Some novel different solutions for Boussinesq-type models including bright, singular, and dark soliton ones

M.T. Darvishi[a,b], M. Najafi[a], H. Rezazadeh[c], S. Rezapour[d,e,*], and M. Inc[f,*]

[a] *Department of Mathematics, Razi University, Kermanshah 67149, Iran.*

[b] *Department of Mathematical Sciences, Kent State University, Kent, OH 44242, USA.*

[c] *Faculty of Engineering Technology, Amol University of Special Modern Technologies, Amol, Iran.*

[d] *Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran.*

[e] *Department of Medical Research, China Medical University, Taichung, Taiwan.*

[f] *Department of Mathematics, Firat University, 23119 Elazig, Turkiye.*

[*] *e-mails: rezapourshahram@yahoo.ca; minc@firat.edu.tr*

Received 13 March 2022; accepted 29 May 2022

Some new different kinds of one-soliton solutions for various forms of Boussinesq-type equations are presented in this paper to describe the nonlinear wave phenomena in coastal and ocean areas such as tsunami waves. These one-soliton solutions include bright, dark, and singular ones. The property of each solution in coastal and ocean engineering is explained.

Keywords: Boussinesq-type equation; singular-solution; dark-soliton solution; bright-soliton solution.

DOI: https://doi.org/10.31349/RevMexFis.70.031306

1. Introduction

Among various mathematical models, nonlinear evolution equations (NLEEs) are very useful. NLEEs can model different kinds of scientific phenomena in different fields including physics, chemistry, marine, coastal and ocean engineering, fluid dynamics, and plasma physics. NLEEs can be seen in "the system of equations for the ion sound wave under the action of the ponderomotive force due to high-frequencies field". Also, the "Langmuir wave" is considered as an important type of NLEEs. As a result, the study of NLEEs has attracted lots of researchers' interests. Finding exact solutions of NLEEs is very important in nonlinear phenomena. There are different categories for solutions of NLEEs such as periodic-, traveling-wave-, cross-kink-wave, and soliton- solutions. Among these solutions, solitons have received much more attention in applied sciences than others, and this is because of their special properties. As a matter of fact, solitons have been appeared in different systems in nature such as shallow water waves, plasma, optical waves, matter-waves in Bose-Einstein condensates, marine, ocean and coastal waves, and ultra-short pulses in nonlinear optics. Solitons are known for keeping their shape and width unaltered even after collision with other similar solitons. There are different kinds of solitons, but we can categorize them into three original types, namely bright, dark, and singular solitons. A bright-soliton is a pulse on a zero intensity background while a dark-soliton appears as an intensity dip in an infinitely extended constant background. Further any explode-decay mode soliton is a singular soliton.

One of the attractive nonlinear evolution equations which is described the movement of water by small-amplitude and long-wave is the Boussinesq equation. Besides, Boussinesq type equations predict wave transformations in coastal fields for time dependent waves as well. For the first time in 1967, a set of equations for variable water depth is presented by Peregrine [1]. These are effective equations for shallow water. As a matter of fact, they are the standard Boussinesq equations that are used in ocean and coastal sciences. These are effective equations to investigate shallow waters. As a matter of fact, they are the standard Boussinesq equations which are used in ocean and coastal engineering. Mathematical modelling of tsunami and tidal oscillations can be done by these equations. Further, these equations are very applicable in study of another subjects in different areas such as: the dynamics of thin slimy layers which have free surfaces, nonlinear strings, shape memory in making metal compositions, paired electrical circuits, continuum limit of lattice dynamics, and wave-propagation in elastic rods (see [2–4]).

Consider the following standard nonlinear Boussinesq equation [5, 6]:

$$-u_{xx}-\left(u^2\right)_{xx}-u_{xxxx}+u_{tt}=0.$$

It's well recognized that the investigation of propagation of waves on the surface of water is a dynamic research field in the nonlinear science. The nonlinear Boussinesq equation describes the physical phenomena in study of the dynamics for thin layers that have viscosity and free surface [5, 6]. This equation and its related type equations are also important to describe some another physical phenomena like the nonlinear lattice-waves, acoustic-waves, ion sound-waves in a plasma, the shape memory in metal compositions, the propagation of waves in elastic rods, the paired electrical circuits, and vibrations in a nonlinear string [7–9]. Moreover, it is useful for the realistic applications in the percolation of water in porous media of a horizontal layer of material, large scale atmospheric and jet stream [**?**, 10–13]. One may encounter with shallow water waves' dynamics in different fields, for example in sea and seashore, in rivers and lakes or the other similar regions [11].

The Boussinesq equation which makes the surface gravity waves is a useful model for simulating wave propagation in the long waves in ocean and seashore regions. The numerical computation is also accomplished by the Boussinesq models in the areas of wave propagation in long waves in shallow water. In ocean and coastal engineering, oceanographers and seashore engineers used the models for simulating of surface water waves in shallow seas and seaports, dune modelling, ocean basin-scale tsunami propagation, wave overtopping and inundation, and near shore wave processes [15-19].

In the last two decades, different kinds of Boussinesq equations were expanded and studied in scientific researches. In the present manuscript, we employ the ansätz method for obtaining closed form soliton solutions of the following four variants of Boussinesq equation:

$$-u_{xx} - (6u^2 u_x + u_{xxx})_x + u_{tt} = 0, \tag{1}$$

$$-u_{xx} - (6u^2 u_x + u_{xtt})_x + u_{tt} = 0, \tag{2}$$

$$-u_{xt} - (6u^2 u_x + u_{xxt})_x + u_{tt} = 0, \tag{3}$$

$$-(6u^2 u_x + u_{xxx})_x + u_{tt} = 0. \tag{4}$$

Indeed, we derive some new families of analytical solutions for the Boussinesq-type models Eqs. (1)–(4). It must be noted that these equations are presented in Ref. [20] for the first time. In addition, Eqs. (1)-(4) are non-integrable ones. Further Eqs. (1) and (4) contain spatial dispersion. In comparison, Eqs. (2) and (3) have spatial and temporal dispersion whereas the second and the third ones contain spatial-temporal dispersion. Besides, in Eq. (1) there are the second order dissipative term u_{xx} and the fourth order spatial term u_{xxxx}. We can have similar descriptions for terms of other equations, for example, the fourth order derivative term u_{xxxx} in Eq. (1) has changed to the fourth order derivative term u_{xxtt} in Eq. (2).

These equations were studied to find their singular- and soliton-solutions, the Hirota's direct method obtained some solutions of these kinds as well, and other methods found more solutions for Eqs. (1)–(4) with different physical bases (see [20-22]).

The structure of this study is organized as follows: The analytical solutions with graphical representations of all solutions are presented in Sec. 2. In Sec. 3, we addressed the physical explanation for the behavior of all reported soliton solutions. The conclusion is given in the end.

2. Analytical solutions

In this present work, different kinds of solitary wave solutions are obtained for Eqs. (1)–(4). To do this, we consider three types of solutions with the general hyperbolic trigonometric forms given by

$$u(\nu) = A\,\mathrm{sech}^p(\nu),\;\; u(\nu) = A\,\tanh^p(\nu),\;\; \text{or}$$
$$u(\nu) = A\,\mathrm{csch}^p(\nu), \tag{5}$$

where $\nu = kx - ct$. By these, we obtain three different families of one-soliton solutions for Eqs. (1)-(4). These families are bright, dark, and singular soliton solutions.

2.1. The 1st model

Consider

$$-u_{xx} - (6\,u^2\,u_x + u_{xxx})_x + u_{tt} = 0. \tag{6}$$

If we apply the soliton wave ansätz $u(x,t) = u(\nu), \nu = k\,x - c\,t$ in Eq. (6), the following relation is obtained

$$(k^2 - c^2)\,u'' + k^2\,(\,6\,u^2\,u' + k^2\,u^{(3)})' = 0. \tag{7}$$

After that, if we integrate Eq. (7) two times and set the integration constants as zero, we have

$$(c^2 - k^2)\,u - k^2\,(2\,u^3 + k^2\,u'') = 0. \tag{8}$$

In Eq. (8) k and c are some constants. It is worthy mention that, our integrations are done with respect to variable ν.

2.1.1. Solutions for the 1st model, bright-soliton ones

To obtain solitary-wave ansätz for the bright-soliton solutions, we use the following assumption:

$$u(\nu) = A\,\mathrm{sech}^p(\nu). \tag{9}$$

The unknown parameter p can find in the process of finding of solutions of Eq. (8). Thus by setting the ansätz Eq. (9) into Eq. (8) we get

$$(k^2 - c^2)A\,\mathrm{sech}^p(\nu) + k^2\,(2\,A^3\mathrm{sech}^{3p}(\nu)$$
$$+\,k^2\,p(\,A\,p\,\mathrm{sech}^p(\nu) - A(p+1)\,\mathrm{sech}^{p+2}(\nu))) = 0. \tag{10}$$

Now, value of one is obtained for p after equating exponents $3\,p$ and $p+2$ in Eq. (10). Further, equating coefficients of functions $\mathrm{sech}(\nu)$ yields the following nonlinear system

$$2\,k^2(A^2 - k^2) = 0,$$
$$-c^2 + k^2 + k^4 = 0,$$

which has the following solution

$$A = k, \quad A = -k, \quad c = k\sqrt{k^2+1},$$
$$c = -k\sqrt{k^2+1}. \tag{11}$$

Using Eq. (11) yields bright-soliton solutions for Eq. (6) as:

$$u_{11}(x,t) = \pm\,k\;\mathrm{sech}(k(x \mp \sqrt{k^2+1}\,t)). \tag{12}$$

2.1.2. *Solutions for the 1st model, dark-soliton ones*

To obtain dark-soliton solutions we set

$$u(\nu) = A\tanh^p(\nu). \tag{13}$$

Similar to previous part, we obtain $p = 1$. So by putting the ansätz Eq. (13) in Eq. (8) for $p = 1$, we have:

$$\begin{aligned}&-2A\,k^2(A^2+k^2)\tanh^3(\nu)\\&+A(c^2-k^2+2k^4)\tanh(\nu)=0.\end{aligned}$$

Equating the coefficients of each pair of functions $\tanh(\nu)$ gives:

$$\begin{aligned}2k^2(A^2+k^2)&=0,\\-c^2+k^2-2k^4&=0,\end{aligned}$$

which has the following solutions

$$A = \pm Ik,\ \ c = \pm k\sqrt{1-2k^2}. \tag{14}$$

Therefore, the following dark-soliton solutions are obtained for Eq. (6) using the values of Eq. (14):

$$u_{12}(x,t) = \pm ik\tanh(k(x\mp\sqrt{1-2k^2}t)). \tag{15}$$

2.1.3. *Solutions for the 1st model, singular-soliton ones*

In order to obtain singular-soliton solutions, we assume that

$$u(\nu) = A\,\mathrm{csch}^p(\nu). \tag{16}$$

As we did before we obtain $p = 1$, by this value of p and setting ansätz Eq. (16) in Eq. (8) gives

$$A(-c^2+k^2+k^4)\cosh^2(\nu)+A(k^2(2A^2+k^2-1)+c^2)=0.$$

Similarly by equating the coefficients of each pair of functions $\cosh(\nu)$, the following system is obtained:

$$\begin{aligned}-c^2+k^2+k^4&=0,\\2\,k^2A^2+c^2-k^2+k^4&=0,\end{aligned}$$

that has the following solutions:

$$A = \pm\,k\,i,\ \ c = \pm k\sqrt{k^2+1}. \tag{17}$$

The values of Eq. (17) give the following solutions for Eq. (6):

$$u_{13}(x,t) = \pm ik\ \mathrm{csch}(k\,x\mp k\sqrt{k^2+1}\,t), \tag{18}$$

which are singular-soliton ones.

2.2. The 2nd model

Next, we investigate the following equation to find its soliton solutions:

$$-u_{xx}-(\,6\,u^2\,u_x+u_{xtt})_x+u_{tt}=0. \tag{19}$$

Here we use the wave ansätz $u(x,t) = u(\nu)$, $\nu = k\,x - c\,t$ in Eq. (19) that yields

$$(c^2-k^2)\,u''-k\,(\,6\,k\,u^2\,u'+k\,c^2\,u^{(iii)})'=0. \tag{20}$$

Besides, with respect to ν we integrate Eq. (20) two times and set zero for integrating constants. All of these yield

$$(c^2-k^2)\,u-2\,k^2\,u^3-k^2\,c^2\,u''=0. \tag{21}$$

In Eq. (21), k and c are some constants.

2.2.1. *Solutions for the 2nd model, bright-soliton ones*

By the assumption

$$u(\nu) = A\,\mathrm{sech}^p(\nu). \tag{22}$$

One may obtain the bright-soliton solutions. Obtaining of the value of parameter p is as same the process which is described in the previous section. Then we substitute the ansätz Eq. (22) in Eq. (21), this yields

$$\begin{aligned}&(c^2-k^2)A\,\mathrm{sech}^p(\nu)-2k^2\,A^3\,\mathrm{sech}^{3p}(\nu)\\&-k^2\,c^2\,p(A\,p\,\mathrm{sech}^p(\nu)-A(p+1)\,\mathrm{sech}^{p+2}(\nu)=0.\end{aligned} \tag{23}$$

To find parameter p, we equate the exponents $3p$ and $p+2$ in Eq. (23). Moreover, we equate the coefficients of each pair of functions $\mathrm{sech}(\nu)$. These result in $p = 1$ and the following system:

$$\begin{aligned}2k^2(A^2-c^2)&=0,\\-c^2+k^2+k^2c^2&=0,\end{aligned}$$

which its solutions are

$$A = \pm\frac{k}{\sqrt{1-k^2}},\ \ c = \pm\,\frac{k}{\sqrt{1-k^2}}. \tag{24}$$

By the values of Eq. (24), we obtain the following solutions for Eq. (19):

$$u_{21}(x,t) = \pm\,\frac{k}{\sqrt{1-k^2}}\,\mathrm{sech}\left(kx\mp\frac{k}{\sqrt{1-k^2}}t\right). \tag{25}$$

Solutions (25) are bright-soliton ones.

2.2.2. *Solutions for the 2nd model, dark-soliton ones*

The dark-soliton solutions express as:

$$u(\nu) = A \tanh^p(\nu).$$

Similar to previous cases, we obtain $p = 1$. Now, setting the ansätz Eq. (13) in Eq. (21) with $p = 1$ gives

$$A(2k^2(A^2 + c^2)) \tanh^3(\nu) + A(k^2(1 - 2c^2) - c^2) \tanh(\nu) = 0.$$

Equating the coefficients of each pair of functions $\tanh(\nu)$ gives:

$$\begin{aligned} 2k^2A^2 + 2k^2c^2 &= 0, \\ -c^2 + k^2 - 2k^2c^2 &= 0. \end{aligned} \tag{26}$$

The solutions of system (26) are

$$A = \pm\frac{k}{\sqrt{-1 - 2k^2}}, \qquad c = \pm\frac{k}{\sqrt{1 + 2k^2}}. \tag{27}$$

The results (27) give the following solutions for Eq. (19) which are dark-soliton ones:

$$u_{22}(x,t) = \pm\frac{k}{\sqrt{-1 - 2\,k^2}} \times \tanh\left(k\,x \mp \frac{k}{\sqrt{1 + 2k^2}}\,t\right). \tag{28}$$

2.2.3. *Solutions for the 2nd model, singular-soliton ones*

To obtain singular-soliton solutions, we set our hypothesis as:

$$u(\nu) = A \operatorname{csch}^p(\nu). \tag{29}$$

Similar to the previous part the value of p is obtained as $p = 1$. Thus by inserting the ansätz Eq. (29) into Eq. (21) and for $p = 1$, we obtain:

$$A(-c^2 + k^2 + k^2c^2) \cosh^2(\nu) + A(2k^2A^2 + c^2 - k^2 + k^2c^2) = 0.$$

Equating the coefficients of each pair of functions $\cosh(\nu)$ gives system:

$$\begin{aligned} -c^2 + k^2 + k^2c^2 &= 0, \\ 2k^2A^2 + c^2 - k^2 + k^2c^2 &= 0. \end{aligned} \tag{30}$$

After solving system (30) we have:

$$A = \pm\frac{k}{\sqrt{k^2 - 1}}, \qquad c = \pm\frac{k}{\sqrt{1 - k^2}}. \tag{31}$$

By the results of Eq. (31) the following solutions are obtained for Eq. (19):

$$u_{23}(x,t) = \pm\frac{k}{\sqrt{k^2 - 1}} \operatorname{csch}\left(k\,x \mp \frac{k}{\sqrt{1 - k^2}}\,t\right), \tag{32}$$

which are singular-soliton ones.

2.3. The 3rd model

The following model is considered in this part

$$u_{tt} - u_{xt} - (6u^2\,u_x + u_{xxt})_x = 0, \tag{33}$$

to obtain its soliton solutions. Applying traveling-wave ansätz $u(x,t) = u(\nu)$, $\nu = k\,x - c\,t$ in Eq. (33) gives:

$$c\,(c + k)\,u'' - k\,(\,6\,k\,u^2\,u' - k^2\,c\,u^{(3)})' = 0. \tag{34}$$

Now, with respect to ν we integrate Eq. (34) two times and set zero for integrating constants. After doing these, we have

$$c\,(c + k)\,u + k^2\,(k\,c\,u'' - 2\,u^3) = 0, \tag{35}$$

for constants k and c.

2.3.1. *Solutions for the 3rd model, bright-soliton ones*

To obtain the solitary-wave ansätz for the bright-soliton solution we use the following hypothesis:

$$u(\nu) = A \operatorname{sech}^p(\nu). \tag{36}$$

Setting the ansätz Eq. (36) to Eq. (35) gives:

$$c(c + k)A \operatorname{sech}^p(\nu) - 2k^2A^3 \operatorname{sech}^{3p}(\nu) + k^3cp(Ap \operatorname{sech}^p(\nu) - A(p + 1) \operatorname{sech}^{p+2}(\nu) = 0. \tag{37}$$

Here, also we obtain $p = 1$ from Eq. (37). In addition equating the exponents $3p$ and $p + 2$, and coefficients of each pair of functions $\operatorname{sech}(\nu)$ yield:

$$\begin{aligned} 2k^2(A^2 + kc) &= 0, \\ -c^2 - ck - k^3c &= 0. \end{aligned} \tag{38}$$

One may obtain the following solutions for Eq. (38)

$$A = \pm k\sqrt{1 + k^2}, \qquad c = -(1 + k^2)\,k. \tag{39}$$

By the values in Eq. (39), the following bright-soliton solutions are obtained for Eq. (33):

$$u_{31}(x,t) = \pm k\sqrt{1 + k^2} \operatorname{sech}\left(kx + (k + k^3)\,t\right). \tag{40}$$

2.3.2. *Solutions for the 3rd model, dark-soliton ones*

To obtain the dark-soliton solutions we assume that:

$$u(\nu) = A \tanh^p(\nu). \tag{41}$$

Here also we obtain $p = 1$. Therefore, setting the ansätz Eq. (41) into Eq. (35) and putting $p = 1$ yields:

$$2k^2A(A^2 - kc) \tanh^3(\nu) + Ac(2k^3 - k - c) \tanh(\nu) = 0.$$

Equating the coefficients of each pair of functions $\tanh(\nu)$ gives:

$$\begin{aligned} 2k^2A^2 - 2k^3c &= 0, \\ -c^2 - ck + 2k^3c &= 0, \end{aligned}$$

that has the following solutions

$$A = \pm\sqrt{2k^2-1}k, \ \ c = 2\,k^3 - k. \tag{42}$$

Using Eq. (42), we obtain the following solutions for Eq. (33) as:

$$u_{32}(x,t) = \pm\sqrt{2k^2-1}k\tanh(kx + (2k^3-k)t), \tag{43}$$

that are dark-soliton ones.

2.3.3. *Solutions for the 3rd model, singular-soliton ones*

We consider the following hypothesis to obtain singular soliton solutions:

$$u(\nu) = A\,\mathrm{csch}^p(\nu). \tag{44}$$

During our computations, we obtain $p = 1$. Besides, by substituting the ansätz Eq. (44) into Eq. (35) and for $p = 1$, one can obtain:

$$-Ac(c+k+k^3)\cosh^2(\nu) \\ + A(k^2(2A^2-kc)+c(c+k)) = 0.$$

As usual by equating the coefficients of each pair of functions $\cosh(\nu)$ gives:

$$-c^2 - ck - k^3c = 0, \\ 2k^2A^2 + c^2 + ck - k^3c = 0,$$

which has the following solutions:

$$A = \pm\sqrt{-1-k^2}\,k, \ \ c = -k - k^3. \tag{45}$$

The following singular-soliton solutions are obtained for (33) using the results Eq. (45) as:

$$u_{33}(x,t) = \pm\sqrt{-1-k^2}k\,\mathrm{csch}(kx + (k+k^3)\,t). \tag{46}$$

2.4. The 4th Boussinesq-type model

In this final part, we find solutions for the following model:

$$-(6u^2u_x + u_{xxx})_x + u_{tt} = 0. \tag{47}$$

Implementing the traveling-wave ansätz $u(x,t) = u(\nu)$, $\nu = k\,x - c\,t$ in Eq. (47) yields:

$$c^2\,u'' - k\,(\,6\,k\,u^2\,u' + k^3\,u^{(3)})' = 0. \tag{48}$$

We integrate Eq. (48) two times with respect to ν and set zero for the integration constants. All of these works give

$$c^2\,u - 2\,k^2\,u^3 - k^4\,u'' = 0, \tag{49}$$

wherein k and c are some constants.

2.4.1. *Solutions for the 4th model, bright-soliton ones*

In order to find the bright-soliton solutions, the following assumption is considered:

$$u(\nu) = A\,\mathrm{sech}^p(\nu), \tag{50}$$

where the unknown parameter p is found during the process of obtaining for the solutions of Eq. (49). Then setting the ansätz Eq. (50) in Eq. (49) gives:

$$c^2A\,\mathrm{sech}^p(\nu) - 2k^2A^3\,\mathrm{sech}^{3p}(\nu) - k^4A\,p(p\,\mathrm{sech}^p(\nu) \\ - (p+1)\,\mathrm{sech}^{p+2}(\nu) = 0.$$

In a similar process which is mentioned in the previous parts, we obtain the value one for p. Besides, we equate the coefficients of each pair of functions $\mathrm{sech}(\nu)$ which gives

$$k^2A^2 - k^4 = 0, \\ -c^2 + k^4 = 0,$$

with the following solutions:

$$A = \pm k \ \ c = \pm\,k^2. \tag{51}$$

Solutions (51) give the following bright-soliton solutions for Eq. (49):

$$u_{41}(x,t) = \pm\,k\,\mathrm{sech}(kx \pm k^2t). \tag{52}$$

2.4.2. *Solutions for the 4th model, dark-soliton ones*

To obtain dark-soliton solutions for Eq. (49) we take

$$u(\nu) = A\tanh^p(\nu). \tag{53}$$

We obtain $p = 1$. For this value of p and by substituting the ansätz Eq. (53) in Eq. (49) yields:

$$-A(2k^2A^2 + 2k^4)\tanh^3(\nu) - A(-c^2 - 2k^4)\tanh(\nu) = 0.$$

Here, also we equate coefficients of each pair of functions $\tanh(\nu)$ which gives

$$2k^2A^2 + 2k^4 = 0, \\ -c^2 - 2k^4 = 0. \tag{54}$$

The solutions of system (54) are

$$A = \pm ik, \ \ c = \pm\sqrt{2}ik^2,$$

which in turn give the following dark-soliton solutions for Eq. (49):

$$u_{42}(x,t) = \pm ik\tanh(kx \mp \sqrt{2}ik^2t). \tag{55}$$

2.4.3. Solutions for the 4^{th} model, singular-soliton ones

Let us now find singular soliton solutions for Eq. (49) where the starting hypothesis is

$$u(\nu) = A\,\mathrm{csch}^p(\nu). \tag{56}$$

The value of its unknown is obtained as $p = 1$. Hence, by substituting the ansätz Eq. (56) into Eq. (49) and for $p = 1$ we obtain:

$$A(k^4 - c^2)\cosh^2(\nu) + A(c^2 + k^2(k^2 + 2A^2)) = 0.$$

Equating the coefficients of each pair of functions $\cosh(\nu)$ results in

$$\begin{aligned} -c^2 + k^4 &= 0, \\ 2\,k^2A^2 + c^2 + k^4 &= 0. \end{aligned} \tag{57}$$

After solving system (57) we have:

$$A = \pm ik, \;\; c = \pm\, k^2. \tag{58}$$

We can obtain the following solutions for Eq. (49) using results (58)

$$u_{43}(x,t) = \pm ik\,\mathrm{csch}(k(x \mp kt)), \tag{59}$$

which are bright-soliton solutions.

3. Physical discussion

It is worthy to have a physical explanation for the presented solutions of all types of Boussinesq equations. The Boussinesq equation is valid for water waves for weakly nonlinear and relatively long waves in fluid dynamics. In computer models the Boussinesq-type equations are applied for simulating of water waves in shallow seashores and seas in coastal sciences. But the Boussinesq equation is applicable to fairy long waves, which means that when the wave length in comparison with its depth of water is not short. In this work, we have obtained some different families for the Boussinesq-type equations. All obtained solutions can be categorized into three types, namely, bright-, dark-, and singular-soliton ones. We encounter a bright-soliton when the energy of the soliton reaches to its maximum value at the center of the wave and then this energy at infinity goes to 0. In this case, soliton group velocity u exceeds some thresholds. Solutions (12), (25), (40), and (52) are bright-soliton ones which are plotted in Figs. 1, 4, 7 and 10.

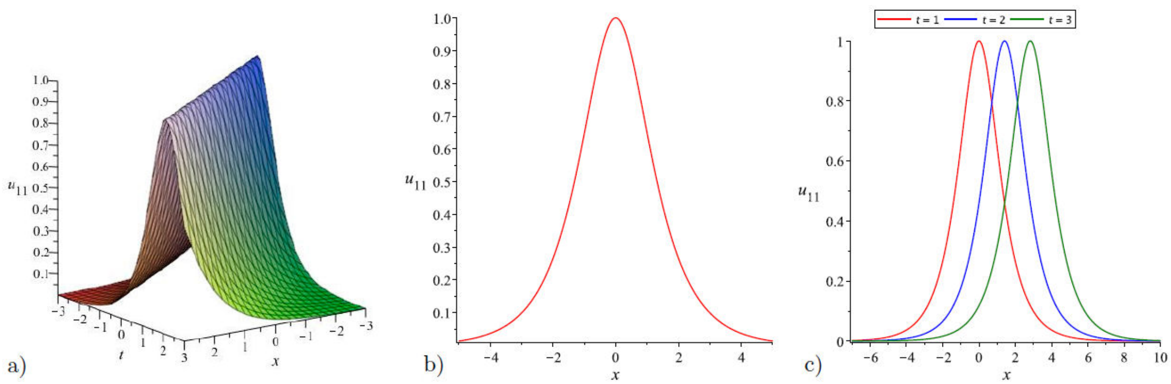

FIGURE 1. a) The bright-soliton solution u_{11} in (12) for $k = 1$, b) the corresponding 2D plot for $t = 0$, c) graphical representation with $-7 \le x \le 10$, and $t = 1, 2, 3$.

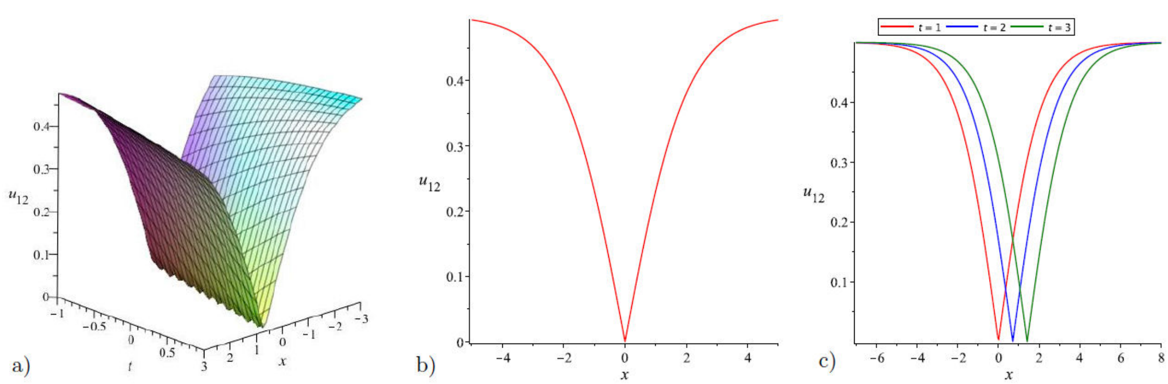

FIGURE 2. a) The dark-soliton solution u_{12} in (15) for $k = 0.5$, b) the corresponding 2D plot for $t = 0$, c) graphical representation with $-7 \le x \le 8$, and $t = 1, 2, 3$.

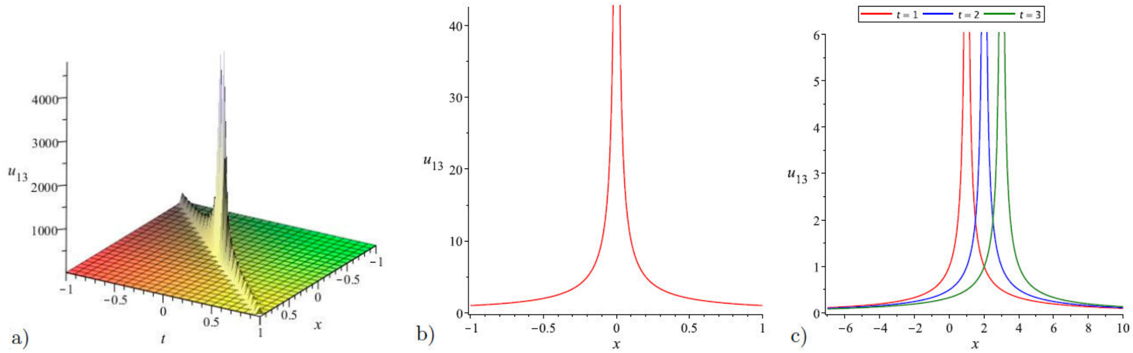

FIGURE 3. a) The singular-soliton solution u_{13} in (18) for $k = 0.1$, b) the corresponding 2D plot for $t = 0$, c) Graphical representation with $-7 \leq x \leq 10$, and $t = 1, 2, 3$.

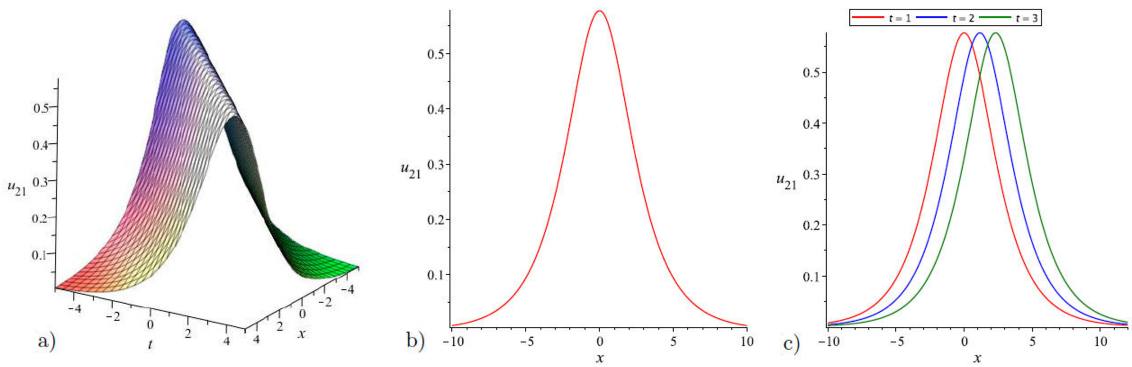

FIGURE 4. a) The bright-soliton solution u_{21} in (25) for $k = 0.5$, b) The corresponding 2D plot for $t = 0$, c) graphical representation with $-10 \leq x \leq 12$, and $t = 1, 2, 3$.

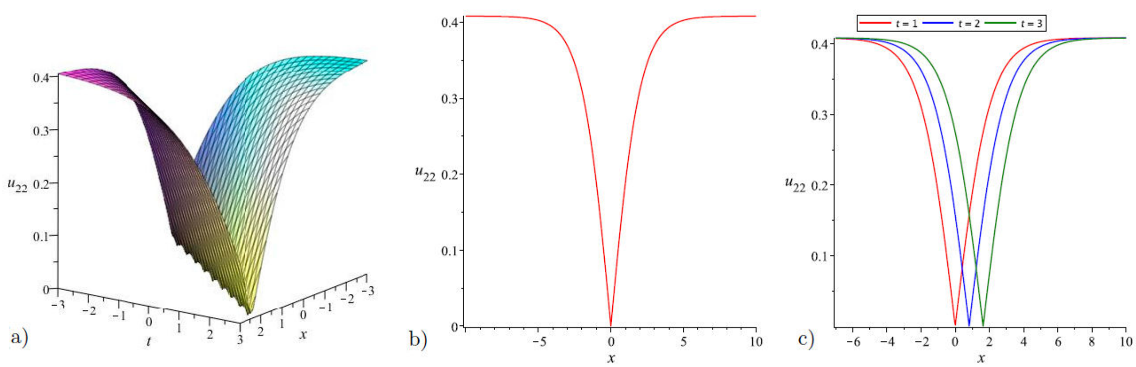

FIGURE 5. a) The dark-soliton solution u_{22} in (25) for $k = 0.5$, b) the corresponding 2D plot for $t = 0$, c) graphical representation with $-7 \leq x \leq 10$, and $t = 1, 2, 3$.

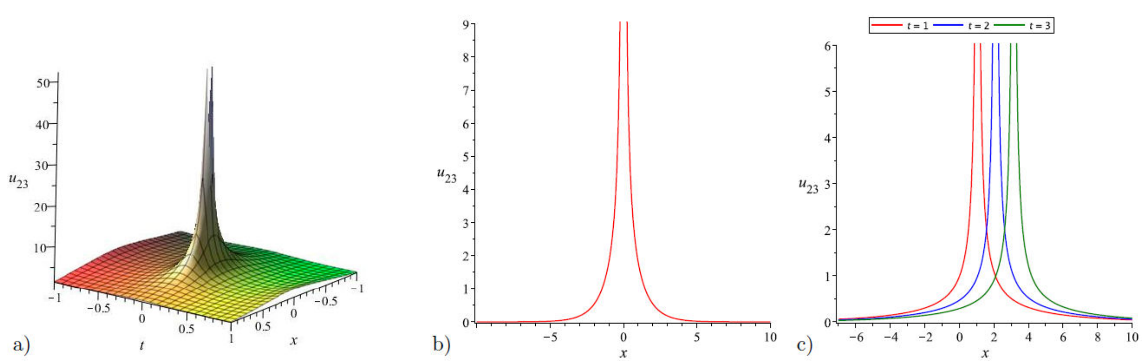

FIGURE 6. a) The singular-soliton solution u_{23} in (32) for $k = 0.5$, b) the corresponding 2D plot for $t = 0$, c) graphical representation with $-15 \leq x \leq 10$, and $t = 1, 2, 3$.

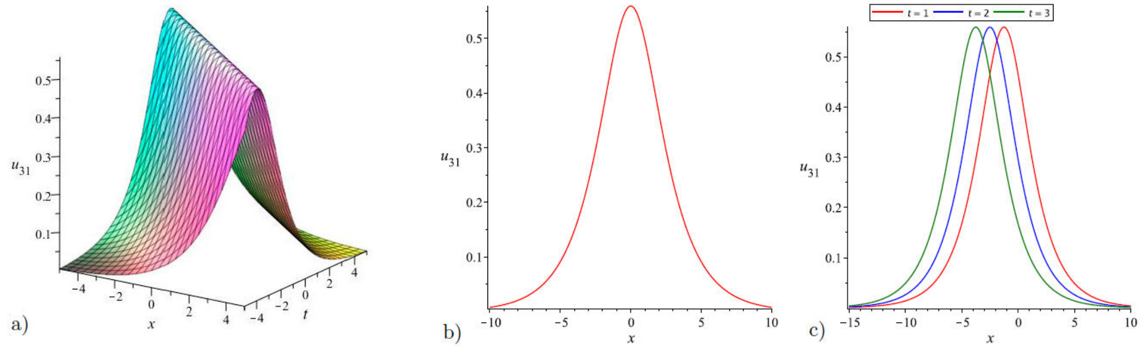

FIGURE 7. a) The bright-soliton solution u_{31} in (32) for $k = 0.5$, b) the corresponding 2D plot for $t = 0$, c) graphical representation with $-10 \leq x \leq 12$, and $t = 1, 2, 3$.

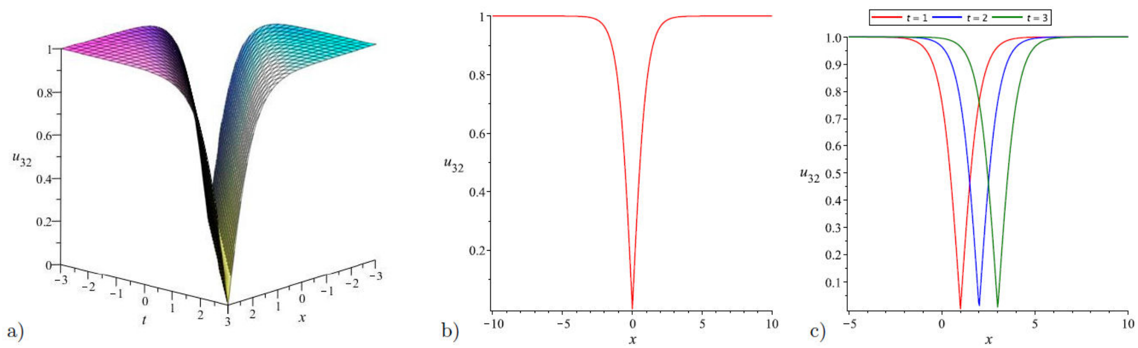

FIGURE 8. a) The dark-soliton solution u_{32} in (43) for $k = 1$, b) the corresponding 2D plot for $t = 0$, c) graphical representation with $-5 \leq x \leq 10$, and $t = 1, 2, 3$.

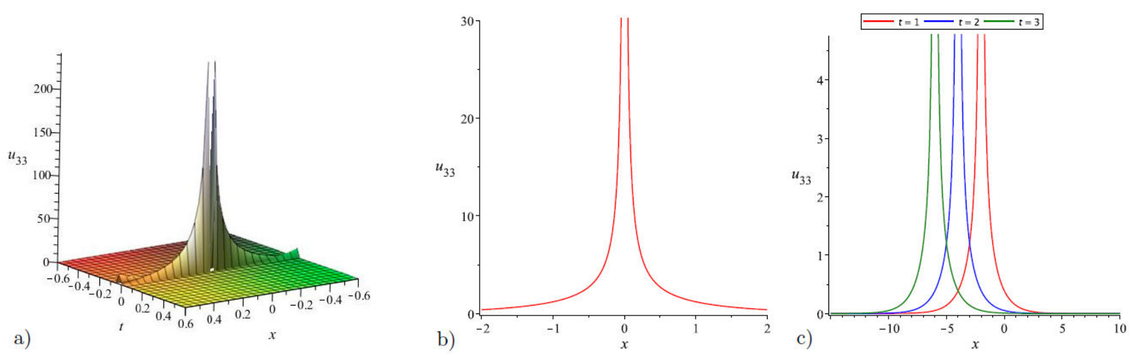

FIGURE 9. a) The singular-soliton solution u_{33} in (46) for $k = 1$, b) the corresponding 2D plot for $t = 0$, c) graphical representation with $-15 \leq x \leq 10$, and $t = 1, 2, 3$.

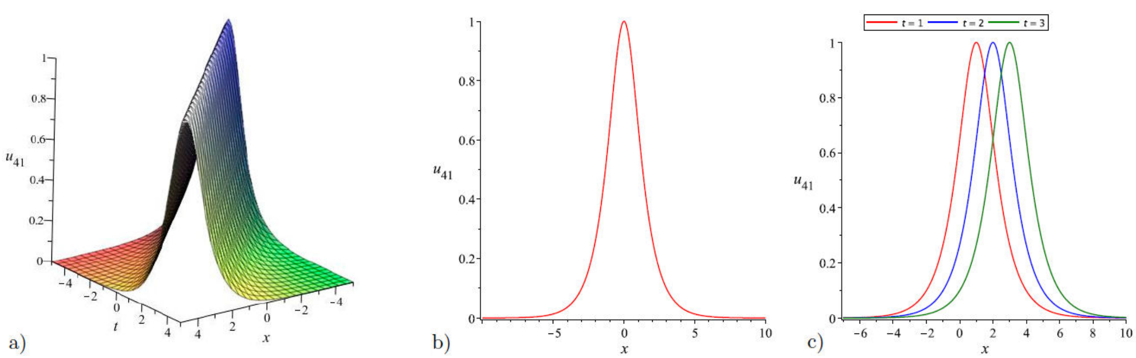

FIGURE 10. a) The bright-soliton solution u_{41} in (52) for $k = 1$, b) the corresponding 2D plot for $t = 0$, c) graphical representation with $-7 \leq x \leq 10$, and $t = 1, 2, 3$.

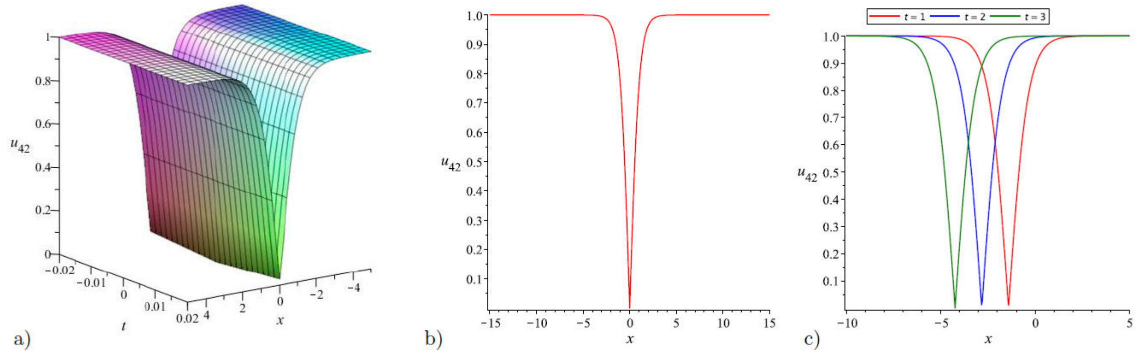

FIGURE 11. a) The dark-soliton solution u_{42} in (55) for $k = 1$, b) the corresponding 2D plot for $t = 0$, c) graphical representation with $-10 \leq x \leq 5$, and $t = 1, 2, 3$.

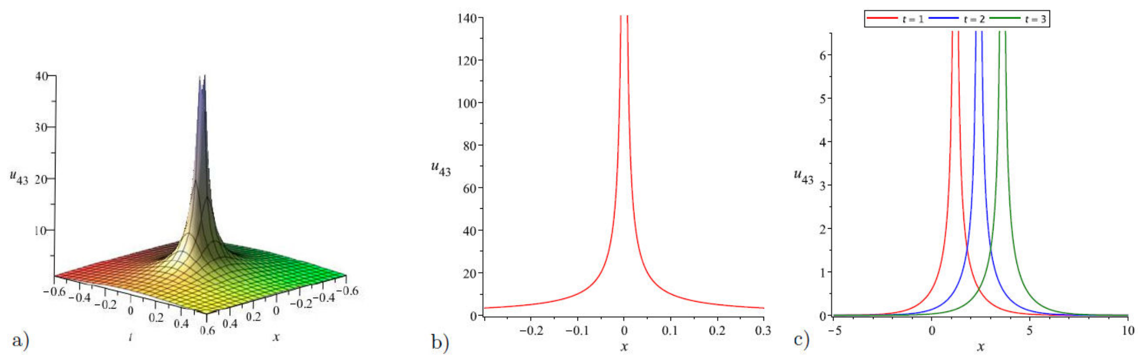

FIGURE 12. a) The singular-soliton solution u_{43} in (59) for $k = 1.2$, b) the corresponding 2D plot for $t = 0$, c) graphical representation with $-5 \leq x \leq 10$, and $t = 1, 2, 3$.

A dark soliton is a solitary wave that is generated by cutting a portion of a continuous wave. As a matter of fact, a dark soliton is an amplitude dip in the continuous wave. It is generally known that dark soliton is a localized surface which is an amplitude dip, that causes a temporary decrease in wave amplitude, and can usually be seen in normal dispersion areas. Furthermore, because of its unique properties in homogeneous optical fibers, a dark-soliton is widely studied. In a homogeneous background, the intensity profile of the dark soliton shows a dip or hole-soliton. Dark-solitons have been found to be both stable and robust to losses. Soliton energy is greatest at infinity in dark-solitons, and there is a gap in the center. All solutions (15), (28), (43), and (55) are dark-soliton ones which are plotted in Figs. 2, 5, 8 and 11.

Nakamura [23] has shown that a certain nonlinear evolution equation except soliton solutions, may also have explode decay mode solutions. This kind of soliton solutions is called a singular-soliton one that can be written by an analytical relation. All solutions Eqs. (18), (32), (46), and (59) are singular-soliton ones that are plotted in Figs. 3, 6, 9 and 12. Furthermore, parts (c) of all Figures demonstrate three plots of our solutions in $2D$ case for a special spatial domain and some values of time which are $t = 1, 2, 3$. As one can see from these figures all of them have soliton properties.

4. A concluding remark

In the present manuscript, the ansätz method is used for constructing singular-, dark-, and bright-soliton solutions for the distinct forms of Boussinesq-type equations. The graphs of soliton solutions with different selections of their parameters were plotted to demonstrate the localizations of the solutions that describe the nonlinear waves in coastal and ocean engineering, like tsunami waves. According to the best of our knowledge, these closed forms of solutions were not presented previously and all of them are new ones and novel. All presented solutions in this paper were tested for satisfy in their relevant equations. Also the obtained results have shown that an excellent performance of the ansätz scheme in using Boussinesq-type equations. Our results can be further extended in future research works by working on various classes of the investigated equation.

Data availability

All generated and analysed results which have obtained during this research are included in the manuscript.

Declarations

The authors declare that they have no conflict of interest.

1. D.H. Peregrine, Long wave on a beach. *J. Fluid Mech.* **27** (1967) 815-827.
2. C.I. Christov, G.A. Maugin, M.G. Velarde, Well-posed Boussinesq paradigm with purely spatial higher-order derivatives. *Phys. Rev. E.* **54** (1996) 3621-3638.
3. A. Biswas, D. Milovic, A. Ranasinghe, Solitary waves of Boussinesq equation in a power law media. *Commun. Nonlinear Sci. Numer. Simul.* **14** (2009) 3738-3742.
4. A.M. Wazwaz, New traveling wave solutions to the Boussinesq and the Klein-Gordon equations. *Commun. Nonlinear Sci. Numer. Simul.* **13** (2008) 889-901.
5. A.M. Wazwaz, Multiple soliton solutions for an integrable couplings of the Boussinesq equation. *Ocean Engineering* **73** (2013) 38-40.
6. M.A. Akbar, L. Akinyemi, S.W. Yao, A. Jhangeer, H. Rezazadeh, M.M. Khater, H. Ahmad, M. Inc, Soliton solutions to the Boussinesq equation through sine-Gordon method and Kudryashov method. *Res. Phys.* **25** (2021) 104228. `https://doi.org/10.1016/j.rinp.2021.104228`.
7. A.M. Wazwaz, Multiple-soliton solutions for the Boussinesq equation. *Appl. Math. Comput.* **192** (2007) 479-486.
8. A.M. Wazwaz, Solitons and singular solitons for a variety of Boussinesq-like equations. *Ocean Engineering* **53** (2012) 1–5.
9. P. Daripa, W. Hua, A numerical study of an ill-posed Boussinesq equation arising in water waves and nonlinear lattices: Filtering and regularization techniques. *Appl. Math. Comput.* **101** (1999) 159–207.
10. D. Adhikari, C. Cao, J. Wu, The 2D Boussinesq equations with vertical viscosity and vertical diffusivity. *Journal of Differential Equations* **249** (2010) 1078–1088.
11. A. Biswas, Solitary waves for power-law regularized long-wave equation and $R(m,n)$ equation. *Nonlinear Dyn.* **59** (2010) 423–426.
12. A. Biswas, C.M. Khalique, Stationary solutions for nonlinear dispersive Schrödinger's equation. *Nonlinear Dyn.* **63** (2011) 623–626.
13. J.L. Bona, M. Chen, J.-C. Saut, Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media: II. The nonlinear theory, *Nonlinearity* **17** (2004) 925.
14. C.I. Christov, G.A. Maugin, M.G. Velarde, Well-posed Boussinesq paradigm with purely spatial higher-order derivatives. *Phys. Rev. E* **54** (1996) 3621.
15. Z. Huazhi, L. Huajun, L. Xiaodong, L. Aixia, The application of a numerical model to coastal surface water waves. *Journal of Ocean University of China* **4** (2005) 177.
16. N. Droenen, R. Deigaard, Adaptation of a Boussinesq wave model for Dune erosion modeling. *Coastal Engineering Proceedings* **1** (2012) 31.
17. J.T. Kirby, Boussinesq models and their application to coastal processes across a wide range of scales. PhD diss., *American Society of Civil Engineers*, (2016).
18. P.J. Lynett, J.A. Melby, D.-H Kim, An application of Boussinesq modeling to hurricane wave overtopping and inundation. *Ocean Engineering* **37** (2010) 135-153.
19. V. Roeber, K.F. Cheung, M.H. Kobayashi, Shock-capturing Boussinesq-type model for nearshore wave processes. *Coastal Engineering* **57** (2010) 407–423.
20. A.M. Wazwaz, Solitons and singular solitons for a variety of Boussinesq-like equations. *Ocean Eng.* **53** (2012) 1–5.
21. M.T. Darvishi, M. Najafi, A.M. Wazwaz, Soliton solutions for Boussinesq-like equations with spatio-temporal dispersion. *Ocean Eng.* **130** (2017) 228–240.
22. M.T. Darvishi, M. Najafi, A.M. Wazwaz, Traveling wave solutions for Boussinesq-like equations with spatial and spatial-temporal dispersion. *Rom. Rep. Phys.* **70** (2018) 108.
23. A. Nakamura, Simple explode-decay mode solutions of a certain one-space dimensional nonlinear evolution equations. *J. Phys. Soc. Japan* **33** (1972) 1456-1458.