

QUANTUM RESTRICTIONS ON THE MEASURABILITY
OF FIELDS IN GRAVITATIONAL THEORY*

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ABSTRACT

The measureability of the gravitational field is discussed from both the classical and quantum points of view. It is shown that classically such a measurement can be performed if one uses test bodies with vanishing small mass. On the other hand, due to the Heisenberg uncertainty principle, it is found that one must use test bodies of extremely large mass in order to obtain accurate results. Thus it appears that an accurate

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determination of the gravitational field is not possible if one takes into account the quantum restrictions. Consequences of this result will be discussed.

I.- INTRODUCTION.

In any attempt to combine the disciplines of quantum mechanics and general relativity, one must discuss what, if any, restrictions the quantum conditions place on the measurability of the gravitational field. Such a study is necessary if one tries to quantize the gravitational field directly, for it is in this manner that we give physical meaning to postulated commutation relations between the field variables. Even if one is not interested in quantizing the gravitational field, quantum restrictions such as the uncertainty relations will still be operative as far as the measuring apparatus is concerned.

In this paper we will analyze the procedures available to us for measuring the gravitational field and discuss the limitations placed on these measurements by the uncertainty relations. We shall treat the gravitational field as a purely classical non-linear field, neglecting any effects which might arise from quantum fluctuations. In treating the effect of imposing quantum conditions on our measuring apparatus, we shall follow closely the methods used by Bohr and Rosenfeld¹ in treating the analagous case in linear electromagnetic theory. Our principal conclusion will be that it is in general impossible to measure the gravitational field within a region of space-time, with arbitrarily high precision.

II.- CLASSICAL MEASUREMENT OF THE GRAVITATIONAL FIELD.

Before we consider quantum effects, we must first discuss how one can measure the gravitational field classically. In electromagnetic theory, it is the existence of the Lorentz force law which enables us to measure the field. In effect, we perform the measurement by actually measuring the change in momentum of a test body occasioned by the force of the field. In gravitational theory, however, there is no Lorentz-type force law. The statement that a material body moves along a geodesic is in general not true, as we shall see.

This lack of a force law in the gravitational theory of general relativity is not a defect in the theory but rather a consequence of it. In order that a force law such as the Lorentz force have meaning, it is necessary that one be able to split the field into two parts, one belonging to the charged body and the other belonging to the remaining field. It is this latter field which appears in the force law. In the case of the electromagnetic field it is possible to perform this splitting (though not in a completely unambiguous manner), since the field equations are inherently linear.

Actually, a Lorentz-type force law is not necessary to the gravitational theory. As Einstein, Infeld and Hoffmann² have shown, if one represents mass points by singularities, then the interactions between two such mass points are really determined by the field equations. Later, Infeld and Schild³ were able to prove that a particle of infinitesimal mass in a given gravitational field will move along a geodesic of the space thus bearing out the old Einstein hypothesis that massive particles move along geodesics. Note, however, that

Infeld's result holds only when the gravitational field of the mass particle is vanishingly small compared to the gravitational field in which it finds itself. Because of the non-linearity of the gravitational field equations, one can separate out the field of the particle only approximately, so that the smaller the mass of the particle, the better the approximation. It is for this reason that it is possible, within the framework of the general theory of relativity, to treat the motion of a planet in the field of the sun satisfactorily by assuming that it moves along a geodesic of the sun's field. If one attempts to apply this same procedure to a genuine two-body problem, like that of a double star, one is led to contradictory results⁴.

With the above facts in mind we see that classically the way to measure the gravitational field is to observe the motion of a test particle whose mass is extremely small, compared to the masses producing the gravitational field^{*}. The important thing for our work then is to realize that the equations of motion for the test body (see note 1) whereby we hope to measure the gravitational field, are only approximate and are rigorously true only in the limit that the mass of the test body vanishes. The situation in gravitational theory is thus radically different from that of electromagnetic theory where the Lorentz force equation is assumed to be valid for any value of the charge whatsoever.

III.- QUANTUM MEASUREMENT OF THE GRAVITATIONAL FIELD.

If, now, we attempt to impose quantum restrictions on

^{*}See Note 1 at the end of this paper.

our measurements, following Bohr and Rosenfeld we immediately run into difficulties. Consider for example a measurement made to determine $\{_{44}^1\}$. We would employ a test particle which initially had no velocity so that $U^i = (0,0,0,1)$. Then the change in time of the momentum mU^i of the test particle would be given by

$$\frac{d(mU^i)}{dt} = \frac{dp^i}{dt} = m \{_{44}^1\} . \quad (1)$$

Thus, to measure $\{_{44}^1\}$, we would let the test particle be accelerated by the gravitational field during a time interval T . At the end of this interval, the test body would have acquired a certain amount of momentum p^i given by

$$p^i = m \{_{44}^1\} T . \quad (2)$$

Thus a measurement of p^i is equivalent to a measurement of $\{_{44}^1\}$.

The uncertainty in the value of $\{_{44}^1\}$ is given by

$$m T \Delta \{_{44}^1\} > \Delta p^i \quad (3)$$

The final momentum p^i can be measured by letting the test body interact with some other object during a time Δt . During this measurement, there will arise an uncontrollable uncertainty in the position of the test body Δx^i given by the uncertainty relation

$$\Delta p^i \Delta x^i \geq \frac{\hbar}{2} \quad (4)$$

Thus, the uncertainty in the measurement of $\{_{44}^1\}$ is given by

$$\Delta \{_{44}^1\} \geq \frac{\hbar}{m T \Delta x} \quad (5)$$

This result is in itself enough to illuminate the nature of our difficulty.

In order to increase the accuracy in our measurement of $\{_{44}^1\}$ we have, according to (5), to increase m accordingly. But then the results of Infeld and Schild are no longer valid and we cannot use (1) to determine $\{_{44}^1\}$

IV.- CONCLUSIONS.

It would be inappropriate at this point to conclude, on the basis of the above analysis, that quantum mechanics and general relativity are incompatible. Yet one must be surprised to find that the most obvious way of measuring the gravitational field classically, is inconsistent with the requirements of quantum mechanics*. We feel justified, therefore, in making the following comments.

Firstly, it is difficult to see how one can proceed with an attempt to quantize the gravitational field in analogy with the electromagnetic field. In order to do so, one must postulate commutation relations between the field variables which in turn must be given physical content through the results of experiments. Bohr and Rosenfeld have shown how to do this for the electromagnetic field. It is obvious that we cannot do

*See Note 2 at the end of this paper.

likewise in the gravitational case, since we cannot define what we mean by the gravitational field quantum mechanically. While it is possible, at least formally, to apply the methods of quantum mechanics to the gravitational field it is no longer clear that such a procedure corresponds to anything physical.

Secondly, we are faced with the question of whether the gravitational field, even treated as a classical concept, is compatible with the requirements of quantum mechanics or vice versa. One might even be tempted to use the above results to argue against one or the other disciplines. However, we know that each theory possess a certain area of applicability. Quantum mechanics has had unparalleled success in explaining atomic phenomena. General relativity, on the other hand, - while not as rich in experimental prediction, has a certain logical consistency and simplicity which we would be reluctant to discard.

Actually, there is at least one possible way out of our difficulties which, in closing, we would briefly like to - describe. As we have mentioned previously, in a non-linear theory such as general relativity, the interaction between sources of the field is implicitly contained in the field equations. And as we also mentioned, nowhere does the concept of the gravitational field enter into the final equations of motions of these sources. One might, therefore, adopt the attitude that the gravitational field is really only a - mathematical construct. The physical reality in the theory would then be the sources of the field, the material particles. The theory tells us how these particles move under their -

mutual interaction. One could then apply the quantum conditions to these particles, i.e., we would quantize their motions.

While such a procedure would be extremely difficult to execute (the equations of motion are obtainable from the general theory of relativity only in approximate form) it does have the following point to its credit. Nowhere in the final equations of motion does the concept of point interaction appear. It is just this concept which many feel to be the basis of our present field theoretic difficulties. It is not completely unreasonable to expect, therefore, that a further examination of the situation in gravitational theory might lead to greater understanding in other field theories.

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NOTE 1.- Strictly speaking the motion of our test particle is governed by the equations for a geodesic

$$\frac{dU^a}{d\tau} = -\{^a_{i\kappa}\} U^i U^\kappa$$

where $\{^a_{i\kappa}\}$ is the Christoffel symbol of the second kind defined by

$$\{^a_{i\kappa}\} = \frac{1}{2} g^{a\beta} (g_{i\beta,\kappa} + g_{\kappa\beta,i} - g_{i\kappa,\beta})$$

and where U^i is the relativistic four-velocity of the test particle. Thus actually the most that we can measure from

observing the motion of our test body are the values of the Christoffel symbols. The $g_{i\kappa}$ are analogous to the potentials of electromagnetic theory while the $\{^a_{i\kappa}\}$ are analogous to the actual E and H fields. When we speak of measuring the gravitational field we really mean a measurement of one of the Christoffel symbols. This change of emphasis from the $g_{i\kappa}$ to the $\{^a_{i\kappa}\}$ as the measurables of the theory should occasion no more difficulty in our present analysis than does the similar shift in electromagnetic theory.

NOTE 2.- One might be tempted to argue that, while for large masses Eq. (1) is invalid, one might be able to obtain from the general theory a more universal force law, valid for masses of all size. This is not too likely however for, as was pointed out previously our equations are non-linear and any such separation of the field into the part to be measured and that of the test body would be to a large extent arbitrary and artificial. This contention is further strengthened by the fact that in the final results of Einstein, Infeld and Hoffmann the gravitational field nowhere appears.

FOOTNOTES.

1. N.Bohr and L.Rosenfeld, Kgl.Danske Vid.Sels, Math.-fys. Medd., 12, No. I (1933).
2. A.Einstein, L.Infeld and B.Hoffmann, Annals of Math., 39, 65 (1938).
3. L.Infeld and A.Schild, Rev.Mod.Phys., 21, 408 (1949).
4. H.P.Robertson, Annals of Math., 39, 101 (1938), footnotes on pp. 103 and 104.