solutions are extracted.

4.1. Rational function solution

The rational solution of converted ODE has the following form

$$V(\zeta) = \frac{\sum_{i=0}^{n} A_i \phi^i(\zeta)}{\sum_{i=0}^{r} B_i \phi^i(\zeta)}, \quad n \geq r, \tag{6}$$

where

$$(\phi'(\zeta))^d = \sum_{i=0}^{dk} b_i \phi^i(\zeta), \quad d = 1, 2, \tag{7}$$

where A_i, B_i and b_i are constants to be determined.

Here n and k are calculated after employing the balancing principle [29].

- **Technique II: GPRE**

  According to Technique II [31], the predicted solution has the form

$$V(\zeta) = a_0 + \sum_{r=1}^{u} s^{r-1}(\xi)\left[a_r s(\zeta) + \beta_r t(\zeta)\right], \tag{8}$$

  where constants a_0, a_r and β_r to be determined. The functions $s(\zeta)$ and $t(\zeta)$ satisfy the ODEs given below

$$s^{'}(\zeta) = e s(\zeta) t(\zeta), \tag{9}$$

$$t^{'}(\zeta) = Y + e t^2(\zeta) - m s(\zeta), \qquad e = \pm 1, \tag{10}$$

  where

$$t^2(\zeta) = -e\left[Y - 2ms(\zeta) + \frac{m^2 + i}{Y} s^2(\zeta)\right], \tag{11}$$

  where the constants $Y > 0$, $i = \pm 1$ and $m \neq$.

  Equations (9-10) give the solutions as below.

  Family 1:

  When $e = -1$, $i = -1$, $Y > 0$,

$$s_1(\zeta) = \frac{Y \operatorname{sech}(\sqrt{Y}\zeta)}{m \operatorname{sech}(\sqrt{Y}\zeta) + 1},$$

$$t_1(\zeta) = \frac{\sqrt{Y} \tanh(\sqrt{Y}\zeta)}{m \operatorname{sech}(\sqrt{Y}\zeta) + 1}$$

  Family 2: For $e = -1$, $i = 1$, $Y > 0$,

$$s_2(\zeta) = \frac{Y \operatorname{csch}(\sqrt{Y}\zeta)}{m \operatorname{csch}(\sqrt{Y}\zeta) + 1},$$

$$t_2(\zeta) = \frac{\sqrt{Y} \coth(\sqrt{Y}\zeta)}{m \operatorname{csch}(\sqrt{Y}\zeta) + 1}.$$

  Inserting Eq. (8) into ODE. Comparing coefficients of similar exponents of $s^a(\zeta)t^b(\zeta)$ equal to zero. Homogenous system of equations are achieved.

  The constants a_0, a_r, β_r are evaluated, after solving the achieved system of equations. Substitution of a_0, a_r and β_r into Eq. (8) provide the required exact solutions.

- **Technique III: $\tan\big(\phi(\zeta)/2\big)$-expansion approach**

  According to $\tan\big(\phi(\zeta)/2\big)$-expansion approach [24], the transformed ODE possess the solution as

$$V(\zeta) = \sum_{r=0}^{m} a_r \left[y + \tan\left(\frac{\phi(\zeta)}{2}\right)\right]^r + \sum_{r=1}^{m} b_r \left[y + \tan\left(\frac{\phi(\zeta)}{2}\right)\right]^{-r}, \tag{12}$$

  where a_r and b_r are constants. $\phi = \phi(\zeta)$ satisfies the underneath ODE

$$\phi'(\zeta) = f\sin(\phi(\zeta)) + g\cos(\phi(\zeta)) + h. \tag{13}$$

  The above differential equation possesses families (family 1-17) of solutions as discussed in Ref. [24].

  Imbedding Eq. (12) in transformed ODE. Setting the coefficients of analogous exponents of $\tan\big(\phi(\zeta)/2\big)$ and $\cot\big(\phi(\zeta)/2\big)$ equal to zero. We gain simultaneous system of equations.

  The solution of these equations provide us the values of unknown constants a_r and b_r.

5. Extraction of solitons for Kaup-Boussinesq System via Unified Method

To solve the fractional systems given in Eq. (2) and Eq. (3), the following transformations

$$u(x,t) = U(\zeta),$$
$$v(x,t) = V(\zeta), \tag{14}$$

are used, where ζ represents the traveling wave variable and obeying the following definition for β-derivative and truncated M-fractional derivative.

For β-**derivative**, ζ is taken as

$$\zeta = \frac{1}{\alpha}\left(x + \frac{1}{\Gamma(\alpha)}\right)^{\alpha} - \frac{\sigma}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{15}$$

For **M-derivative**, ζ is taken as

$$\zeta = \frac{\Gamma(\lambda+1)}{\alpha}(x^{\alpha} - \sigma t^{\alpha}), \tag{16}$$

where σ be soliton's speed. Using Eq. (14) along with Eq. (15) and Eq. (16) on Eq. (2)-(3), the following ODEs are obtained as,

$$-\sigma U^{'} - V^{'''} - 2VU^{'} - 2UV^{'} = 0, \tag{17}$$
$$-\sigma V^{'} - U^{'} - 2VV^{'} = 0. \tag{18}$$

The above coupled system (17)-(18) becomes

$$U = -\sigma V - V^2. \tag{19}$$

Putting Eq. (19) into Eq. (17) and integrating once, gives

$$V^{''} - \sigma^2 V - 3\sigma V^2 - 2V^3 = 0. \tag{20}$$

The Eq. (20) has been solved in the following sections via three proposed analytical techniques to drive new soliton solutions for the proposed model.

6. Polynomial function solution

Balancing principle applied to Eq. (20) gives $n = k - 1$, for all $k \geq 2$.

Two cases, when $k = 2,\ \ d = 1$ and $k = 2,\ \ d = 2$ are discussed in the following subsections.

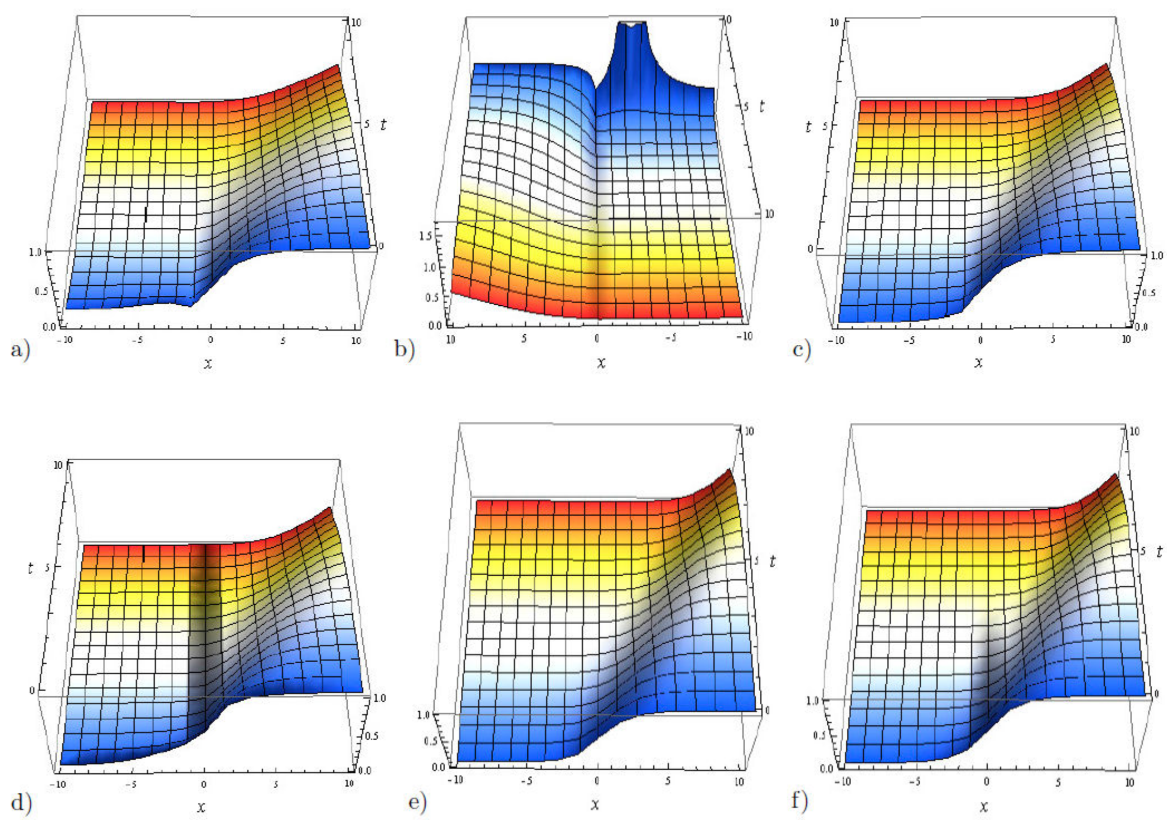

FIGURE 1. 3D-graphs of Eq. (24) for $\sigma = 1$, $C = 0$, $\lambda = 0.8$. a) presents the 3D-plot with β-derivative for $\alpha = 0.5$. b)shows the 3D-plot with M-derivative for $\alpha = 0.5$. c) presents the 3D-plot with β-derivative for $\alpha = 0.7$. d) presents the 3D-plot with M-derivative for $\alpha = 0.7$. e) shows the 3D-plot with β-derivative for $\alpha = 0.9$. f) presents the 3D-plot with M-derivative for $\alpha = 0.9$.

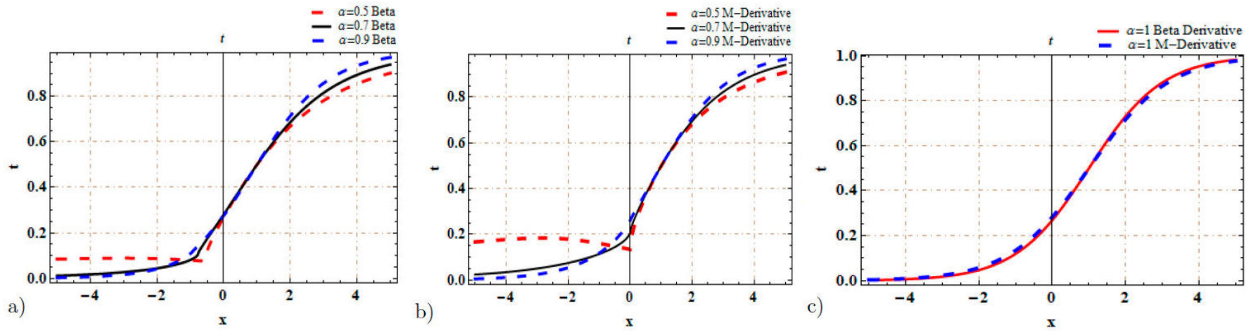

FIGURE 2. Line-plots for Eq. (24) with $\sigma = 1, C = 0, \lambda = 0.8$. a) shows the 2D-graphs of Eq. (24) with β-derivatives for $\alpha = 0.5,\ 0.7,\ 0.9$ at $t = 1$. b) shows the 2D-plots of Eq.(24) with M-derivatives for $\alpha = 0.5,\ 0.7,\ 0.9$ at $t = 1$. c) depicts the line-plots of Eq. (24) with β-derivatives and M-derivative for $\alpha = 1$ at $t = 1$.

6.1. Soliton solution when $k = 2,\ \ d = 1$

For $k = 2, d = 1$, the polynomial function solution of Eq. (20) has the form

$$V(\zeta) = a_0 + a_1\phi(\zeta), \tag{21}$$

$$\phi'(\zeta) = \sum_{i=0}^{2} b_i\phi^i(\zeta). \tag{22}$$

Using Eqs. (21)-(22) into Eq. (20) yields an algebraic equations in ϕ. The homogenous system of equations are achieved by setting the coefficients of $\phi(\zeta)$ equal to zero. The constants are obtained as

$$b_0 = \pm\frac{a_0(a_0+1)}{a_1},\ \ b_1 = \pm(2a_0+1),\ \ b_2 = \pm a_1. \tag{23}$$

Inserting Eq. (23) in Eqs. (21)-(22) and solving the auxiliary equation Eq. (22), the solution of Kaup-Boussinesq system is obtained as

$$v(x,t) = -\frac{1}{2}\sigma\left(1 + \tanh\left[\frac{1}{2}\sigma(\zeta + C)\right]\right), \tag{24}$$

$$u(x,t) = \frac{1}{4}\sigma^2\,\mathrm{sech}^2\Big(\frac{1}{2}\sigma[\zeta + C]\Big). \tag{25}$$

6.2. Optical solitary wave solution when $k = 2, d = 2$

For $k = 2, d = 2$, the optical solitary wave solutions of Eq. (20) has the following form

$$V(\zeta) = a_0 + a_1\phi(\zeta), \tag{26}$$

$$\phi'(\zeta) = \phi(\zeta)\sqrt{b_0 + b_1\phi(\zeta) + b_2\phi^2(\zeta)}. \tag{27}$$

Using Eqs. (26)-(27) into Eq. (20) yields an algebraic equations in $\phi(\zeta)$. The homogenous system of equations are achieved by setting the coefficients of $\phi(\zeta)$ equal to zero. The following values of unknown constants are given below

$$a_0 = -\sigma,\ \ a_1 = -\frac{b_1}{2\sigma},\ \ b_0 = \sigma^2,\ \ b_2 = \frac{b_1^2}{4\sigma^2}. \tag{28}$$

Inserting Eq. (28) in Eq. (26)-(27) and solving the auxiliary equation Eq. (27), the solution of Kaup-Boussinesq system is obtained as

$$v(x,t) = \frac{1}{-1 + 2b_1(\cosh[\sigma\zeta] + \sinh[\sigma\zeta])}, \tag{29}$$

$$u(x,t) = -\frac{2b_1\sigma^2(\cosh[\sigma\zeta] + \sinh[\sigma\zeta])}{(1 - 2b_1(\cosh[\sigma\zeta] + \sinh[\sigma\zeta]))^2}. \tag{30}$$

6.2.1. *Optical elliptic wave solution*

For $k = 2,\ d = 2$, the optical elliptic wave solution of Eq. (20) has been obtained as,

$$V(\zeta) = a_0 + a_1\phi(\zeta), \tag{31}$$

$$\phi'(\tau) = \sqrt{b_0 + b_2\phi^2(\tau) + b_4\phi^4(\tau)}. \tag{32}$$

Using Eq. (31)-(32) into Eq. (20) yields an algebraic equations in $\phi(\zeta)$. The homogenous system of equations are achieved by setting the coefficients of $\phi(\zeta)$ equal to zero. The following values of unknown constants are given below

$$a_1 = \pm\sqrt{b_4},\quad a_0 = -\frac{\sigma}{2},\quad \sigma^2 = -2b_2. \tag{33}$$

Inserting Eq. (33) in Eq. (31)-(32) and solving the auxiliary equation Eq. (32), the solution of Kaup-Boussinesq system is obtained as

$$v(x,t) = -\frac{\sigma}{2} \pm \sqrt{b_4}\phi(\zeta), \tag{34}$$

It is to mentioned here that $b_i, i = 0, 2, 4$ are arbitrary constants and by taking particular values of b_i, different Jacobi elliptic functions solutions have been obtained.

Upon taking $b_0 = -(1 - m_1^2)^2/4$, $b_2 = (1 + m_1^2)/2$ and $b_4 = -1/4$, then

$$\phi(\zeta) = m_1 cn(\zeta, m_1) + dn(\zeta, m_1).$$

Eq. (34) takes the following form

$$v(x,t) = -\frac{\sigma}{2} \pm \sqrt{b_4}\left(m_1 cn(\zeta, m_1) + dn(\zeta, m_1)\right), \tag{35}$$

$$u(x,t) = \frac{\sigma^2}{2} \mp \sigma\sqrt{b_4}\left(m_1 cn(\zeta, m_1) + dn(\zeta, m_1)\right) - \left(-\frac{\sigma}{2} \pm \sqrt{b_4}\left(m_1 cn(\zeta, m_1) + dn(\zeta, m_1)\right)\right)^2, \tag{36}$$

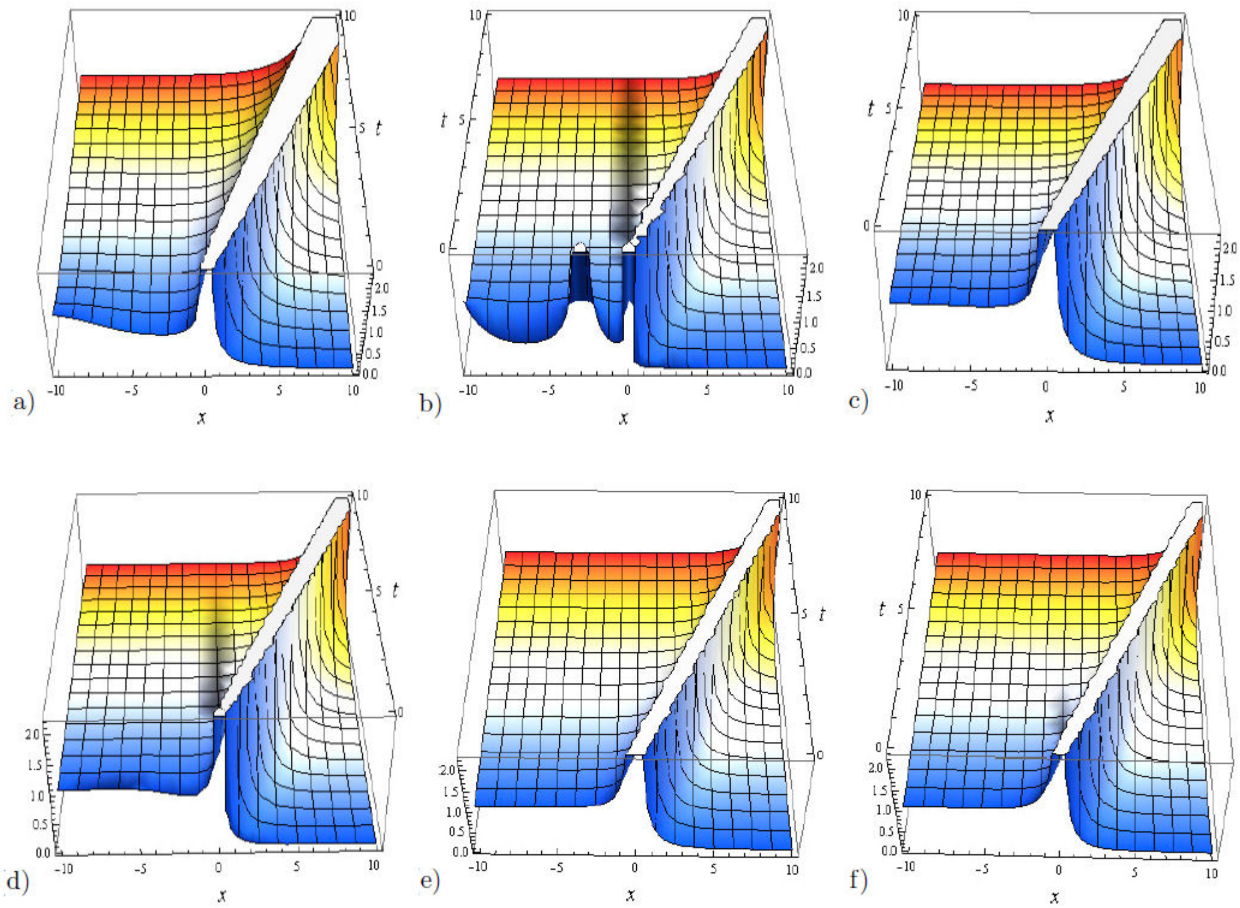

FIGURE 3. 3D-plots of Eq. (29) for $\sigma = 1$, $b_1 = 1$, $\lambda = 0.8$. a) shows the surface plot with β-derivative for $\alpha = 0.5$. b) shows the 3D-plot with M-derivative for $\alpha = 0.5$. c) shows the 3D-plot with β-derivative for $\alpha = 0.7$. d) depicts the surface plot with M-derivative for $\alpha = 0.7$. e) shows the 3D-plot with β-derivative for $\alpha = 0.9$. f) shows the 3D-plot with M-derivative for $\alpha = 0.9$.

where $0 < m_1 < 1$ is the modulus of the Jacobi elliptic functions.

7. Rational function solution

For finding the rational function solution of Kaup-Boussinesq system, the solution of Eq. (20) is taken as

$$V(\zeta) = \frac{\sum_{i=0}^{n} A_i \phi^i(\zeta)}{\sum_{i=0}^{r} B_i \phi^i(\zeta)}, \quad n \geq r, \tag{37}$$

$$(\phi'(\zeta))^\sigma = \sum_{i=0}^{\sigma k} b_i \phi^i(\zeta), \tag{38}$$

where unknown constants A_i, B_i and b_i are to be determined. The balance principle to Eq. (20), grants $k = 1$ and relation $n = r$, that allows free choice of n.

In the following subsections two possibilities depending on $k = 1, \;\; d = 2$ are discussed.

7.1. Optical soliton rational solutions

For the first case, the Eq. (37)-Eq. (38) can be converted into

$$V(\zeta) = \frac{A_0 + A_1\phi(\tau)}{B_0 + B_1\phi(\tau)}, \tag{39}$$

$$\phi'(\tau) = \sqrt{b_0 + b_1\phi(\tau) + b_2\phi^2(\tau)}. \tag{40}$$

Using Eq. (39)-(40) into Eq. (20) yields an algebraic equations in ϕ. For finding the values of $p_0, p_1, q_0, q_1, b_0, b_1, b_2$ and σ, equating the coefficients of ϕ equal to zero, homogenous system of equations are achieved. After solving the system of equations, following values of unknown constants are obtained as

$$A_0 = -\frac{b_1 B_1}{2\sigma}, \;\; A_1 = -B_1\sigma, \;\; b_2 = \sigma^2, \;\; b_0 = \frac{b_1^2}{4\sigma^2}. \tag{41}$$

Inserting Eq. (41) in Eq. (39)-(40) and solving the auxiliary equation Eq. (40), the solution of Kaup-Boussinesq system is obtained as

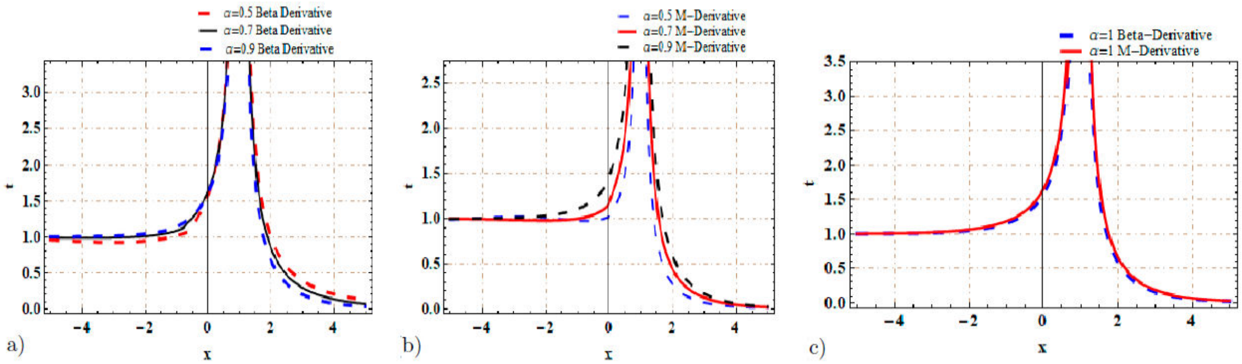

FIGURE 4. 2D-plots for Eq. (29) with $\sigma = 1,\ b_1 = 1,\ \lambda = 0.8$. a) shows the 2D-plots of Eq. (29) with β-derivatives for $\alpha = 0.5,\ 0.7,\ 0.9$ at $t = 1$. b) depicts the line-plots of Eq. (29) with M-derivatives for $\alpha = 0.5,\ 0.7,\ 0.9$ at $t = 1$. c) shows the 2D-plots of Eq. (29) with β-derivatives and M-derivative for $\alpha = 1$ at $t = 1$.

$$v(x,t) = \frac{B_1\sigma\left(\cosh[\sigma\zeta] + \sinh[\sigma\zeta]\right)}{2b_1B_1 - 4B_0\sigma^2 - \left(\cosh[\sigma\zeta] + \sinh[\sigma\zeta]\right)B_1}, \tag{42}$$

$$u(x,t) = \frac{2B_1\sigma^2\left(\cosh[\sigma\zeta]+\sinh[\sigma\zeta]\right)\left(-b_1B_1+2B_0\sigma^2\right)}{\left(-2b_1B_1+4B_0\sigma^2+(\cosh[\sigma\zeta]+\sinh[\sigma\zeta])B_1\right)^2}. \tag{43}$$

7.2. Optical periodic rational solution

$$V(\zeta) = \frac{A_0 + A_1\phi(\zeta)}{B_0 + B_1\phi(\zeta)}, \tag{44}$$

$$\phi'(\zeta) = \sqrt{b_0^2 - b_2^2\phi^2(\zeta)}. \tag{45}$$

Using Eqs. (44)-(45) into Eq. (20) yields an algebraic equations in $\phi(\zeta)$. For finding the values of $A_0, A_1, B_0, B_1, b_0, b_2$ and σ, equating the coefficients of $\phi(\zeta)$ to zero, simultaneous equations is procured. The following values of unknown constants are given as

$$A_0 = -b_0B_1, \qquad A_1 = -\frac{B_1\sigma}{2}, \qquad b_2 = -\frac{1}{\sqrt{2}}, \qquad B_0 = 0. \tag{46}$$

Inserting Eq. (46) in Eqs. (44)-(45) and solving the auxiliary equation (45), the solution of Kaup-Boussinesq system is obtained as

$$v(x,t) = -\frac{1}{2}\sigma\left(1 + \sqrt{2}\csc[\frac{\sigma(\zeta + C)}{\sqrt{2}}]\right), \tag{47}$$

$$u(x,t) = \frac{1}{4}\sigma^2\left(1 - 2\csc^2[\frac{\sigma(\zeta + C)}{\sqrt{2}}]\right). \tag{48}$$

In all the above solutions obtained by Unified approach, ζ defined by Eq. (15) for β-derivative and Eq. (16) for truncated M-fractional derivative. C be the constant of integration.

8. Extraction of soliton solutions via GPRE

Taking the homogeneous balance in Eq. (20), we obtain $u = 1$. Equation (8) becomes

$$V(\zeta) = a_0 + a_1s(\zeta) + \beta_1t(\zeta), \tag{49}$$

where a_0, a_1 and β_1 are unknowns constants. For $e = -1$, GPRE extracts, following sets of solutions.

SET 1

$$a_0 = \pm\frac{\sigma}{2}, \qquad a_1 = \mp\frac{\sqrt{m^2 + i}}{2\sigma}, \qquad \beta_1 = -\frac{1}{2}, \qquad Y = \sigma^2.$$

Taking **Family 1**, the achieved solitons for **SET 1** are given below

$$v(x,t) = \pm\frac{\sigma}{2} \mp \frac{\sigma\sqrt{m^2-1}\,\mathrm{sech}(\sigma\zeta)}{2(m\,\mathrm{sech}(\sigma\zeta)+1)} - \frac{\sigma\tanh(\sigma\zeta)}{2(m\,\mathrm{sech}(\sigma\zeta)+1)}, \tag{50}$$

$$u(x,t) = -\sigma\left(\pm\frac{\sigma}{2} \mp \frac{\sigma\sqrt{m^2-1}\,\mathrm{sech}(\sigma\zeta)}{2(m\,\mathrm{sech}(\sigma\zeta)+1)} - \frac{\sigma\tanh(\sigma\zeta)}{2(m\,\mathrm{sech}(\sigma\zeta)+1)}\right) - \left(\pm\frac{\sigma}{2} \mp \frac{\sigma\sqrt{m^2-1}\,\mathrm{sech}(\sigma\zeta)}{2(m\,\mathrm{sech}(\sigma\zeta)+1)} - \frac{\sigma\tanh(\sigma\zeta)}{2(m\,\mathrm{sech}(\sigma\zeta)+1)}\right)^2. \tag{51}$$

The above solitons exist if $(m^2-1)>0$.

Taking **Family 2**, the attained solitons for **SET 1** are given below

$$v(x,t) = \pm\frac{\sigma}{2} \mp \frac{\sigma\sqrt{m^2+1}\,\mathrm{csch}(\sigma\zeta)}{2(m\,\mathrm{csch}(\sigma\zeta)+1)} - \frac{\sigma\coth(\sigma\zeta)}{2(m\,\mathrm{csch}(\sigma\zeta)+1)}, \tag{52}$$

$$u(x,t) = -\sigma\left(\pm\frac{\sigma}{2} \mp \frac{\sigma\sqrt{m^2+1}\,\mathrm{csch}(\sigma\zeta)}{2(m\,\mathrm{csch}(\sigma\zeta)+1)} - \frac{\sigma\coth(\sigma\zeta)}{2(m\,\mathrm{csch}(\sigma\zeta)+1)}\right) - \left(\pm\frac{\sigma}{2} \mp \frac{\sigma\sqrt{m^2+1}\,\mathrm{csch}(\sigma\zeta)}{2(m\,\mathrm{csch}(\sigma\zeta)+1)} - \frac{\sigma\coth(\sigma\zeta)}{2(m\,\mathrm{csch}(\sigma\zeta)+1)}\right)^2. \tag{53}$$

The above soliton solutions exist if $(m^2+1)>0$.

9. Extraction of solitons via improved $\tan\left(\phi(\zeta)/2\right)$-expansion approach

Using the improved $\tan\left(\Phi(\zeta)/2\right)$-expansion approach, the solution for $y=0$ of Eq. (12) becomes

$$D(\mu) = a_0 + a_1\left[\tan\left(\frac{\phi(\mu)}{2}\right)\right] + b_1\left[\tan\left(\frac{\phi(\mu)}{2}\right)\right]^{-1}, \tag{54}$$

where $a_0,\ a_1$ and b_1 are to be calculated. Inserting Eq. (54) and its derivatives into Eq. (20 provides the ensuing solution sets as

SET 1

$$a_0 = -\frac{\sigma}{2}, \quad b_1 = \frac{\sigma^2}{16a_1}, \quad f=0, \quad g = -a_1 - \frac{\sigma^2}{16a_1}, \quad h = a_1 - \frac{\sigma^2}{16a_1}. \tag{55}$$

SET 2

$$a_0 = \frac{f-\sigma}{2}, \quad a_1 = 0, \quad g = \sqrt{-f^2+h^2+\sigma^2}, \quad b_1 = \frac{h+\sqrt{-f^2+h^2+\sigma^2}}{2}.$$

SET 3

$$a_0 = \frac{f-\sigma}{2}, \quad a_1 = \frac{1}{2}\left(h+\sqrt{-f^2+h^2+\sigma^2}\right), \quad g = -\sqrt{-f^2+h^2+\sigma^2}, \quad b_1 = 0.$$

Using Eq. (54) and **SET 1**, the following solutions are extracted from family 2 of the proposed method [24].

$$v_2(x,t) = -\frac{\sigma}{2} - \frac{\sigma}{4}\left(\tanh\left[\frac{\sigma}{4}\hat{\zeta}\right] + \coth\left[\frac{\sigma}{4}\hat{\zeta}\right]\right), \tag{56}$$

$$u_2(x,t) = -\sigma\left(-\frac{\sigma}{2} - \frac{\sigma}{4}\left[\tanh\left\{\frac{\sigma}{4}\hat{\zeta}\right\} + \coth\left\{\frac{\sigma}{4}\hat{\zeta}\right\}\right]\right) - \left(-\frac{\sigma}{2} - \frac{\sigma}{4}\left[\tanh\left\{\frac{\sigma}{4}\hat{\zeta}\right\} + \coth\left\{\frac{\sigma}{4}\hat{\zeta}\right\}\right]\right)^2. \tag{57}$$

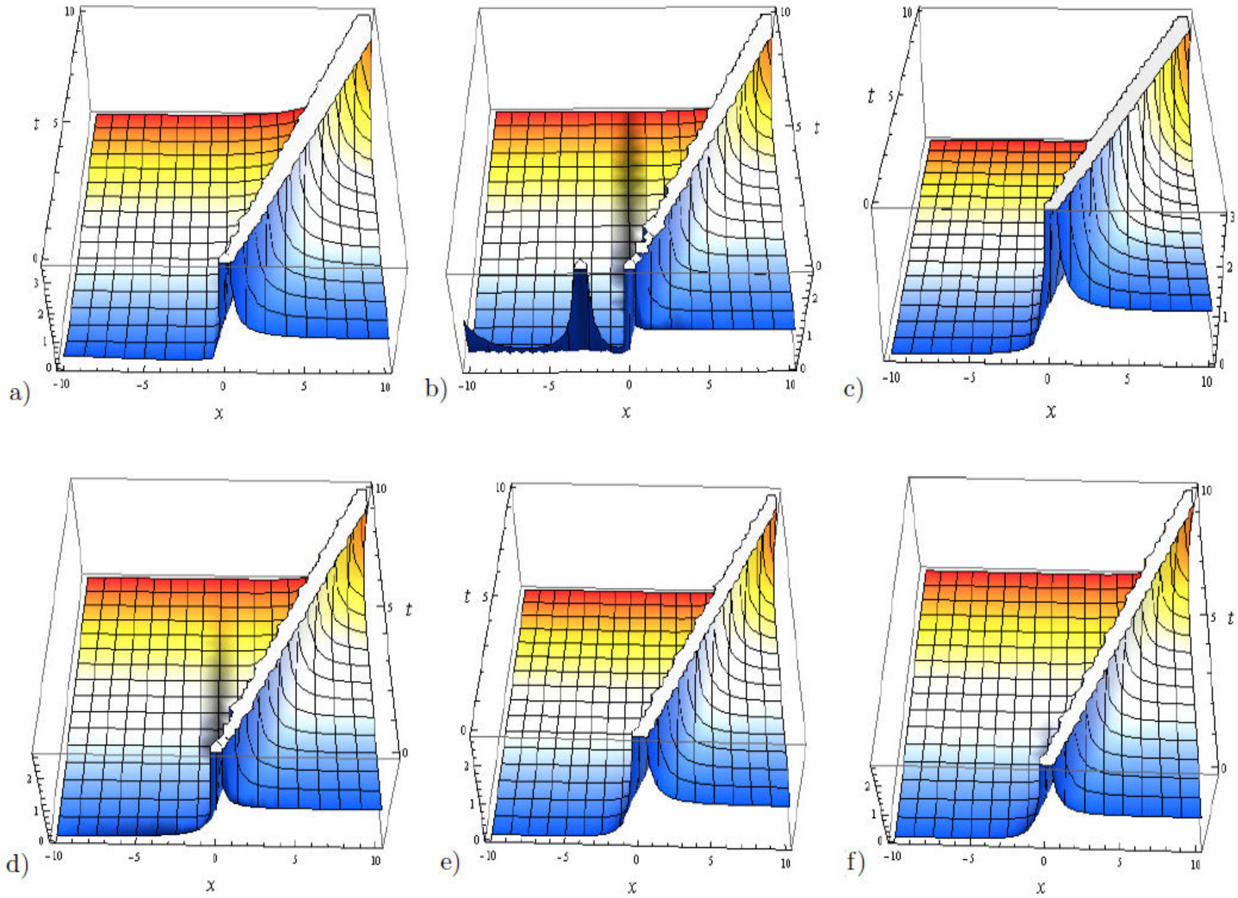

FIGURE 5. 3D-plots of Eq. (55) with $\sigma = 1$, $\lambda = 0.8$. a) exhibits the 3D-plot with β-derivative for $\alpha = 0.5$. b) exhibits the 3D-plot with M-derivative for $\alpha = 0.5$. c) exhibits the 3D-plot with β-derivative for $\alpha = 0.7$. d) exhibits the 3D-plot with M-derivative for $\alpha = 0.7$. e) exhibits the 3D-plot with β-derivative for $\alpha = 0.9$. f) exhibits the 3D-plot with M-derivative for $\alpha = 0.9$.

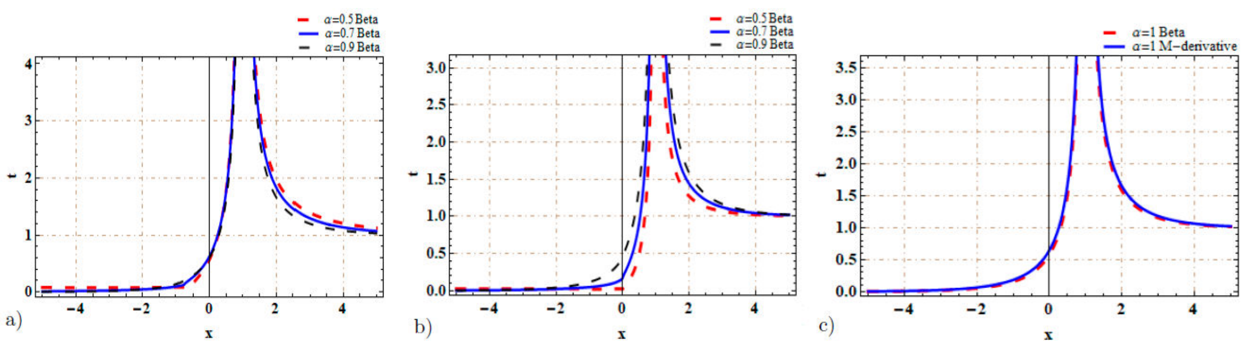

FIGURE 6. 2D-plots for Eq. (55) with $\sigma = 1$, $b_1 = 1$, $\lambda = 0.8$. a) shows the 2D-plots of Eq. (55) with β-derivatives for $\alpha = 0.5,\ 0.7,\ 0.9$ at $t = 1$. b) shows the 2D-plots of Eq. (55) with M-derivatives for $\alpha = 0.5,\ 0.7,\ 0.9$ at $t = 1$. c) shows the 2D-plots of Eq. (55) with β-derivatives and M-derivative for $\alpha = 1$ at $t = 1$.

Using Eq. (54) and **SET 2**, the following solutions are extracted from family 2, 5 and 6 of the proposed method [24].

$$v_2(x,t) = \frac{f-\sigma}{2} + \frac{\sigma^2 - f^2}{2\left(f + \sigma \tanh\left[\frac{\sigma}{2}\hat{\zeta}\right]\right)}, \tag{58}$$

$$u_2(x,t) = -\sigma \left(\frac{f-\sigma}{2} + \frac{\sigma^2 - f^2}{2\left[f + \sigma \tanh\left\{\frac{\sigma}{2}\hat{\zeta}\right\}\right]} \right) - \left(\frac{f-\sigma}{2} + \frac{\sigma^2 - f^2}{2\left[f + \sigma \tanh\left\{\frac{\sigma}{2}\hat{\zeta}\right\}\right]} \right)^2, \tag{59}$$

$$v_5(x,t) = \frac{1}{2}\sigma \coth\left(\frac{\sigma\hat{\zeta}}{2}\right), \tag{60}$$

$$u_5(x,t) = -\sigma \left(\frac{1}{2}\sigma \coth\left(\frac{\sigma\hat{\zeta}}{2}\right)\right) - \left(\frac{1}{2}\sigma \coth\left(\frac{\sigma\hat{\zeta}}{2}\right)\right)^2, \tag{61}$$

$$v_6(x,t) = -\frac{\sigma}{2}\left(1 - \cot\left\{\frac{1}{2}\arctan\left(\frac{e^{2\sigma\hat{\zeta}}-1}{e^{2\sigma\hat{\zeta}}+1}, \frac{2e^{\sigma\hat{\zeta}}}{e^{2\sigma\hat{\zeta}}+1}\right)\right\}\right), \tag{62}$$

$$u_6(x,t) = \frac{\sigma^2}{2}\left(1 - \cot\left\{\frac{1}{2}\arctan\left(\frac{e^{2\sigma\hat{\zeta}}-1}{e^{2\sigma\hat{\zeta}}+1}, \frac{2e^{\sigma\hat{\zeta}}}{e^{2\sigma\hat{\zeta}}+1}\right)\right\}\right) - \frac{\sigma^2}{4}\left(1 - \cot\left\{\frac{1}{2}\arctan\left(\frac{e^{2\sigma\hat{\zeta}}-1}{e^{2\sigma\hat{\zeta}}+1}, \frac{2e^{\sigma\hat{\zeta}}}{e^{2\sigma\hat{\zeta}}+1}\right)\right\}\right)^2. \tag{63}$$

Using Eq. (54) and **SET 3**, the following solutions are extracted from family 2, 5 and 6 of the proposed method [24].

$$v_2(x,t) = \frac{f-\sigma}{2} + \frac{1}{2}\left(-f - \sigma\tanh\left(\frac{\sigma}{2}\hat{\zeta}\right)\right), \tag{64}$$

$$u_2(x,t) = \frac{\sigma^2}{2 + 2\cosh\left(\sigma\hat{\zeta}\right)}, \tag{65}$$

$$v_5(x,t) = -\frac{\sigma}{2}\left(1 + \tanh\left(\frac{\sigma}{2}\hat{\zeta}\right)\right), \tag{66}$$

$$u_5(x,t) = \frac{\sigma^2}{2 + 2\cosh\left(\sigma\hat{\zeta}\right)}, \tag{67}$$

$$v_6(x,t) = -\frac{\sigma}{2}\left(1 - \tan\left\{\frac{1}{2}\arctan\left(\frac{e^{-2\sigma\hat{\mu}}-1}{e^{-2\sigma\hat{\zeta}}+1}, \frac{2e^{-\sigma\hat{\zeta}}}{e^{-2\sigma\hat{\zeta}}+1}\right)\right\}\right), \tag{68}$$

$$u_6(x,t) = \frac{\sigma^2}{2}\left(1 - \tan\left\{\frac{1}{2}\arctan\left(\frac{e^{-2\sigma\hat{\zeta}}-1}{e^{-2\sigma\hat{\zeta}}+1}, \frac{2e^{-\sigma\hat{\zeta}}}{e^{-2\sigma\hat{\zeta}}+1}\right)\right\}\right) - \frac{\sigma^2}{4}\left(1 - \tan\left\{\frac{1}{2}\arctan\left(\frac{e^{-2\sigma\hat{\zeta}}-1}{e^{-2\sigma\hat{\zeta}}+1}, \frac{2e^{-\sigma\hat{\zeta}}}{e^{-2\sigma\hat{\zeta}}+1}\right)\right\}\right)^2, \tag{69}$$

where $\hat{\zeta} = \zeta + C$ defined by Eq. (15) for β-derivative, Eq. (16) for M-truncated derivative. C be the constant of integration.

10. Comparative investigation with β and M-Truncated fractional derivatives

The two-dimensional and three-dimensional plots of Eq. (24), Eq. (29) and Eq. (55) are presented in this paper to illustrate the dynamics of obtained solutions. Figures 1 and 2 show the 3D-plots and 2D-plots of Eq. (24) for both definitions of fractional derivatives with distinct values of fractional parameter α. The 2D-plots of Eq. (24) are presented in Fig. 2 for $\alpha = 0.5,\ 0.7\ 0.9,\ 1$ by taking independent variable $t = 1$ and $-5 \leq x \leq 5$. It has also been observed that upon taking $\alpha = 1$, both the fractional derivative overlap. Moreover, Figs. 3 and 4 show the 3D-plots and 2D-plots of Eq. (29) for both definitions of fractional derivatives with distinct values of fractional parameter α. The 2D-plots of Eq. (29) are presented in Fig. 4 for

$\alpha = 0.5,\ 0.7\ 0.9,\ 1$ by taking independent variable $t = 1$ and $-5 \leq x \leq 5$. It has also been observed that upon taking $\alpha = 1$, both the fractional derivative overlap. Figures 5 and 6 show the 3D-plots and 2D-plots of Eq. (55) for β derivative and M-derivative taking $\alpha = 0.5,\ 0.7,\ 0.9$. The 2D-plots of Eq. (24) are presented in Fig. 2 for $\alpha = 0.5,\ 0.7\ 0.9,\ 1$ by taking independent variable $t = 1$ and $-5 \leq x \leq 5$. It has also been observed that upon taking $\alpha = 1$, both the fractional derivative overlap.

11. Results and discussion

The fractional KB model is investigated in this article via three integrating schemes. The β-derivative and truncated M-fractional derivative are applied for investigating fractional nature of FKB model. Three most popular and novel approaches namely unified approach, GPRE method and improved $\tan\left(\phi(\zeta)/2\right)$-expansion method are used for obtaining different types of solitons such as bright, dark, singular soliton, combo soliton and periodic solutions. The graphical representation of few obtained solutions are also provided in this article. In this paper, two fractional derivatives are employed for discussing the fractional effects of the obtained solutions of the governing model. It has been noticed for assigning various values of fractional parameter $\alpha, 0 < \alpha < 1$, the β- derivative approaches the classical derivative (for $\alpha = 1$) faster than M-truncated derivative.

12. Conclusion

In this article, we have developed soliton solutions of fractional Kaup-Boussinesq system that depict the propagation of long waves at the surface of a perfect fluid. The full spectrum of soliton solution includes dark, bright, singular soliton, combo soliton and periodic solutions. This has been obtained by applying Technique I, Technique II and Technique III. These three techniques are efficient integration tools for finding exact solitary wave solutions of various NLPDEs arising in nonlinear sciences. The suitable choice of parametric values allow us to discuss the fractional behavior of the attained solutions. 2D and 3D graphs illustrate the physical importance of procured results. Moreover, the comparison between β-derivative and M-derivative is provided for different fractional parametric values. On comparing our obtained results with [36–39], it has been observed that our obtained solutions for the proposed model are new. Only the solutions found in Eq. (24) and Eq. (25) are similar with the solutions obtained in [38, 39]. The acquired solutions may give a decent enhancement to theory of water waves. All the three techniques are very effective in finding the full spectrum of solitary wave solutions which possess a significant part in many real-world problems occurring in mathematical physics and engineering.

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