DIFFRACTION IN TIME

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RESUMEN

Doctor Marcos Moshinsky[8] and the author[4,9] have been independently interested in the subject. Recently, Doctor Moshinsky[10] has shown the equivalence between his own treatment and the author's.

Our method in this problem is essentially based on Minkowski's space-time geometry and Fresnel's intuitive - treatment of diffraction. In this way, the fundamental - formulae of wave mechanics come out with a complete symmetry between space and time, and Feynman's[6] interpretation of negative energy states is very clearly illustrated.

I. INTRODUCTION.

Let us consider an incident wave, asociated with one relativistic particle, and such that one space-like - component of the four-current, say j_1, has a definite sign say:

$$j_1 > a > 0 \quad , \quad a \text{ fixed number.} \qquad (1)$$

Moreover, the magnitude of the current is supposed to become negligible outside a world current tube, whose space-like section is entirely at a finite distance

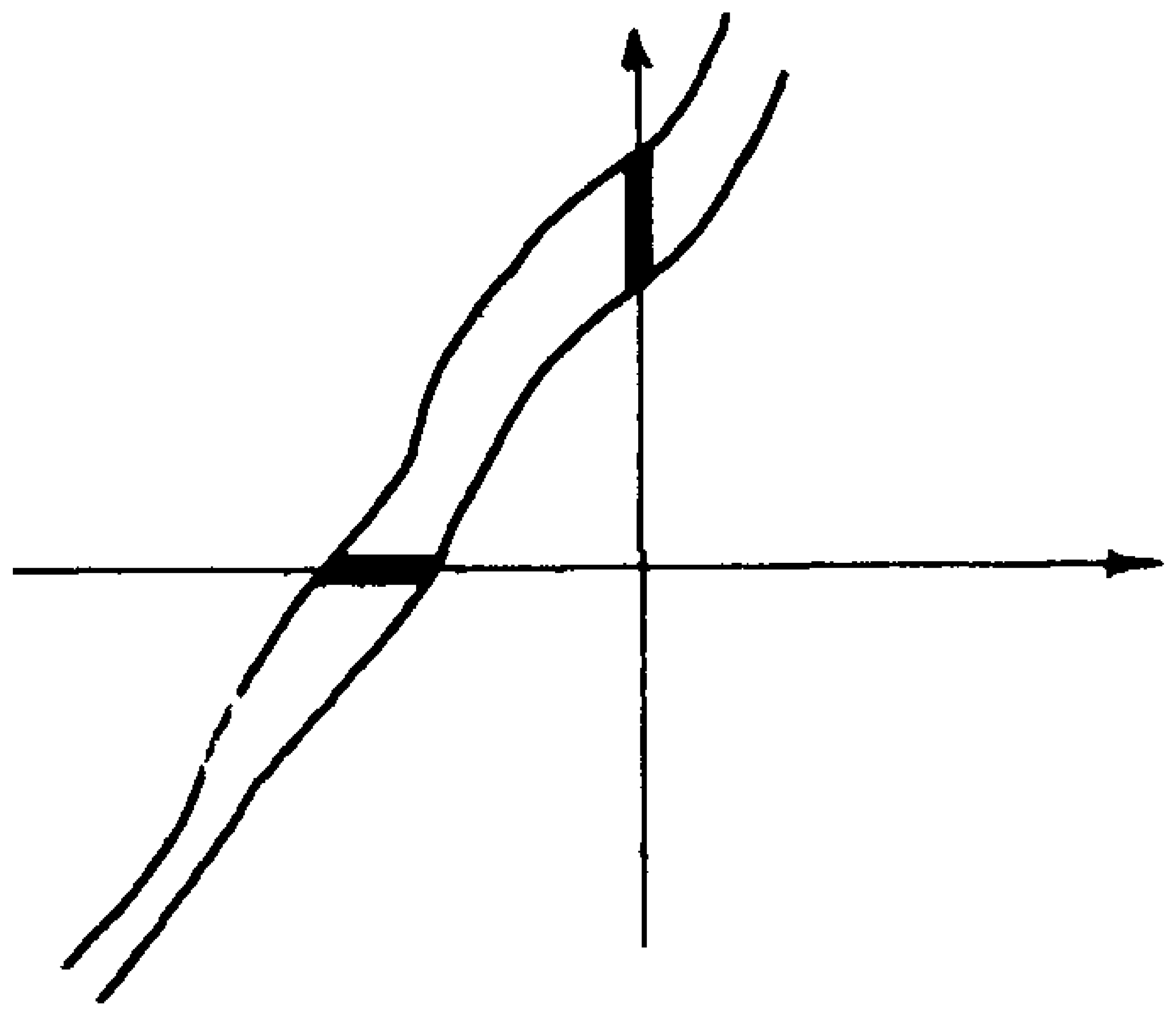

Fig. 1

Under these assumptions, and owing to the continuity equation:

$$\partial_\lambda j^\lambda = 0 \quad , \qquad (\lambda, \mu, \nu, \rho = 1,2,3,4) \qquad (2)$$

the wave function is "equivalenty" normalized by the formula:

$$\frac{1}{ic}\iiint_{x_4=0} j_4 \, dx_1 dx_2 dx_3 = 1, \quad (x_4 = ict) \qquad (3)$$

or by the formula;

$$\iiint_{x_1=0} j_1 \, dx_2 \, dx_3 \, dt = 1 \quad . \qquad (4)$$

Physically, the above assumptions will be met if the particle is emitted on a "practically" plane monocromatic wave, limited in space and time, say by some "collimator" operated during a finite period.

Suppose that the plane $x_1 = 0$ of three dimensional space is materialized by an impenetrable and perfectly absorbing screen with a hole in it, which is initially and finally shut. At any instant, the contour of this hole will be described by equations of the form:

$$x_1 = 0 \,, \qquad \mathfrak{C}(x_2, x_3, t) = 0 \quad \text{or} \quad \mathfrak{C}(x_2, x_3, x_4) = 0. \qquad (5)$$

In space-time, the geometrised image of the evolution of this diaphragm will be a three dimensional screen parallel to the x_2, x_3 and x_4 axes, with a hole $\mathfrak{D}$ in it, according to equations (5). The simplest possible type of hole is limited, in the x_2, x_4 and the x_3, x_4 planes by two time-like semi-contours, intersecting each other in an initial I and a final F space-time point; the vector $F-I$ being time-like, the x_4 axis may be taken along it.

Thus, it is clear that the time-diffraction of the incident wave, due to the opening and shutting of the

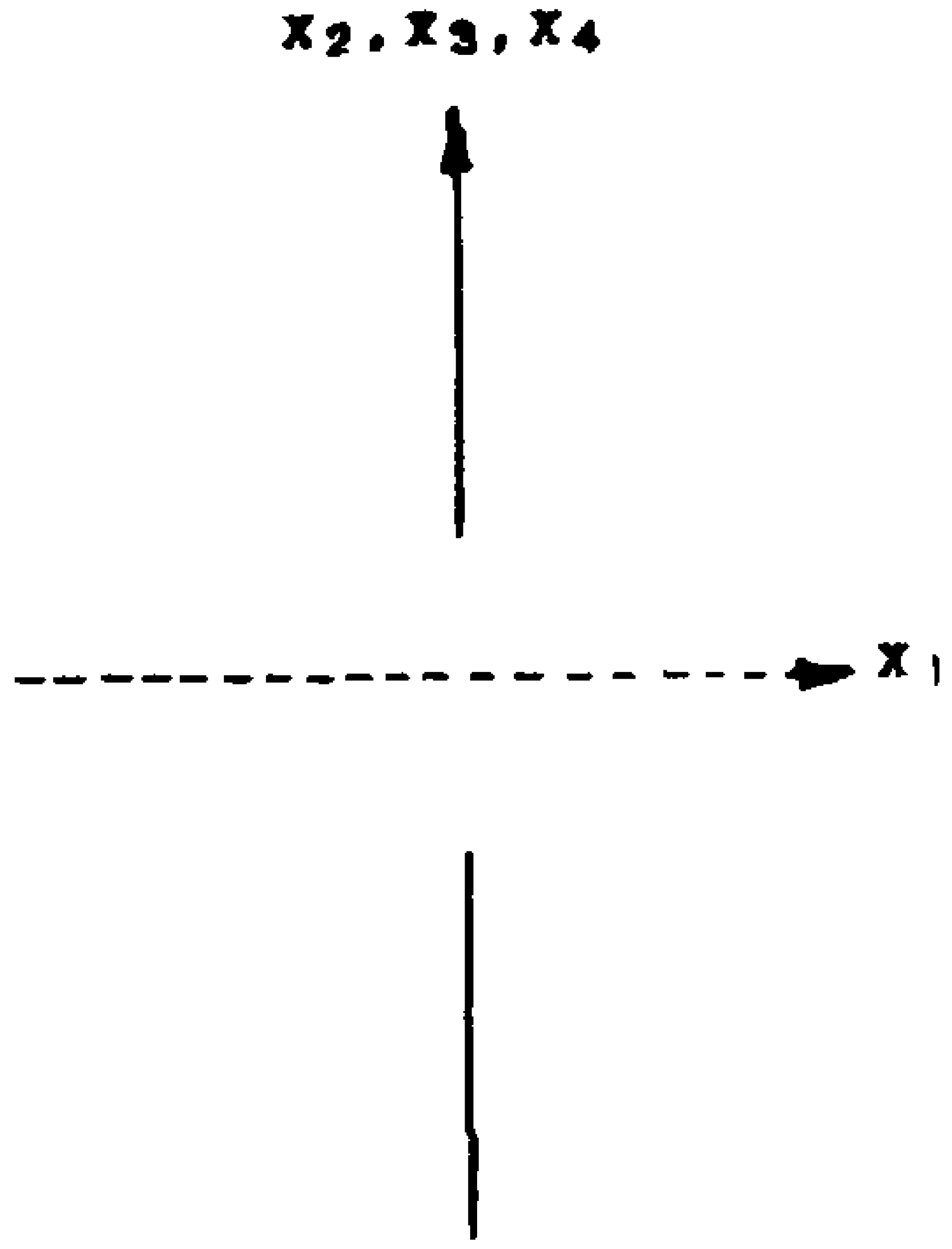

Fig.2. Scheme of the 3-dimensional screen.

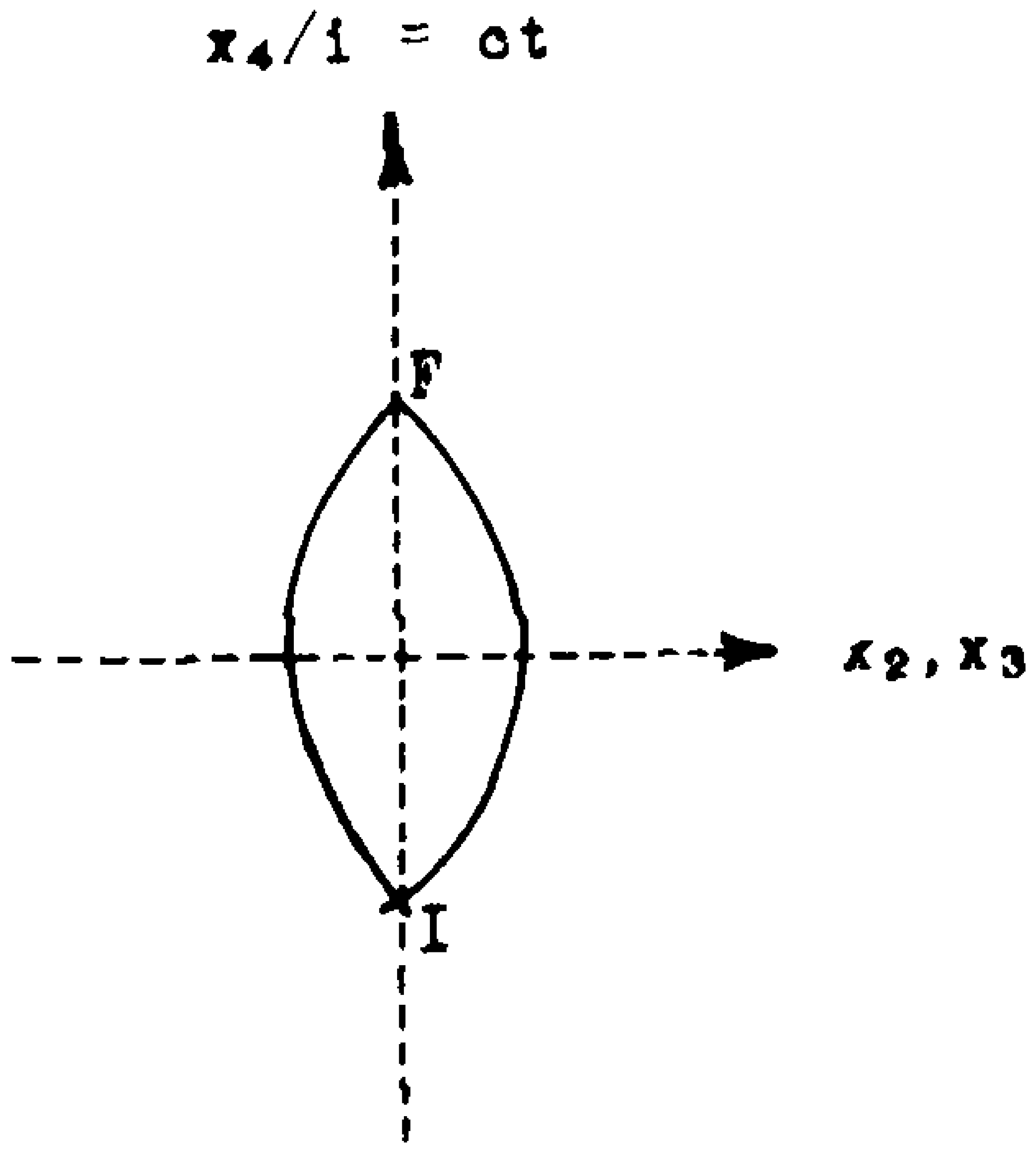

Fig.3. Scheme of the 3-dimensional hole.

diaphragm, is equivalent to its space-diffraction, due to the limited dimensions of the hole. The 2nd., 3rd. and 4th. uncertainty relations will come out in a perfectly symmetrical way. Of course, the variation of momentum and energy of the particle caused by this space and time diffraction must be given or taken by the heavy macroscopic screen and its powerful diaphragm-mechanism.

If we select the cases where the incident particle effectively falls in the open hole $\mathfrak{D}$, the normalization condition (4) becomes:

$$\iiint_{\mathfrak{D}} j_1 \, dx_2 dx_3 dt = 1 \quad . \tag{6}$$

Once the effect of the absorbing screen is taken into account, the equation of the diffracted wave in the plane $x_1 = 0$ reads:

$$\psi(x_1 = 0, x_2, x_3, x_4) = \begin{cases} 0 \quad \text{outside of } \mathfrak{D}, \\ \psi_0(x_1 = 0, x_2, x_3, x_4) \ \text{inside of } \mathfrak{D}; \end{cases} \tag{7}$$

ψ_0 represents the primitive incident wave.

It becomes already clear that, in our problem, the natural evolution parameter is not t or x_4, but x_1. Along the increasing values of this parameter, we will speak of "before" and "after" the screen, for instance, always using then quotation marks.

II. COVARIANT THEORY OF FOURIER TRANSFORMS.

The corresponding formulae have been derived by the author[9] following an initial hint by Marcel Riesz[3]; a complete deduction has been given recently[11] and is here summarized.

A.- *Formulae associated with the second order Gordon equation.*

$\mathfrak{H}$ meaning the two sheet hyperboloid in momentum 4-space:

$$k^\lambda k_\lambda + k_0^2 = 0 \quad , \tag{8}$$

$i\delta v_\lambda$ the volume element on it (which is colinear to k_λ)

and δv (>0) the lenght of $i\delta v_\lambda$ according to the definition*:

$$k_0 \delta v_\lambda = k_\lambda \delta v \quad . \tag{9}$$

The direct covariant Fourier transform of the wave function reads

$$(2\pi)^{3/2} \psi(x) = \iiint_{\mathfrak{H}} e^{ik^\nu x_\nu} \zeta(k) \delta v \quad . \tag{10}$$

The inverse covariant Fourier transform of ψ reads

$$(2\pi)^{3/2} \zeta(k) = \frac{i}{2k_0} \iiint_{\mathfrak{S}} \epsilon(k) e^{-ik^\nu x_\nu} (ik^\lambda + \partial^\lambda) \psi(x) \delta u_\lambda ; \tag{11}$$

$\mathfrak{S}$ meaning any space-like surface in space-time, $\epsilon(k)$ the well-known sign commutator

$$\epsilon(k) = \frac{k_4/i}{|k_4/i|} \quad , \tag{12}$$

$i\delta u_\lambda$ the volume element on $\mathfrak{S}$. One verifies that, according to Gordon's equation

$$(\partial_\lambda^\lambda - k_0^2) \psi(x) = 0 \quad \text{or} \quad (k^\lambda k_\lambda + k_0^2) \zeta(k) = 0 , \tag{13}$$

$\zeta(k)$ is defined by (11) independently of $\mathfrak{S}$**.

*With $\lambda = 4$, one recognizes the well-known relativist invariant $id^3\vec{k}/k_4$.

**Write the difference of two expressions (11) with two different $\mathfrak{S}$'s, and convert in a 4-fold integral.

The presence of the normal derivative $\partial^\lambda \psi \delta u_\lambda$ in (11) was of course expected.

Carrying (11) in (10) gives the formal solution of - Cauchy's problem:

$$\psi(x') = \iiint_{\mathfrak{S}} S^\lambda(x'-x)\ \psi(x)\ \delta u_\lambda(x) \quad , \tag{14}$$

where:

$$S^\lambda(x) = D(x)(ik^\lambda + \partial^\lambda) \quad , \tag{15}$$

and $D(x)$ is the propagation function, first defined by Stuckelberg[3]:

$$D(x) = \frac{i}{2k_0(2\pi)^3} \iiint_{\mathfrak{H}} \epsilon(k)\ e^{ik^\nu x_\nu}\ \delta v \quad ; \tag{16}$$

this D, equal to $D_{ret}-D_{adv}$, is nil outside the light cone. The above formulae (14) to (16) are substantially the same as those given by Schwinger[5] for the photon.

$[\partial^\lambda] = \overrightarrow{\partial}^\lambda - \overleftarrow{\partial}^\lambda$ meaning the well-known operator appearing in Schrodinger's 3-current and Gordon's 4-current, and $\overline{\psi}$ or $\overline{\zeta}$ the complex conjugate of the scalar ψ or ζ, Parseval's covariant equality reads:

$$\frac{i}{2k_0} \iiint_{\mathfrak{S}} \overline{\psi}_1 [\partial^\lambda] \psi_2 \delta u_\lambda = \iiint_{\mathfrak{H}} \epsilon(k) \overline{\zeta}_1 \zeta_2\ \delta v \quad , \tag{17}$$

Of course, according to the continuity equation, the left-hand side in (17) is independent of $\mathfrak{S}$.

Putting $\zeta_1 = \zeta_2 = \zeta$, the right-hand side of (17) is the so-called *distribution function* of 4-momentum, and putting:

$$\zeta_1 = \zeta \quad , \qquad \zeta_2 = \zeta \, e^{ik^\nu y_\nu} \quad , \qquad (18)$$

one obtains the so-called *characteristic function* of 4-momentum:

$$\frac{i}{2k_0} \iiint_{\mathfrak{S}} \bar{\psi}(x) [\partial^\lambda] \psi(x+y) \delta u_\lambda = \iiint_{\mathfrak{H}} \epsilon(k) \bar{\zeta} \, \zeta e^{ik^\nu y_\nu} \, \delta v. \qquad (19)$$

B. ***Formulae associated with the first order spinning particle equation.***

The Gordon equation (13) is always a consequence of the spinning particle equation:

$$(a^\lambda \, \partial_\lambda + k_0)\psi(x) = 0 \quad \text{or} \quad (a^\lambda \, k_\lambda - ik_0)\zeta(k) = 0 \, . \qquad (20)$$

Using (20) and (9), one may transform (10) into:

$$(2\pi)^{3/2} \, \psi(x) = -i \iiint_{\mathfrak{H}} a^\lambda \delta v_\lambda \, e^{ik^\nu x_\nu} \zeta(k) \, . \qquad (21)$$

Of much wider interest is the transformation of (11): replacing in it $\partial^\lambda \psi$ according to (20), one obtains[11], with the Dirac equation:

$$(2\pi)^{3/2} \, \zeta_{(1/2)}(k) = \frac{i}{2k_0} \iiint_{\mathfrak{S}} \epsilon(k) e^{-ik^\nu x_\nu} (i\gamma^\mu k_\mu - k_0) \gamma^\lambda \psi(x) \delta u_\lambda \, , \qquad (22)$$

and, with the Duffin-Kemmer equation:

$$(2\pi)^{3/2}\ \zeta_{(1)}(k) = \frac{i}{2k_0} \iiint_{\mathfrak{S}} \epsilon(k) e^{-ik^\nu x_\nu} [ik^\lambda + ik_\mu(\beta^\mu\beta^\lambda - \beta^\lambda\beta^\mu) - k_0\beta^\lambda]\psi(x)\,\delta u_\lambda; \quad (23)$$

instead of the normal derivative, one has now a linear combination of the ψ components. The corresponding expressions for $S^\lambda(x)$ in (14) are:

$$S^\lambda_{(1/2)}(x) = (\gamma^\mu \partial_\mu - k_0)\ \gamma^\lambda\ D(x)\ , \quad (24)$$

$$S^\lambda_{(1)}(x) = [\partial^\lambda + \partial_\mu(\beta^\mu\beta^\lambda - \beta^\lambda\beta^\mu) - k_0\ \beta^\lambda]\ D(x)\ ; \quad (25)$$

(24) is the expression given by Schwinger[6] in the electron case.

Both sides of the Parseval equality (17) may be transformed[11] according to the Gordon decomposition of the 4-current*, and there comes $(\overline{\psi} = \psi^+\beta)$:

$$\iiint_{\mathfrak{S}} \overline{\psi}_1\ a^\lambda\ \psi_2\ \delta u_\lambda = \iiint_{\mathfrak{H}} \epsilon(k)\overline{\zeta}_1\ a^\lambda\ \zeta_2\ \delta v_\lambda\ . \quad (26)$$

The corresponding form of the characteristic funtion (19) is easily written; it has also been derived in another way by - the author[7].

*The total current $\overline{\psi}\ a^\lambda\psi$ and the Gordon current $-\overline{\psi}[\partial^\lambda]\psi/2k_0$ are integrally equivalent in the sense of equations (17) and (26), for their difference has the form $\partial_\mu\ m^{\lambda\mu}$, where $m^{\lambda\mu}$ is a skew symmetric tensor; when integrated, this last term may be transformed in a 2-fold vanishing contour integral.

III. SPACE-TIME FRESNEL THEORY.

According to the hypotheses and definitions of Section I, all the formulae of Section II will still hold with the space-like surface $\mathfrak{S}$ replaced by the time-like surface $\mathfrak{D}$ of Section 1. The momentum amplitude of the diffracted wave will be given by formulae (7), and (11), (22) or (23) and its space-time amplitude by formulae (7), (14) and (15), (24) or (25)*.

As was said above, the natural evolution parameter in our problem is the x_1 coordinate and we shall use quotation marks every time we speak of it in terms of time. Given, on the surface $\mathfrak{D}$ of the space-time hole, the "initial" distribution (7), one has the following four classes of diffracted plane waves:

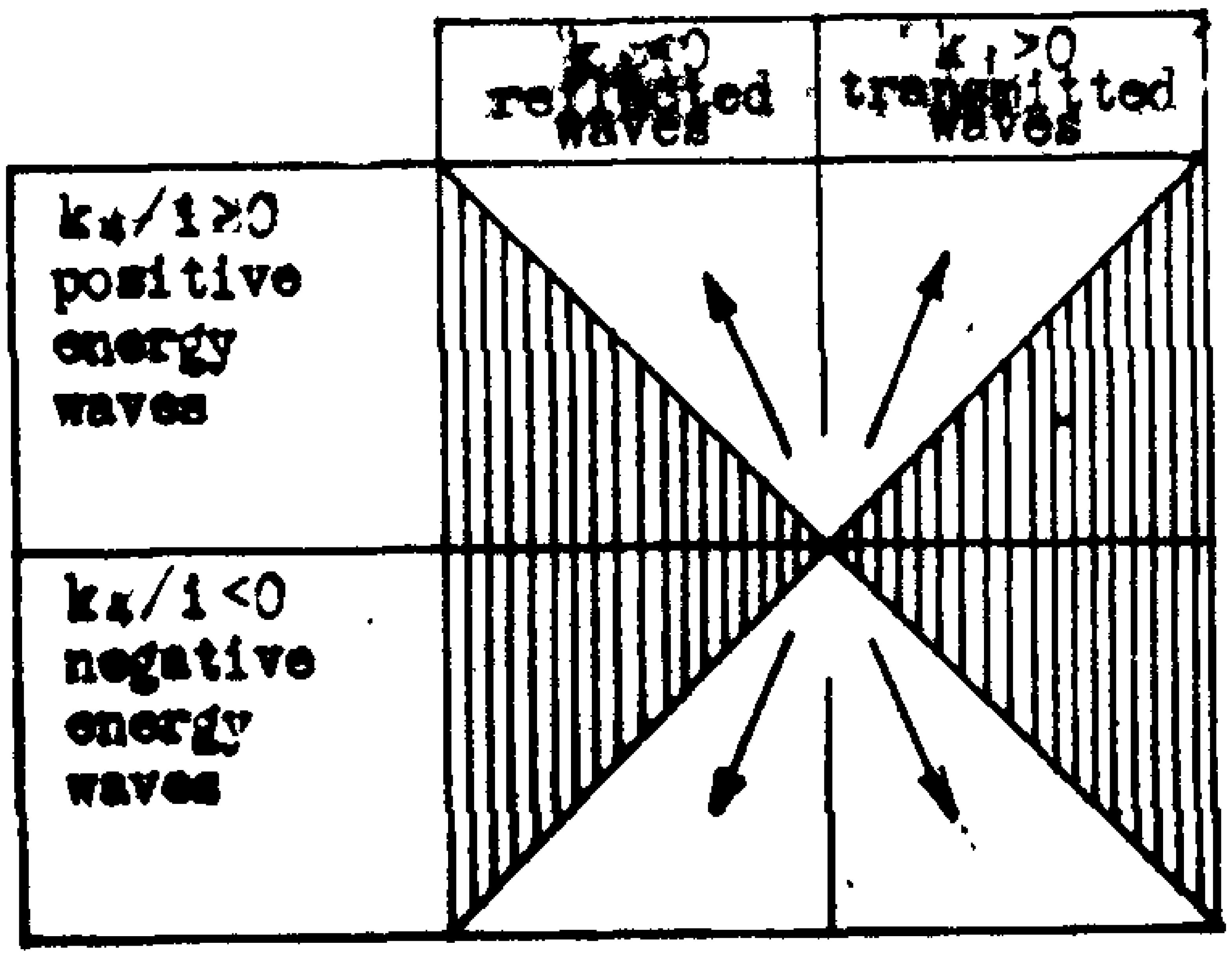

Fig.4 The four types of diffracted waves.

*Formulae (17) or (26) written with $\mathfrak{D}$ instead of $\mathfrak{S}$ (or equivalently, formula (4)), show that the Fresnel criterion $|\psi|^2$ for the *intensity* of a wave, is an over-simplification of the right formulae; this remains true if one uses the momentum or energy flux per surface and time units, rather than the particle number flux.

This symmetry with regard to the $k_1 = 0$ plane strongly suggests that this direct application of the formulae corresponds to the case of a perfectly reflecting screen; in fact this was demonstrated in a particular case by Dr. Moshinsky[10], at least as regards the $x_1 > 0$, $x_4/i > 0$ region; but the whole problem deserves some more reflection.

Here, we are interested in the case of a perfectly absorbing screen. In Fresnel's static case of a screen with a hole in it, the reflected $k_1 < 0$ waves are well known to exist formally, but not really. one must simply rub them out, and so will we do here. Thus, in formulae (11), (22) or (23), we assume that only the $k_1 > 0$ plane waves have a not-nil amplitude.

Now, the above quoted formulae will give, in all space-time, an expression for the diffracted ψ. This, in the $x_1 > 0$ region, will represent the proper "transmitted" wave. In the $x_1 < 0$ region, it will represent a fictitious "ingoing" wave, differing from the real "incident" wave by the suppression of all the Fourier components that the screen annihilates. Given the "initial" distribution (7), with the above restriction to $k_1 > 0$ plane components, this ψ will correspond in the $x_1 > 0$ region to a "prediction" problem, and in the $x_1 < 0$ region to a "retrodiction" problem.

Let us consider now the future and past light cones respectively associated with the space-time points I and F of Fig. 3; the region o, outside the two light cones, is forbidden to any particle going through the hole $\mathfrak{D}$. At any point of the regions 1 and 3 situated far enough (in space and time) from $\mathfrak{D}$, the influence of one particular possitive energy plane wave is largely predominating, as is shown by

Fresnel's argument of stationary phase. Similarly, at a

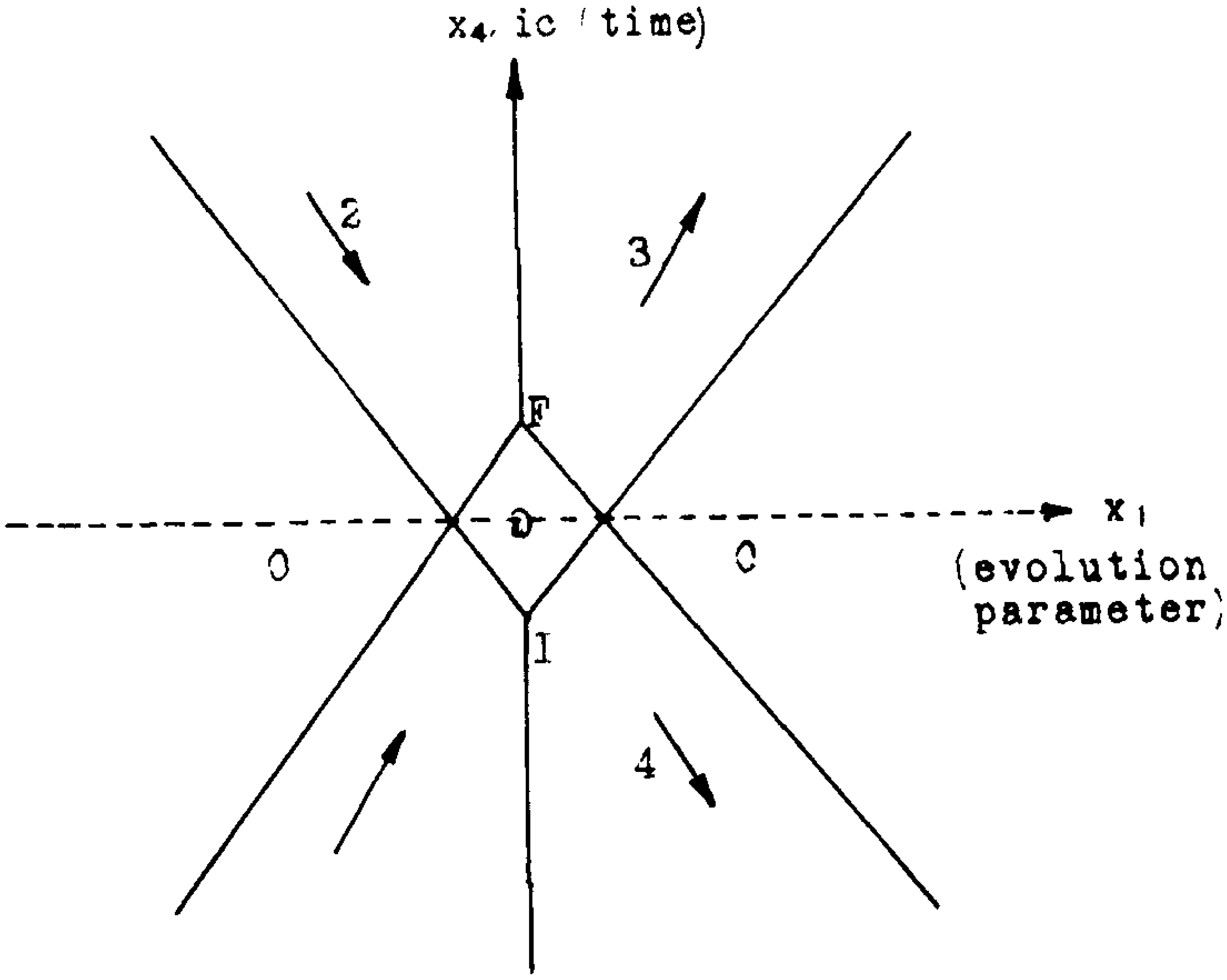

Fig. 5

distant point of the regions 2 and 4 the influence of a negative energy plane wave is largely predominating.

Now comes the problem of the negative energy states interpretation. It is obvious that the association of negative energy plane waves with the "macroscopic" ideas concerning the identificability of particles will entail the paradox of an advanced causality: the operation of the shutter mechanism will telegraph into the past, and call from there negative energy particles.

The solution of the paradox lies essentially in the use of the Fermi-Dirac or the Bose-Einstein statistics. Let

$$P_{12} = P_{21} \tag{27}$$

be the probability associated by the formulae of Section II with the transition between an "initial" state 1 to a "final" state 2, which is always equal to that of the reciprocal transition "2 → 1". In the case of a one particle transition, this probability must be superquantized according to

$$n_1 P_{12} n_2 = n_2 P_{21} n_1 \qquad (28)$$

where n_1 is the "initial" occupation number of the "initial" state 1 and n_2 the "final" occupation number of the "final" state 2 in the hypothesis of the transition taking place. In Fermi-Dirac statistics, $n_1, n_2 = 0, 1$, and, for instance, $n_2 = 0$ if the "final" state 2 is "initially" occupied. In the Bose-Einstein statistics, $n_1, n_2 = 0, 1, 2, \ldots$

In other words, it is essentially the new feature of quantum statistics, i.e. the explicitly symmetric role of the occupation numbers of the initial and the final state, which bars out the paradox of advanced causality. The association of the concept of negative energy waves with Maxwell-Boltzmann statistics would be paradoxical. For a relativistic presentation of quantum theory, quantum statistics are no less unavoidable than the representation of spin; perhaps this is one more argument in favor of the essential connexion between spin and quantum statistics.

The above interpretation is, of course, the Feynman[6] interpretation of negative energy states. One has 4 subcases: 1, ordinary diffraction of an electron; 2, annihilation of a pair; 3, creation of a pair; 4, ordinary diffraction of a positron. In Feynman's papers, the occupation numbers are not explicitly introduced but in the fermion case, there is the word *if*: P_{12} is the transition probability *if* an

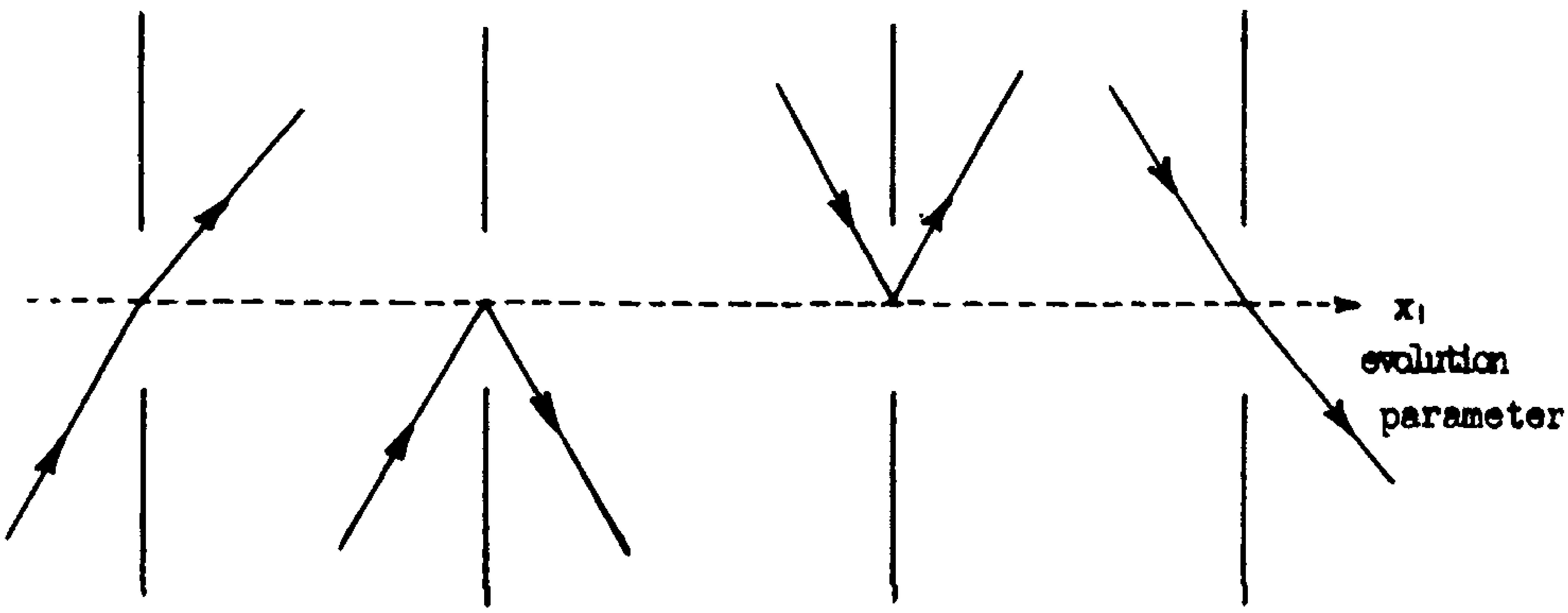

Fig. 6. Four types of electron diffraction.

initial state is initially occupied or *if* a final state is initially empty (initial and final, in the ordinary sense or, as well, in the "x_1 sense"); the symmetric assertions hold in a retrodiction problem.

IV. REMARK: TIME GRATING.

Any heavy periodical physical system, for instance an oscillator or a rotator, may be considered as a time-like grating in Minkowski's space-time.

In an ordinary grating, let k_0 denote the number of lines per lenght unit, and k_i the incident wave number in the same direction (the wave planes, of course, must be parallel to the grating lines). The diffracted wave numbers in the same direction are given by:

$$k_d = k_i \pm n\, k_0 , \qquad n = 0, 1, 2, \ldots \tag{29}$$

This holds also with a time like grating, and is identical, for instance, with Fermi's formula for the diffraction of a wave by a rotator. A typical physical example would be that of the Raman effect in molecular rotation spectra, with very heavy moleculas.

V. CONCLUSION.

As was clearly postulated in De Broglie's wave-mechanics of 1925[1] the Minkowski symmetry between space and time is an essential feature of quantum phenomena; this is no surprise, if one notices that the wave theory is included at the very basis of Relativity Theory on the one hand and of the Quantum and Complementarity Theory on the other hand. But, to formulate explicitly this essential symmetry between space and time some more information than that known in 1925 was requested. Anyhow, a quite elementary approach to covariant Quantum Theory (which has been so brilliantly developed by Tomonaga, Schwinger, Feynman, Dyson), is possible.

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